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# Miller's theorem for cell-like embedding relations

by

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**Abstract.** Let $G$ be an uppersemicontinuous cell-like decomposition of a boundaryless manifold $M^n$ ($n \geqslant 5$) and $g$ be the identification map. If we denote the inverse of $g$ by $R$, then $R$ is a relation which assigns a cell-like set to each point of the decomposition space. J. W. Cannon called $R$ a cell-like embedding relation. We obtain a generalization of the approximation theorem of R. T. Miller for embeddings of codimension three disks to a theorem for cell-like embeddings of codimension three disks. We give applications to decomposition space theory.

**0. Introduction.** Much progress in the study of decompositions of manifolds resulted from J. W. Cannon's novel idea of studying decompositions of manifolds "in reverse". Suppose $M^n$ is a topological $n$-manifold ($n \geqslant 5$) without boundary and $G$ is an uppersemicontinuous cell-like decomposition of $M^n$. Cannon considered the inverse relation $\pi^{-1}\colon (M^n/G) \to M^n$. The image of each point, $\pi^{-1}(y)$, is a cell-like set; also if $x \neq y$ then $(\pi^{-1}(x) \cap \pi^{-1}(y)) = \emptyset$. Appropriately, Cannon called these objects *cell-like embedding relations* and noted that they in many respects like functions. He developed this idea into a theory; he used an approach in which results for functions are generalized to results for cell-like relations ([Ca$^1$, Appendix I]). This theory has been quite fruitful. F. Ancel and Cannon exploited it in using Stanko's process ([St$^2$]) to prove a 1-LCC approximation theorem for embeddings of codimension one manifolds ([An$^1$] and [An-Ca]). D. L. Everett also used this notion in obtaining embedding and product theorems for cell-like decompositions ([Ev]).

At the same time Cannon was aware of a close relationship between taming theory for embeddings and decompostion space theory. This meant that the 1-LC property, which is crucial for taming embeddings, would be quite important also. Cannon generalized the 1-LC taming theorem for embeddings of $S^{n-1}$ in $S^n$ to obtain the following:

THEOREM ([Ca$^1$, Theorem 55]). *If $R$: $S^{n-1} \to S^n$ is a cell-like embedding relation such that $S^n - R(S^{n-1})$ is 1-LC at each point-image of $R$, then* $R$

*extends to a cell-like embedding relation $R^*\colon S^n \to S^n$ with $(R^*|(S^n - S^{n-1}))$ a function.*

He then used $R^*$ to shrink the corresponding decomposition of $S^n$.

Ultimately, Cannon conjectured that a necessary and sufficient condition for $M^n$ and $M^n/G$ to be homeomorphic is that $M^n/G$ satisfy the *disjoint disk property* (singular 2-disks in $M^n/G$ can be adjusted slightly so that they do not intersect). This condition is obviously necessary. Cannon showed it to be also sufficient in the case where the nonmanifold part of $E^n/G$ has dimension $\leqslant n-3$ provided the following is true:

1-LCC *shrinking conjecture* ([Ca³]): Let $X$ be a locally-compact separable metric space of dimension $\leqslant n-3$, let $R\colon X \to E^n$ denote a 1-LCC cell-like embedding relation onto a closed subset of $E^n$, and let $G_R$ denote the decompostion of $E^n$ whose nondegenerate elements are the nondegenerate point images of $R$. Then $E^n/G_R$ and $E^n$ are homeomorphic.

We showed ([Ti]) that this shrinking conjecture is true if either $(2\cdot\dim(X))+2 \leqslant n$ or $X$ is a polyhedron. Our method was first to generalize the following taming theorems of J. L. Bryant — C. L. Seebeck and Bryant for embeddings:

THEOREM ([Br-Se²], Theorem 2). *Suppose $f\colon D^k \to E^n$ is a 1-LCC embedding with $n \geqslant 5$ and $k \leqslant n-3$. Then $f$ extends to a homeomorphism $f^*\colon E^n \to E^n$.*

THEOREM ([Br¹, Theorem 1]). *Suppose $X \subset E^n$ with the inclusion a 1-LCC embedding and with $\dim(X) = k$, $2k+2 \leqslant n$, and $n \geqslant 5$. Suppose also that $f\colon X \to E^n$ is a 1-LCC embedding. Then $f$ extends to a homeomorphism $f^*\colon E^n \to E^n$.*

Our versions follow:

THEOREM 4.1. *Suppose $R\colon D^k \to E^n$ is a 1-LCC cell-like embedding relation with $k \leqslant n-3$ and $n \geqslant 5$. Then $R$ extends to a cell-like embedding relation $R^*\colon E^n \to E^n$ with $(R^*|(E^n - D^k)$ a function.*

THEOREM 4.2. *Suppose $X \subset E^n$ with the inclusion a 1-LCC embedding and with $\dim(X) = k$, $2k+2 \leqslant n$, and $n \geqslant 5$. Suppose also that $R\colon X \to E^n$ is a 1-LCC cell-like embedding relation. Then $R$ extends to a cell-like embedding relation $R^*$ with $(R^*|(E^n - X))$ a function.*

We then used $R^*$ to shrink the decomposition.

A crucial step in the proof given by Bryant-Seebeck was the PL approximation theorem which Miller subsequently proved.

THEOREM ([Mi³, Theorem 1]). *Suppose $f\colon D^k \to E^n$ is an embedding with $k \leqslant n-3$. Then for each $\varepsilon > 0$ there is a PL embedding $g\colon D^k \to E^n$ with $d(f, g) < \varepsilon$.*

In our proof of Theorem 4.1 it was necessary to generalize Miller's theorem to the cell-like embedding relation case.



THEOREM 3.0. *Suppose $R\colon D^k \to E^n$ is a cell-like embedding relation with $k \leqslant n-3$. Then for each $\varepsilon > 0$ there is a PL embedding $g\colon D^k \to E^n$ with $g \subset \varepsilon \circ R \circ \varepsilon \subset D^k \times E^n$ (think of $\varepsilon \circ R \circ \varepsilon$ as being a small neighborhood of $R$ in $D^k \times E^n$).*

Shortly after the announcement of these results, Edwards ([Ed]) proved the disjoint disk property to be a sufficient condition for $M^n$ being homeomorphic to $M^n/G$ (with no restriction on the dimension of the nonmanifold set). A key step in his proof is the verification of the 1-LCC shrinking conjecture for any codimension three set $X$. However, he notes that our Theorem 4.2 suffices for $n \geqslant 6$.

Also, subsequent to our announcement, Cannon, Bryant, and R. C. Lacher solved the trivial range resolution problem for generalized manifolds and in the process gave an alternate proof of Theorem 4.2 ([Ca-Br-La]).

Thus, we shall restrict our attention to Theorem 3.0. Interesting in itself, this theorem depends on neither the results of Edwards nor Cannon, et al. In addition, Cannon shows how it can be used (as part of Theorem 4.1) in his proof of the Double Suspension Problem ([Ca²]). Finally, it is a good illustration of how well the theory of cell-like relations works, even in a rather complicated situation.

**1. Basic definition and theorems.** Let $E^n$ denote Euclidean $n$-space with the PL metric (derivated from the "sup" norm). Define $D^n = ([0, 1])^k = ([0, 1])^k \times \{0\}^{n-k} \subset E^n$. For any subset $X$ of $E^n$, $N_\varepsilon(X)$ is the open $\varepsilon$-neighborhood of $X$ with its closure abbreviated by $N_{\bar{\varepsilon}}(X)$.

A relation $R\colon X \to Y$ is a subset of $X \times Y$. Define the point-image $R(X) = \{y \in Y|\ (x, y) \in R\}$. The inverse of $R$ is denoted by $R^{-1} = \{(y, x) \in Y \times X|\ (x, y) \in R\}$. A relation is continuous if the inverse of each closed set is closed; it is proper if the inverse of each compact set is compact. We record the useful fact that a closed subset of a proper, continuous relation is continuous and proper. Also, if $R\colon X \to Y$ and $S\colon Y \to Z$ are two relations their composition, $S \circ R\colon X \to Z$, is defined naturally as $\{(x, z) \in X \times Z|\ (x, y) \in R$ and $(y, z) \in S$ for some $y \in Y\}$. Finally, we note that the composition of continuous relations is continuous.

Our goal is to generalize several theorems about functions to theorems about relations. Using the fact that any locally compact, finite-dimensional metric space embeds in some Euclidean space, we can restrict our attention to relations $R\colon E^m \to E^n$ (some $R(X)$'s may be empty). A topology on these is gotten from neighborhoods in $E^m \times E^n$. Of particular interest are "$\varepsilon$-neighborhoods." With each map $\varepsilon\colon E^n \to (0, \infty)$ is associated the relation $\varepsilon\colon E^n \to E^n$ where $\varepsilon = \{(x, y) \in E^n \times E^n|\ y \in N_{\varepsilon(x)}(\{x\})\}$. The closure of $\varepsilon$ in $E^n \times E^n$ is denoted by $\bar{\varepsilon}$. Thus if $\varepsilon'\colon E^m \to (0, \infty)$ is another map, then the composition $\varepsilon \circ R \circ \varepsilon'\colon E^m \to E^m \to E^n \to E^n$ is an open neighborhood of $R$. In particular if $R$ has both compact point-images and point-preimages, then for any neighborhood $U$ of $R$ in $E^m \times E^n$, there exist $\varepsilon$, $\varepsilon'$ such that $\varepsilon \circ R \circ \varepsilon' \subset U$; if $R$ itself is compact, then $\varepsilon$ and $\varepsilon'$ may be chosen to be the same constants. When written as part of a composition (e.g., $\varepsilon \circ R \circ \varepsilon$) a positive real will always represent the associated relation.

The following oft-used theorem (not stated here in its full generality) provides a clean method of handling the details of relations proofs.

THEOREM 1.1 ([Ca[1], Theorem A12], the Composition Theorem). *Suppose $R\colon E^m \to E^n$ and $S\colon E^n \to E^q$ are continuous, proper relations, both with compact point-images. Then if $U$ is any neighborhood of $S \circ R$ in $E^m \times E^q$, there exist neighborhoods $V$ of $R$ in $E^m \times E^n$ and $W$ of $S$ in $E^n \times E^q$ such that the composition $W \circ V \subset U$. If either of $R$ or $S$ is also compact, then there exists a constant $\delta > 0$, such that $(\delta \circ S \circ \delta) \circ (\delta \circ R \circ \delta) \subset U$.*

The support of a relation $R\colon E^m \to E^n$ (Support $(R)$) is the set of all $x \in E^m$ with $R(x)$ non-empty. The image of $R$ (Image $(R)$) is the union of all point-images of $R$. We say $R$ is *cell-like* if for each $x \in \text{Support}(R)$ $R(x)$ is compact and contracts in each neighborhood of itself. The following theorem is well-known in the decomposition space context. We state it in a form convenient for this paper.

THEOREM 1.2 ([Ca[1], Chapter II, Theorem 14], the Approximation Theorem for cell-like relations). *Suppose $R\colon E^m \to E^n$ is a continuous cell-like relation with* Support$(R) = X$, *a compactum. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any closed subset $C$ of $N_{\bar{\delta}}(X)$ and map $f\colon C \to E^n$ with $f \subset \bar{\delta} \circ R \circ \bar{\delta}$, there is a map $g\colon N_{\bar{\delta}}(X) \to E^n$ which extends $f$ such that $g \subset \varepsilon \circ R \circ \varepsilon$.*

Since many of the applications are made in the case where Support$(R)$ $= X \times I$, we state the following corollary.

COROLLARY 1.2. *Suppose $R\colon E^m \times I \to E^n$ is a continuous cell-like relation with* Support$(R) = X \times I$ *for some compactum $X$. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any closed subset $C$ of $(N_{\bar{\delta}}(X)) \times I$ and map $F\colon C \to E^n$ with $F \subset \bar{\delta} \circ R \circ \bar{\delta}$ there is a map $G\colon (N_{\bar{\delta}}(X)) \times I \to E^n$ which extends $F$ such that $G \subset \varepsilon \circ R \circ \varepsilon$.*

A cell-like embedding relation $R\colon E^m \to E^n$ is a cell-like relation with closed image in $E^n$ such that $R(x) \cap R(y) = \emptyset$ for all $x \neq y$, $\{x, y\}$ in Support$(R)$. The link to decomposition theory is that $R$ yields a corresponding uppersemicontinuous cell-like decomposition of $E^n$ with nondegenerate elements precisely the nondegenerate point-images of $R$ and with the identification map $\pi$ obtained naturally from $R^{-1}$. The saturation relation, $R \circ R^{-1}\colon E^n \to E^n$, is a cell-like relation containing (as a subset) the inclusion map of Image$(R)$ into $E^n$.

**2. Piecewise-linear PL topology.** We assume an understanding of regular neighborhood theory and general position. A reference is [R-S]. We will explain any notation which may be non-standard.



This discussion of collapsing polyhedra serves as a preface for the next section. Some of it taken directly from Miller's paper ([Mi[3]]) so we can model our proofs after his. However, the material is quite standard.

Suppose a complex $Y$ in $E^n$ collapses by a sequence of elementary simplicial collapses to a subcomplex $X$. Let $\sigma$ be a simplex of $Y$ that collapses through a face $\tau$. Subdivide $\sigma$ by starring at $b(\tau)$, the barycenter of $\tau$. Call this subdivision $\sigma'$. Let $r$ be the simplicial retraction from $\sigma'$ onto $\operatorname{clos}((\operatorname{bdy}\sigma) - \tau)$ which maps $b(\tau)$ to the vertex of $\sigma$ opposite $\tau$. We obtain a PL deformation retraction as follows. Take the cell complex $\sigma' \times [0, 1]$ and subdivide to a simplicial complex, $K$, without adding any vertices ([R-S, Proposition 2.9]). Let $F_0^\sigma\colon \sigma' \times \{0\} \to \sigma'$ be the identity and $F_1^\sigma\colon \sigma' \times \{1\} \to \sigma'$ be $r$. The maps $F_0^\sigma$ and $F_1^\sigma$ determine a simplicial map $F^\sigma\colon K \equiv \sigma \times [0, 1] \to \sigma$. We use the $F^\sigma$'s in the same order that the simplexes collapse and then reparametrize to obtain a PL deformation retraction $F\colon Y \times [0, 1] \to Y$ which accomplishes the collapse.

Let $Y$, $X$, and $F$ be as above. We denote the collapse by $C$ with the understanding that $|C|$ is the underlying complex $Y$ and $C\colon X \times I \to Y$ is the deformation retraction $F$. If $Z$ is a subcomplex of $Y$ then define $\text{track}_C(Z)$ $= C(Z, [0, 1])$ and $\text{image}_C(Z) = C(Z, 1)$. By convention $C(t)$ will mean $C(Y, t)$, the image of $C$ at time $t$. Also, $C$ may be denoted $Y \searrow X$. Also define the boundary of $C$ ($\operatorname{bdy}|C|$) to be the collection of "free faces" of $|C|$, those simplices $\tau$ of $|C|$ with star $(\tau, |C|) = \sigma$ where $\sigma$ collapses through $\tau$.

It is well-known that a collapse in a PL manifold $M$ gives rise to an ambient isotopy of $M$ which "follows" $C$. We use the following theorem of Miller which is stated in both [Mi[2]] and [Mi[3]] and proved in [Mi[1]]. The $r$'th derived subdivision of a complex $Q$ is written as $Q'^{\cdots\prime}$($r$ primes). Also, let $N(X, Q'^{\cdots\prime})$ be the simplicial neighborhood of $X$ in $Q'^{\cdots\prime}$.

THEOREM 2.1 (Mi[2], Proposition 1] or [Mi[3], Theorem 7]). *Suppose that $Q$ is a PL $q$-manifold and $Y$ and $X$ are subcomplexes of $Q$. If $Y$ collapses simplicially to $X$, then there exists an isotopy $\varphi$ of $N(Y, Q'')$ into itself with*

$$\varphi_1(N(Y, Q'')) = N(X, Q'')$$

*and with*

$$\varphi_t \text{ fixed outside } N\big(N(\text{vertici in } (Y-X)', Q'')Q'''\big)$$

*such that if $Z$ is a subcomplex of $Y$, then*

$$\varphi_1(N(Z, Q'')) \subset N(\text{image}_{Y\searrow X}(Z), Q'').$$

*If, in addition, $(Y-X)$ is in the interior of $Q$, then $\varphi_t$ extends to an ambient isotopy of $Q$ (also called $\varphi$) for which*

$$\varphi_t\big(N(N(Z, Q''), Q''')\big) \subset N\big(N(\text{track}_{Y\searrow X}(Z), Q''), Q'''\big)$$

*and where $\varphi_t$ is fixed on the same set as above.*

**3. Approximating codimension 3 cell-like embedding relations.** In this chapter we generalize the approximation theorem of Miller ([Mi$^3$, Theorem]) for embeddings of codimension three polyhedra to the cell-like embedding relation case. The new statement follows.

THEOREM 3.0 ([Mi$^3$, Theorem 1]). *Suppose $k \leqslant n-3$ and $R\colon D^k \to E^n$ is a cell-like embedding relation. Then for each $\varepsilon > 0$ there exists a* PL *embedding $p\colon D^k \to E^n$ with $p \subset \varepsilon \circ R \circ \varepsilon$.*

Our proof follows from a sequence of lemmas analogous to those of Miller. The essential difference is that we must express controls in terms of neighborhoods of relations rather than simple distances between functions. However, the use of $\varepsilon$-neighborhoods allows statements of the lemmas to be the same for both the relation and function cases.

Let $R\colon D^k \to E^n$ be a cell-like embedding relation where $D^k$ is included naturally in $E^n$. Let $\theta^j\colon D^k \times I \to D^k$ be the deformation retraction of $D^k$ along the $j$'th factor given by

$$\theta^j_t(x_1, \ldots, x_n) = \begin{cases} \text{identity} & \text{for } t \leqslant x_j, \\ (x_1, \ldots x_{j-1}, t, x_{j+1}, \ldots, x_k) & \text{for } t > x_j. \end{cases}$$

Define the cell-like relation $S^j$ by the composition

$$S^j = R \circ \theta^j \circ (R^{-1} \times \mathrm{id})\colon E^n \times I \to D^k \times I \to D^k \to E^n.$$

We wish to construct homotopies that are close in some sense to $S^j$. In other words, if a point $y$ in $E^n$ is close to a point-image $R(x)$ of $R$, then its track should stay close to the track of a nearby point-image $R(x')$ under $S^j$. Specifically, a proper homotopy $F\colon X \times I \to E^n$ is a $(j, \delta)$-homotopy if the composite relation

$$F' = F \circ (F_0^{-1} \times \mathrm{id})\colon E^n \times I \to X \times I \to E^n$$

is contained in $\delta \circ S^j \circ \delta$, the $\delta$-neighborhood of $S^j$ in $E^n \times I \times E^n$. A collapse $C$ in a PL submanifold $M$ of $E^n$ is a $(j, \delta)$-collapse if the associated deformation retraction $C\colon |C| \times I \to M \subset E^n$ is a $(j, \delta)$-homotopy.

LEMMA 3.1 ([Mi$^3$, Theorem 8]). *For each $\varepsilon > 0$ there exists a $\delta > 0$ so that if $M$ is a* PL *submanifold of $R^n$ triangulated with mesh $< \delta$, $C$ is a $(j, \delta)$-collapse in $M \cap N_\delta(R(D^j))$ with $|C|$ a subcomplex of the triangulation, and $p_{j-1}\colon D^{j-1} \to (\mathrm{bdy}\, N(|C|, M''))$ is a* PL *embedding with $p_{j-1} \subset \delta \circ (R|D^{j-1}) \circ \delta$ then there exists a* PL *embedding $p_j\colon D^j \to N(|C|, M'')$ with $p_j \subset \varepsilon \circ (R|D^j) \circ \varepsilon$.*

Proof. We use the same inductive construction of $p_j$ as Miller used. The idea is to let Theorem 2.1 "stretch" $p_{j-1}$ out along the tracks of $C$ to obtain $p_j$. A detailed description is included.

We choose $\gamma = 1/N$ for a large, positive integer $N$; we then choose a much smaller $\delta > 0$. Without loss, we can adjust the parametrization of $C$ so that $C(w\delta)$ is a subcomplex of $M$ for each integer $w$, $0 \leqslant w \leqslant N$.



We inductively construct a sequence of PL embeddings $G^j_w\colon D^{j-1} \times [-1, w\gamma] \to M$ which satisfy the following three conditions:

$(1_w)$ $\quad G^j_w(D^{j-1} \times \{w\gamma\}) = G^j_w(D^{j-1} \times [-1, w\gamma]) \cap (\mathrm{bdy}\, N(C(w\gamma), M''))$,

$(2_w)$ $\quad G^j_w((p_{j-1})^{-1}(N(\sigma, M'') \times \{w\gamma\}) \subset N(\mathrm{image}_{C \searrow C(w\gamma)}\, \sigma, M''))$ for each $\sigma \in |C|$,

$(3_w)$ $\quad G^j_w|(D^{j-1} \times [0, w\gamma])$ is an $\varepsilon$-approximation of $R|(D^{j-1} \times [0, w\gamma])$, i.e., $G^j_w|(D^{j-1} \times [0, w\gamma]) \subset \varepsilon \circ (R|(D^{j-1} \times [0, w\gamma]) \circ \varepsilon$.

First, $G^j_0$ is constructed to start the induction. Using regular neighborhood theory we obtain a PL homeomorphism

$$\mu\colon (\mathrm{bdy}\, N(|C|, M'')) \times [-1, 0] \to \mathrm{clos}(N(N(|C|, M''), M''') - N(|C|, M''))$$

such that $(\mu|(\mathrm{bdy}\, N(|C|, M'') \times \{0\})) = \text{identity}$. If we take

$$G^j_0 = \mu \circ (p_{j-1} \times \mathrm{id})\colon D^{j-1} \times [-1, 0] \to (\mathrm{bdy}\, N(|C|, M'')) \times I \to M$$

then it is easy to check that $(1_0)$, $(2_0)$, and $(3_0)$ hold.

Suppose $G^j_{w-1}$ has been constructed satisfying the three conditions. For an arbitrarily small $\gamma'$, property $(1_{w-1})$ and regular neighborhood theory allow us to choose a different third derived (leaving the first two alone) so that

$$G^j_{w-1}(D^{j-1} \times [-1, (w-1)\gamma]) \cap \\ \cap \mathrm{clos}\big(N(N(C((w-1)\gamma), M''') - N(C((w-1)\gamma), M'')\big)$$

is equal to $G^j_{w-1}(D^{j-1} \times [(x-1)\gamma - \gamma', (w-1)\gamma])$. We obtain a PL ambient isotopy $\varphi$ of $M$ satisfying the conclusion to Theorem 2.1 for the collapse $C((w-1\gamma) \searrow C(w\gamma)$ and the newly chosen third derived subdivision of $M$. Now suppose $h\colon D^{j-1} \times [-1, w\gamma] \to D^{j-1} \times [-1, (w-1)\gamma]$ is defined on the first factor by the identity and on the second factor by sending 0 to 0, $(w-1)\gamma$ to $((w-1)\gamma - \gamma')$ and $w\gamma$ to $(w-1)\gamma$ and then extending linearly. Thus, $h$ is a PL homeomorphism with

$$h(D^{j-1} \times [(w-1)\gamma, w\gamma]) = D^{j-1} \times [((w-1)\gamma - \gamma'), (w-1)\gamma].$$

We now complete the induction step by defining $G^j_w = \varphi \circ G^j_{w-1} \circ h$.

We need only check that the three conditions listed above hold for $G^j_w$. The first two follow easily from the properties of $\varphi$. We give the details for the third.

We check that $(3_w)$ holds. By choosing $\delta'$ very small, $(G^j_w|(D^{j-1} \times [0, (w-1)\gamma]))$ can be made arbitrarily close (pointwise) to $G^j_{w-1}$. Therefore we can assume that $(G^j_w|(D^{j-1} \times [0, (w-1)\gamma])) \subset \varepsilon \circ R \circ \varepsilon$. Thus, our goal is to show that $(G^j_w|(D^{j-1} \times [(w-1)\gamma, w\gamma])) \subset \varepsilon \circ R \circ \varepsilon$.

First, Condition $(2_w)$ implies that

$$G^j_{w-1}\big(x, [((w-1)\gamma-\gamma'), (w-1)\gamma]\big) \subset N\big(N((\mathrm{image}_{C\searrow C(w-1)\gamma}\,\sigma), M''), M'''\big)$$

for each $x \in p^{j-1}$ where $p_{j-1}(x) \in N(\sigma, M'')$ and $\sigma \subset |C|$. Also for $t \in [(w-1)\gamma, w\gamma]$ we have by definition that $G^j_w(x, t) = \varphi \circ G^j_{w-1} \circ h(x, t)$. Thus, by the above and Theorem 2.1 we have

$$\begin{aligned} G^j_w(x, t) &\subset \varphi\circ G^j_{w-1}\big(D^{j-1}\times[((w-1)\gamma-\gamma'), (w-1)\gamma]\big)\\ &\subset \varphi\circ\big(N(N(\mathrm{image}_{C\searrow C((w-1)\gamma)}\,\sigma), M''), M'''\big)\\ &\subset N\big(N((\mathrm{track}_{C((w-1)\gamma)\searrow C(w\gamma)}(\mathrm{image}_{C\searrow C((w-1)\gamma)}\,\sigma)), M''), M'''\big). \end{aligned}$$

We conclude by showing that for appropriate choices of $\gamma$ and $\delta$, the relation sending $(x, t)$ to the last set above is a subset of $\varepsilon\circ(R|D^j)\circ\varepsilon$.

Here, the style of proof is different from the embedding case. Since cell-like sets can be quite large, two collapses close to $S^j$ can be quite far apart (point-wise). We must handle the epsilontics using neighborhoods of relations. Since this is the first such proof, we include all the details.

These choices of $\gamma$ and $\delta$ are made through the use of the Composition Theorem (Theorem 1.1). Start by considering Diagram 3.1. It is easy to check that this diagram commutes. Note that all the relations are continuous with compact point images and proper so that we are in a position to use the Composition Theorem freely.

$$\begin{tikzcd}
D^{j-1}\times I \arrow[r] \arrow[d, "(\mathrm{id}\times\mathrm{id})"'] & E^n \\
D^{j-1}\times I \arrow[ur] \arrow[d, "((R|D^{j-1})\times\mathrm{id})"'] & \\
E^n\times I \arrow[d, "(\mathrm{id}\times\mathrm{id})"'] & \\
E^n\times I \arrow[r, "S^j"] & E^n \arrow[uuu, "(\mathrm{id})"']
\end{tikzcd}$$

**Diagram 3.1**

We begin with the top part of Diagram 3.1. Fix $\varepsilon > 0$; then by the Composition Theorem choose $\gamma, \varepsilon' > 0$ so that

$$\varepsilon'\circ(R|D^j)\circ\varepsilon'\circ\gamma\circ(\mathrm{id}\times\mathrm{id})\circ\gamma \subset \varepsilon\circ(R|D^j)\circ\varepsilon.$$

Similarly, the bottom Diagram 3.1 is completed by finding $\delta$, $0 < \delta < \gamma$, so that the composition of relations

$$\delta\circ(\mathrm{id})\circ\delta\circ 2\delta\circ S^j\circ 2\delta\circ 3\delta\circ(\mathrm{id}\times\mathrm{id})\circ 3\delta\circ\delta\circ((R|D^{j-1})\times\mathrm{id})\circ\delta$$

is contained in $\varepsilon'\circ(R|D^j)\circ\varepsilon'$.



We now describe a set of relations whose composition is contained in $\varepsilon\circ(R|D^j)\circ\varepsilon$ and contains the embedding $G^j_w$. This will complete the induction. Recall that the mesh of the triangulation of the manifold $M$ is smaller than the $\delta$ we just chose.

Diagram 3.1 depicts the pattern of the following choices; simply follow the arrows around the outside of the diagram.

First, define a relation $R_\gamma\colon D^{j-1}\times I \to D^{j-1}\times I$ by

$$R_\gamma(x, t) = (x, [(w-1)\gamma, w\gamma]).$$

We see that

$$(R_\gamma|D^{j-1}\times[(w-1)\gamma, w\gamma]) \subset \gamma\circ(\mathrm{id}\times\mathrm{id})\circ\gamma\colon D^{j-1}\times I \to D^{j-1}\times I.$$

For the second step we have that

$$(p_{j-1}\times\mathrm{id}) \subset \delta\circ((R|D^{j-1})\times\mathrm{id})\circ\delta\colon D^{j-1}\times I \to E^n\times I$$

since

$$\delta\circ((R|D^{j-1})\times\mathrm{id})\circ\delta = (\delta\circ(R|D^{j-1})\circ\delta)\times(\delta\circ(\mathrm{id})\circ\delta)$$

and since by hypothesis $p_{j-1} \subset \delta\circ(R|D^{j-1})\circ\delta$. For the next step recall that by hypothesis the image of $p_{j-1}$ lies in $\mathrm{bdy}(N(C, M''))$. Thus, for each $x \in D^{j-1}$ there is a simplex $\sigma_x \in |C|$ with $p_{j-1}(x) \in N(\sigma_x, M'')$. Also, the diameter of $N(\sigma_x, M'')$ is less than $3\delta$. So if $R_\sigma\colon E^n\times I \to E^n\times I$ is defined by

$$R_\sigma(x, t) = (\sigma_x, t) \quad \text{for} \quad x \in p_{j-1}(D^{j-1})$$

then

$$R_\sigma \subset 3\delta\circ(\mathrm{id}\times\mathrm{id})\circ 3\delta\colon E^n\times I \to E^n\times I.$$

For the fourth step, since $C(x, t) \subset \delta\circ S^j\circ\delta$ and $|C|$ has mesh $< \delta$ we can readjust the parametrization of $C$ so that $C(w\gamma)$ is a subcomplex $(0 \leqslant w \leqslant N)$ and $C \subset 2\delta\circ S^j\circ 2\delta$. A quick inspection shows that $C\circ R_\sigma\circ(p_{j-1}\times\mathrm{id})\circ R_\gamma(x, t)$ is equal to the set $\{\mathrm{track}_{C((w-1)\gamma)\searrow C(w\gamma)}(\mathrm{image}_{C\searrow C((w-1)\gamma)}\,\sigma_x)\}$. This fact, the fact that the mesh of the triangulated $M$ is less than $\delta$, and Diagram 3.1 imply that the relation from $D^{j-1}\times I$ to $E^n$ sending the point $(x, t)$ to the set

$$N\big(N(\mathrm{track}_{C(w-1)\gamma)\searrow C(w\gamma)}(\mathrm{image}_{C\searrow C((w-1)\gamma)}\,\sigma_x), M''), M'''\big)$$

is contained in $\varepsilon\circ(R|D^j)\circ\varepsilon$.

Lemma 1 is completed by setting $p_j = (G^j_N|D^j)$.

Lemma 3.2 ([Mi[3], Corollary 10]) $(0 \leqslant j \leqslant k \leqslant n-3)$. *Hypothesis*: *Suppose*

(1) $\{\mathcal{M}(\delta)\}$, $\delta > 0$, *is a set of collections of* PL *m-manifolds without boundary such that* $M \in \mathcal{M}(\delta)$ *implies* $M \subset N_\delta(R(D^j))$.

(2) *For each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *satisfying the following*:

*If* $\delta \geqslant \delta^* > 0$ *and* $M^* \in \mathcal{M}(\delta^*)$, *then for each complex* $Z$ *with*

$\dim(Z) \leq m-4$ *and* $Z \subset M^*$, *there is an* $M \in \mathcal{M}(\varepsilon)$ *with* $M \cap N_\delta^*(R(D^j)) = M^*$ *and a* $(j, \varepsilon)$*-homotopy of* $Z$ *contained in* $M$.

*Conclusion*: *Suppose* $\varepsilon > 0$ *is given, there exists a* $\delta > 0$ *satisfying*:

*If* $\delta \geq \delta^* \leq 0$ *and* $M^* \in \mathcal{M}(\delta^*)$, *then for each* $(j, \delta)$*-collapse* $C^*$ *in* $M^*$ *with* $\dim(|C^*|) \leq m-3$ *and each complex* $X$ *in* $M^*$ *with* $\dim(X) \leq m-4$ *there exists* $M \in \mathcal{M}(\varepsilon)$ *with* $M \cap N_\delta^*(R(D^j)) = M^*$ *and* $C$, *a* $(j, \varepsilon)$*-collapse, with* $|C| \supset (|C^*| \cup X)$, $M \supset |C|$, *and* $\dim(|C|-|C^*|) \leq (\dim(X)+1)$.

*Addendum*: *If in addition* $X \subset (M^* - N_\varepsilon(R(D^{j-1})))$, *then* $C$ *can be chosen so that* $|C|-|C^*|$ *lies outside* $N_\delta(R(D^{j-1}))$.

Proof. This lemma builds collapses from homotopies much in the same manner as in radial engulfing. The complicated hypotheses are needed because instead of working in a single manifold, we work in a nested (by inclusion) sequence of manifolds. As in Lemma 1, our proof consists of checking to see that Miller's ideas generalize to the cell-like embedding relation case.

We induct on $r = \dim(X)$. For $r = -1$, let $C = C^*$. This starts the induction.

Now, our goal is to enlarge a $(j, \delta)$-collapse, $C^*$, to a $(j, \varepsilon)$-collapse, $C$, with $|C|$ containing an $(m-4)$-complex $X$. An outline follows.

Step 1. We use induction to enlarge $|C^*|$ to $|C1|$ to contain $X^{(r-1)}$, the $(r-1)$-skeleton of $X$. Call this enlargement $C1$.

Step 2. We now have all but a finite collection of $r$-simplexes of $X$ in $|C1|$. Let $\sigma$ be one such simplex. Then $\mathrm{bdy}(\sigma) \subset |C1|$. Let $Z_\sigma = \sigma \cup_{(\mathrm{bdy}\,\sigma)} (\mathrm{track}_{C1}(\mathrm{bdy}\,\sigma))$. We apply the hypothesis of Lemma 3.2 to get a homotopy $F_\sigma\colon Z \times I \to E^n$ of $Z$ with $F$ "close" to $S^j$.

Step 3. We construct a collapse $c_\delta$ of $Z_\sigma \times I$ to $((Z_\sigma \times \{1\}) \cup \cup((\mathrm{image}_{C1}\,\mathrm{bdy}\,\sigma)) \times I)$ so that $F_\sigma \circ c_\sigma\colon Z_\sigma \times \mathrm{I} \to Z_\sigma \times I \to E^n$ is close to $S^j$ and $c_\sigma|(\mathrm{track}_{C1}(\mathrm{bdy}\,\sigma) \times \{0\}) = C1$. Then let $W = \bigcup_\sigma (Z_\sigma \times I)$ where the union is taken along $\mathrm{track}_{C1}(\mathrm{bdy}\,\sigma)$. By construction the $c_\sigma$'s piece together to form a collapse $c$ of $W$ and the $F_\sigma$'s piece together to form a map $F\colon W \to E^n$ so that $F \circ c$ is close to $S^j$. The desired collapse $C$ will eventually be constructed in part from $F(W)$.

Step 4. We use induction and general position to construct a collapse $C2$ so that $F(W) \cap |C2|$ is "saturated" with respect to $c$, i.e.,

$$F(\mathrm{track}_c(f^{-1}(F(W) \cap |C2|))) = F(W) \cap |C2|.$$

Also, $F(\mathrm{track}_c(F^{-1}(S(F)))) \subset |C2|$ where $S(F)$ is the singular set of $F$.

Step 5. The underlying polyhedron of the promised collapse is then given by $|C| = F(W) \cup |C2|$. The two conditions described in Step 4 allow us to show that $|C|$ actually collapses.



We now fill in the details. We are given $\varepsilon > 0$. The positive number $\delta > 0$ is chosen as follows.

Choice (0). Choose $0 < \delta(0) < \varepsilon$ so that if $C_1$ and $C_2$ are any two $(j, \delta(0))$-collapses, then the following property holds. If $(x, t) \in E^n \times I$ with $C_1(x, t) \in |C_2|$ then $((x, t), C_2(C_1(x, t), t')) \in (\varepsilon \circ S^j \circ \varepsilon)$ for all $t \geq t'$.

Choice (i). Choose $0 < \delta(1) < \delta(0)$ so the relation

$$(\delta(1) \circ S^j \circ \delta(1)) \circ (\mathrm{id} \times \delta(1))\colon E^n \times I \to E^n \times I \to E^n$$

is contained in $\delta(0) \circ S^j \circ \delta(0)$.

Choice (ii). Choose $0 < \delta(2) < \delta(1)$ so that if $C_1$ and $C_2$ are any two $(j, \delta(2))$-collapses then

$$C_1(t) \cap C_2(t+\delta(1)) \subset C_1(t+\delta(1)|2) \quad \text{for} \quad t \in [0, 1-\delta(1)].$$

Choices (0), (i), and (ii) are needed in Step 5 above in order to fit $F(W)$ and $|C2|$ together so that the union collapses properly.

Choice (iii). We use induction as follows. For $i = 3, 4, \ldots, r+2$ let $\varepsilon(\mathrm{ind}) = \delta(i-1)$ and choose $\delta(i) = \delta(\mathrm{ind})$. These choices give us the necessary power and control to complete Step 4 above.

Choice (iv). Choose $0 < \delta(r+3) < \delta(r+2)$ so that if $C$ is a collapse which is a $(j, \delta(r+3))$-homotopy of $\mathrm{bdy}(C)$ then $C$ is a $(j, \delta(r+2))$-collapse. This insures that the collapse we are about to construct will in the end be a $(j, \delta(r+2))$-collapse.

Choice (v). The following are technical conditions that supply the controls necessary for the description of $c$ in Step 3 above. There exist $\delta'$, $\delta''$, and $\delta(r+4)$ with $\delta(r+4) < \delta'' < \delta' < \delta(r+3)$ such that if $K$ is a $(j, \delta(r+4))$-collapse and $F$ is a $(j, \delta(r+4))$-homotopy of $|K|$ with $(|K| \cup (\mathrm{Image}(F))) \subset N_{\delta(r+4)}(R(D^j))$, then the following hold:

a. $\delta(r+4) < \delta(r+3)/2$.

b. $(x, t, F(K(N_{\delta'}(x), [t-\delta', t+\delta']), [0, \delta'])) \subset \delta(r+3) \circ S^j \circ \delta(r+3)$.

c. $(x, t, F(K(N_{\delta'}(x), [t-3\delta', t+\delta']), [0, t+2\delta'])) \subset \delta(r+3) \circ S^j \circ \delta(r+3)$.

d.
$$(x, t, F(K(N_{\delta'}(x), [t^*-\delta', t^*+\delta']), [t-\delta', t+\delta'])) \subset \delta(r+3) \circ S^j \circ \delta(r+3)$$

for each $x$ and $t$ where $t^* \in [0, t]$.

e. Let $Z$ be a subcomplex of $(|K| \cap N_{\delta'}(R(D^{j-1}) \times \bar{t}))$; for $x \in Z$ and $t \in [0, \bar{t}]$ we have

$$(x, t, F(K(N_{\delta'}(x), [0, \bar{t}]), [0, \bar{t}+2\delta'])) \subset \delta(r+3) \circ S^j \circ \delta(r+3).$$

f. Let $Z$ be a subcomplex of $(|K|\cap N_{\delta''}(R(D^{j-1}\times\bar{t})))$; then for $t\in[\bar{t}, 1-\delta']$, $K(Z, [0, t])\cap K(Z, t+\delta')=\emptyset$.

g. If $Z\subset N_{\overline{\delta(r+4)}}(R(D^j))$ and $\operatorname{dia}(Z)<\delta(r+4)$, then

$$Z\subset N_{\delta''}(R(D^{j-1}\times\bar{t}))\quad \text{for some } \bar{t}\in[0, 1].$$

Choice (vi). We now use the hypothesis of Lemma 3.2 to supply the necessary homotopies of the $r$-simplexes of $X$. Let $\varepsilon(\mathrm{Hyp})=\delta(r+4)$ and choose $\delta(r+5)=\delta(\mathrm{Hyp})$.

Choice (vii). Choose $\delta(r+6)<\delta(r+5)/2$.

Choice (viii). We once again use the inductive hypothesis; here we engulf the $(r-1)$-skeleton of $X$. Let $\varepsilon(\mathrm{ind})=\delta(r+6)$ and choose $\delta(r+7)=\delta(\mathrm{ind})$.

We now show that $\delta=\delta(r+7)$ is the proper choice. To this end suppose that $\delta$, $\delta^*$, $C^*$, $M^*$, and $X$ are as in the conclusion to Lemma 2; we need to find $C$ and $M$. The proof follows the previous outline.

Step 1. We apply induction with $\varepsilon(\mathrm{ind})=\delta(r+6)$. Choice (viii) yields $\delta=\delta(r+7)=\delta(\mathrm{ind})$. Then for $M^*(\mathrm{ind})=M^*$, $C^*(\mathrm{ind})=C^*$, $\delta^*(\mathrm{ind})=\delta^*$, and $X(\mathrm{ind})=X^{(r-1)}$ (the $(r-1)$-skeleton of $X$), induction yields $M(r+6)=M(\mathrm{ind})\in\mathscr{M}(\delta(r+6))$ and $C(r+6)=C(\mathrm{ind})$ (a $(j, \delta(r+6))$-collapse) with $(|C^*|\cup X^{(r-1)})\subset|C(r+6)|\subset M(r+6)$, $M(r+6)\cap N_\delta(R(D^j))=M^*$ and $\dim(|C(r+6)|-|C^*|)\leqslant r$.

Step 2. Now the only part of $X$ not in $C(r+6)$ consists of a finite collection of $r$-simplexes of $X$, each of whose boundaries lies in $|C(r+6)|$. Using Choice (vi) we obtain homotopies of these simplexes that will eventually become part of $C$, the collapse promised by Lemma 3.2. For the remainder of Step 2 and all of Step 3 let $K=C(r+6)$ for ease of notation. Let $\sigma$ be an $r$-simplex of $X$ and $Z_\sigma=\sigma\cup_{(\mathrm{bdy}\,\sigma)}(\mathrm{track}_K(\mathrm{bdy}\,\sigma))$ (the union is the abstract topological union taken along the boundaries of the $\sigma$'s). Notice that in general $\mathrm{track}_K(\mathrm{bdy}\,\sigma)\cap\operatorname{int}(\sigma)\neq\emptyset$ so that $Z_\sigma$ is not embedded in $E^n$. However, there is a PL map $f_\sigma\colon Z_\sigma\to M(r+6)\cap N_{\sigma(r+6)}(R(D^j))$ obtained from the inclusions of each of $\sigma$ and $\mathrm{track}_K(\mathrm{bdy}\,\sigma)$ separately into $M(r+6)\cap N_{\delta(r+6)}(R(D^j))$. Let $Z=\bigcup_{\sigma\in X}f_\sigma(Z_\sigma)$. Now, $Z$ does lie in $M(r+6)$ since $X$ and $|K|=|C(r+6)|$ do. Notice that $\dim(Z)=r\leqslant(m-4)$. If we let $\varepsilon=\delta(r+4)$ in the hypothesis of Lemma 3.2, then Choice (vi) and Lemma 3.2 yield $\delta(r+5)$. Then by letting $\delta^*(\mathrm{hyp})=\delta(r+6)$, $M^*(\mathrm{hyp})=M(r+6)$ and $Z(\mathrm{hyp})=Z$, we obtain $M(r+4)=M(\mathrm{hyp})\in\mathscr{M}(\delta(r+4)$ and a $(j, \delta(r+4))$-homotopy $f\colon Z\times I\to M(r+4)$ with $(M(r+4)\cap N_{\delta(r+6)}(R(D^j)))=M(r+6)$.

For each $\sigma\subset X^{(r)}$ define a $(j, \delta(r+4))$-homotopy $F_\sigma\colon Z_\sigma\times I\to M(r+4)$ by $F_\sigma=f\circ(f_\sigma\times\mathrm{id})$.

Step 3. The goal is to construct a collapse $c_\sigma$ of $(Z_\sigma\times I)$ to $((Z_\sigma\times\{1\})\cup\mathrm{image}_K(\mathrm{bdy}\,\sigma\times I))$ so that $F_\sigma\circ c_\sigma$ is a $(j, \delta(r+3))$-homotopy of $\sigma$ and $(F_\sigma\circ c_\sigma|\mathrm{bdy}\,\sigma)$ has the same tracks as $(K|\mathrm{bdy}\,\sigma)$. We note that, in fact, $c_\sigma$ yields a homotopy of $\sigma\times I$, not $\sigma$, so we identify $c_\sigma$ with $c_\sigma\circ i_\sigma\colon\sigma\times I\to(Z_\sigma\times I)\times I\to Z_\sigma\times I$ where $i_\sigma(x, t)=(x, o, t)\in((Z_\sigma\times I)\times I)$.



Following Miller ([Mi³, p. 410]), we describe $c$ with four steps. Pick $\beta$, $0<\beta<1$.

1. $\sigma\times[0, \beta]\searrow((\mathrm{bdy}\,\sigma)\times[0, \beta]\cup\sigma\times\{\beta\})$ through $\sigma\times\{0\}$.
2. $((\mathrm{track}_K(\mathrm{bdy}\,\sigma))\times[0, \beta])$
$$\searrow$$
$((\mathrm{track}_K(\mathrm{bdy}\,\sigma))\times\{0, \beta\})\cup(\mathrm{image}_K(\mathrm{bdy}\,\sigma))\times[0, \beta])$.

This is accomplished by copying the collapse of $\mathrm{track}_K(\mathrm{bdy}\,\sigma)$; in other words if $\tau$ is a simplex of $\mathrm{track}_K(\mathrm{bdy}\,\sigma)$ and $\tau$ collapses through a face $\tau'$, then we collapse the cell $\tau\times[0, \beta]$ through the face, $\tau'\times[0, \beta]$.

3. $((\mathrm{track}_K(\mathrm{bdy}\,\sigma))\times\{0\})\searrow((\mathrm{image}_K(\mathrm{bdy}\,\sigma))\times\{0\})$.
4. $(Z_\sigma\times[\beta, 1])\searrow(Z_\sigma\times\{1\}\cup\mathrm{image}_K(\mathrm{bdy}\,\sigma)\times I)$ through $Z_\sigma\times\{\beta\}$.

The collapse $c_\sigma$ will have the same tracks as the one just described, but the order in which the various stages occur will be altered carefully to yield the desired properties. Recall that $\mathrm{track}_K(\mathrm{bdy}\,\sigma)$ is a subcomplex of $|K|$ and so is triangulated with mesh less than $\delta(r+6)$. Extend this triangulation to one of $Z$ with mesh less than $\delta(r+6)$. Choose $\beta$, $0<\beta<\delta(r+6)$, so that $N\cdot\beta=1$ for some integer $N$. Choice (vii) allows us to adjust the parametrization of $K$ so that $K(l\cdot\beta)$ is a subcomplex of $|K|$ for each $l$, $0\leqslant l\leqslant N$, and $K$ is still a $(j, \delta(r+5))$-collapse.

We give a precise description of the collapse $c$ as a sequence of $N$ moves, each consisting of a collapse of PL cells and each taking place while $t$ varies over a subinterval of length $\beta$. The case $N=3$ is illustrated in Figure 3.2.

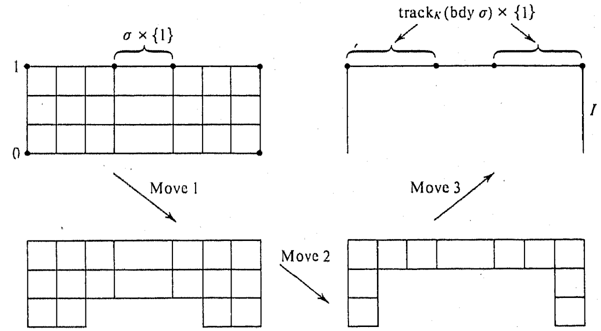


**Figure 3.2**

Move 1. This move occurs from $t=0$ until $t=\beta$. It can be described in three distinct submoves.

a. $(\sigma\times[0,\beta])\searrow(((\mathrm{bdy}\,\sigma)\times[0,\beta])\cup(\sigma\times\{\beta\}))$.

b. $(\mathrm{track}_K(\mathrm{bdy}\,\sigma)\times[0,\beta])\searrow((K^\sigma(\beta)\times[0,\beta))\cup(\mathrm{track}_K(\mathrm{bdy}\,\sigma)\times\{0,\beta\}))$ where $K^\sigma=(K|\mathrm{track}_K(\mathrm{bdy}\,\sigma))$. This is the first part of the collapse of $(\mathrm{track}_K(\mathrm{bdy}\,\sigma)\times[0,\beta])$ which mimicks $K^\sigma$.

c. $(\mathrm{track}_K(\mathrm{bdy}\,\sigma)\times\{0\})\searrow K^\sigma(\beta)\times\{0\}$.

We now give the $J$'th move for each integer $J$, $2\leqslant J\leqslant N$.

Move $J$. Each move takes place from $t=(J-1)\beta$ until $t=J\cdot\beta$. Each one can be described in four submoves. We note that for $s<t$ $K(\mathrm{bdy}\,\sigma,[s,t])=\mathrm{track}_{K(s)\searrow K(t)}(\mathrm{bdy}\,\sigma)$.

a. $$\big((\sigma\cup K(\mathrm{bdy}\,\sigma,[0,(J-1)\beta]))\times[(J-1)\beta,J\cdot\beta]\big)$$
$$\searrow$$
$$\Big(\big((\sigma\cup K(\mathrm{bdy}\,\sigma,[0,(J-1)\beta))\times\{J\cdot\beta\}\big)\cup$$
$$\cup\big((K(\mathrm{bdy}\,\sigma,(J-1)\beta))\times[(J-1)\beta,J\cdot\beta]\big)\Big).$$

b. $(K^\sigma((J-1)\beta)\times[0,\beta])\searrow\big((K^\sigma((J-1)\beta)\times\{0,\beta)\cup(K^\sigma(J\cdot\beta)\times[0,\beta])\big)$.

c. $K^\sigma((J-1)\beta)\times\{0\}\searrow K^\sigma(J\cdot\beta)\times\{0\}$.

d. $$(K(\mathrm{bdy}\,\sigma,[(J-1)\beta,J\cdot\beta])\times[\beta,J\cdot\beta])$$
$$\searrow$$
$$((K(\mathrm{bdy}\,\sigma,[(J-1)\beta,J\cdot\beta])\times\{J\cdot\beta\})\cup(K(\mathrm{bdy}\,\sigma,J\cdot\beta)\times[\beta,J\cdot\beta])).$$

We use Choice (v) to show that $((x,t),F_\sigma\circ c_\sigma(x,t))\subset\delta(r+3)\circ S^j\circ\delta(r+3)$ for each $x\in\sigma$ and $t\in I$. We assume that $c_\sigma$ has been subdivided to be simplicial ([Wh[1], Theorem 7]).

We split the proof into cases. To give the reader the flavor, we include details for case 1. Assume $(c_\sigma|[0,t_0])$ is just move 1(a). Let $x\in\mathrm{int}(\tau)$ where $\tau$ is a simplex of $\sigma$ subdivided.

Case 1. $c_\sigma(x,t_0)\notin((\mathrm{bdy}\,\sigma)\times[0,\beta))$.

Since we are collapsing along product lines, $c_\sigma(x,t)\in\tau\times I$ for all $t\in I$. Since $\mathrm{dia}(\tau)<\delta$ and $\beta<\delta$,
$$F_\sigma\circ c_\sigma(x,t)\subset(F_\sigma\circ\delta)(x,t)\colon E^n\times I\to E^n\times I\to E^n.$$
But by Choice (vi) $F_\sigma\subset\delta(r+4)\circ S^j\circ\delta(r+4)$. Thus, since $\delta<\delta(r+4)$ we have by Choice (v(a)) that
$$F_\sigma\circ\delta\subset\delta(r+4)\circ S^j\circ\delta(r+4)\circ\delta\subset\delta(r+3)\circ S^j\circ\delta(r+3).$$
The reader will find Choice (v) helpful for

Case 2. $c_\sigma(x,t)\in((\mathrm{track}_K(\mathrm{bdy}\,\sigma))\times(0,\beta))$ for all $t\in[t_0,1]$.

Case 3. $c_\sigma(x,t')\in\mathrm{track}_K(\mathrm{bdy}\,\sigma)\times\{0\}$ for some $t'\in I$.

Case 4. $c_\sigma(x,t')\in\mathrm{track}_K(\mathrm{bdy}\,\sigma)\times\{\beta\}$ for some $t'\in I$.



We leave these as exercises.

We complete Miller's argument (for cell-like embedding relations) by letting $W=\bigcup_{\sigma\in X}Z_\sigma\times I$ where identifications are made along the tracks of the boundaries at the 0-level. Note that for each $\sigma$, $\mathrm{track}_K(\mathrm{bdy}\,\sigma)$ actually resides in $M(r+4)$ and $(F_\sigma\circ c_\sigma|((\mathrm{track}_K(\mathrm{bdy}\,\sigma))\times\{0\}))$ is inclusion. We let $F=\bigcup_{\sigma\in X}F_\sigma$ and $c=\bigcup_{\sigma\in X}c_\sigma$. Then it is easy to check that $F$ and $c$ are well-defined, that $c$ is a collapse of $W$, and that $F\circ c$ provides a $(j,\delta(r+3))$-homotopy of $X^{(r)}$ in $M(r+4)$.

Step 4. The argument is precisely that of Miller ([Mi[3], p. 411–412]). We use Choice (iii) and induction on Lemma 2.

We have $F\circ c\colon W\to M(r+4)$, a $(j,\delta(r+3))$-homotopy of $X$. After a general position move we may also assume that $\dim(S(F))\leqslant r-2$ and $\dim(F\cdot C)\leqslant r-2$ where $S(F)$ is the singular set of $F$ and
$$F\cdot C=\mathrm{clos}\,\{w\in(W-\bigcup_\sigma((\mathrm{track}_{C(r+6)}(\mathrm{bdy}\,\sigma))\times\{0\}))\}$$
such that $F(w)\in|C(r+6)|$.

For $T\subset F(W)$, let $\mathrm{SAT}(T)=F(\mathrm{track}_c(F^{-1}(T))$ and call $T$ saturated if $\mathrm{SAT}(T)=T$. Construct inductively collapses $C(r+1),C(r),\ldots,C(2)$ such that for $r+1\geqslant 1\geqslant 2$, $C(q)$ is a $(j,\delta(q))$-collapse in $M(q)\in\mathcal{M}(\delta(q))$,
$$|C(q)|\supset(\mathrm{SAT}(|C(q+1)|\cap F(W))\cup|C(q+1)|),$$
$(|C(q)|-(|C(q+1)|\cup\mathrm{SAT}(|C(q+1)|\cap F(W))))$ is in general position with respect to $F(W)$ and to each simplex of $X$, $\dim(|C(q)|-|C(q+1)|)\leqslant(q-1)$, and $M(q)\cap N_{\delta(q+1)}(R(D^j))=M(q+1)$. For $q=r+1$, let $C(q+1)\equiv C(r+6)$ and replace $\mathrm{SAT}(|C(q+1)|\cap F(W))$ by the set $(\mathrm{SAT}(F(F\cdot C))\cup F(S(F)))$.

It is easy to check that $(|C(2)|\cap F(W))$ is saturated, the desired property.

Step 5. The desired collapse $C$ is given by $|C|=(F(W)\cup|C(2)|)$. Since $(|C(2)|\cap F(W))$ is saturated, the collapse $c$ provides a collapse $c^*$ of $W$ to
$$((F^{-1}(|C(2)|))\cup\bigcup_\sigma((Z_\sigma\times\{1\})\cup(\mathrm{image}_{C(r+6)}(\mathrm{bdy}\,\sigma)\times[0,1]))).$$

But $(F|(|C|-|C(2)|))$ is an embedding so that $(F\circ c^*\circ F^{-1}|(|C|-|C(2)|))$ is a collapse. We define the time parametrization of $C$ as follows.
$$C(x,t)=\begin{cases}F\circ c^*\circ(F^{-1}\times\mathrm{id})(x,t) & \text{for } x\in(|C|-|C(2)|),\\ C(2)(x,t-\delta(1)) & \text{for } x\in|C(2)|,\ t\in[\delta(1),1],\\ (x,t) & \text{for } x\in|C(2)|,\ t\in[0,\delta(1)].\end{cases}$$

The time delay removes obstructions blocking the collapse of $C(2)$ (Choice (ii)). The track of a point $x$ may be included entirely in either $F(W)$ or $|C(2)|$ or switch from $F(W)$ to $|C(2)|$. Choices (i) and (0) respectively insure that in either case, $((x,t),C(x,t))\subset\varepsilon\circ S^j\circ\varepsilon$, completing the proof of Lemma 2.

The keys to generalizing Lemma 2 to the relation case are of course Choices (0)-(viii) some of which require proof. As an example we verify Choice v(d) and then leave the remaining proofs as exercises.

We show that for $\varepsilon > 0$ there is a $\delta > 0$ such that if $C$ is a $(j, \delta)$-collapse and $F$ is a $(j, \delta)$-homotopy of $|C|$ then

$$\big((x, t), F\big(C(N_\delta(x), [t^*-\delta, t^*+\delta]), [t-\delta, t+\delta]\big)\big) \subset \varepsilon \circ S^j \circ \varepsilon \quad \text{for} \quad t^* \in [0, t].$$

Consider the composite relation

$$S^j \circ (S^j \times \mathrm{id}) \circ \varrho\colon E^n \times I \to (E^n \times J) \times I \to E^n \times I \to E^n$$

where $J \equiv [0, 1]$ and $\varrho(x, t) = (x, [0, t], t)$. Note that $S^j \circ (S^j \times \mathrm{id}) \circ \varrho = S^j$: $E^n \times I \to E^n$. We apply the Composition Theorem to obtain $\delta > 0$ with $(\delta \circ S^j \circ 2\delta) \circ ((\delta \circ S^j \circ 2\delta) \times \mathrm{id}) \circ \varrho \subset \varepsilon \circ S^j \circ \varepsilon$. Thus, $F \circ \delta \circ ((C \circ \delta) \times \mathrm{id}) \circ \varrho \subset \varepsilon \circ S^j \circ \varepsilon$. But we also have that

$$\big((x, t), F\big(C(N_\delta(x), [t^*-\delta, t^*+\delta]), [t-\delta, t+\delta]\big)\big) \subset F \circ \delta \circ ((C \circ \delta) \times \mathrm{id}) \circ \varrho.$$

The following yields the Addendum.

Sublemma 3.2. *Suppose $0 \leqslant h \leqslant j \leqslant k$. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $X$ lies outside $N_\varepsilon(R(D^{j-1}))$ and $F$ is an $(h, \delta)$-homotopy of $X$, then $F(X \times I)$ lies outside $N_\delta(R(D^{j-1}))$.*

Proof. For $\gamma > 0$ we have $S^h \circ \pi^{-1}(R(D^{j-1} \times [\gamma, 1])) \cap R(D^{j-1}) = \emptyset$ where $\pi\colon E^n \times I \to E^n$ is the natural projection. By compactness of $D^j$ there exists a $\gamma > 0$ such that $N_\gamma(R(D^{j-1} \times [0, \gamma])) \subset N_\varepsilon(R(D^{j-1}))$. The Composition Theorem yields a $\delta$, $0 < \delta < \gamma$, with $(\delta \circ S^j \circ \delta \circ \pi^{-1}) \circ \delta \circ R(D^{j-1} \times [\gamma, 1]) \cap \delta \circ R(D^{j-1}) = \emptyset$. So if $F$ and $X$ are as above, then it is also true that $X \cap N_\gamma(R(D^{j-1} \times [0, \gamma])) = \emptyset$ and $X \subset N_\delta(R(D^{j-1} \times [\gamma, 1]))$; so we have $F \circ \pi^{-1}(X) \cap N_\delta(R(D^{j-1})) = \emptyset$ and thus $F(X \times I) \cap N_\delta(R(D^{j-1})) = \emptyset$.

Lemma 3.3 ([Mi³, Lemma 11]) $(0 \leqslant j \leqslant k)$. *Suppose $\varepsilon > 0$ is given. There exists a $\delta > 0$ such that if $\delta \geqslant \delta^* > 0$ and if $C^k, \ldots, C^j$ and $B_k, \ldots, B_j$ are sequences of complexes in $N_\delta{}^*(R(D^k))$ satisfying*

1. *$C^r$ is an $(r, \delta^*)$-collapse in $(\mathrm{bdy}(B_{r+1})) \cap N_{\delta^*}(R(D^r))$ with $\dim(|C^r|) \leqslant r$.*

2. *$B_k = N(|C^k|, E^{n''})$ and $B_r = N(|C^r|, (\mathrm{bdy}(B_{r+1}))'')$ for $j \leqslant r \leqslant k-1$, then for $h \leqslant j$ and each complex $X$ in $(\mathrm{bdy}(B_{j+1})) \cap N_{\delta^*}(R(D^j))$ with $\dim(X) < h$, there are new complexes $C_X^k, \ldots, C_X^j$ and $B_k^X, \ldots, B_j^X$ satisfying (1) and (2) for $\varepsilon$ in place of $\delta^*$ and which also satisfy*

3. *$|C^j| \subset |C_X^j|$ and $X \subset |C_X^j|$.*

4. *If $h < j$ then $B_j^X$ contains an $(h, \varepsilon)$-homotopy of $X$.*

5. *If $X$ is outside $N_\varepsilon(R(D^{j-1}))$ then $|C_X^r| \cap N_\delta(R(D^{j-1})) = |C^r| \cap N_\delta(R(D^{j-1}))$ and $B_r^X \cap N_\delta(R(D^{j-1})) = B_r \cap N_\delta(R(D^{j-1}))$ for $j \leqslant r \leqslant k$.*

Proof. The proof is by downward induction on $j$ beginning with $j = k$. In each case Lemma 2 is used to finish the argument.



Case 1. $j = k$. In Lemma 2, let $\{M(\delta)\} = \{N_\delta(R(D^k))\}$ for $\delta > 0$. We verify the hypothesis in this case. We apply the Approximation Theorem (1.2) to $S^k\colon E^n \times I \to E^n$, taking $\delta(\mathrm{hyp}) = \delta(1.2)$ for $\varepsilon(1.2) = \varepsilon(\mathrm{hyp})$. For $Z \subset N_{\delta^*}(R(D^k))(\delta(\mathrm{hyp}) \geqslant \delta^* > 0)$, define $f = (\mathrm{INC}|(Z \times \{0\}))\colon E^n \times \{0\} \to E^n$. Theorem 1.2 yields an extension $g\colon N_{\overline{\delta(\mathrm{hyp})}}(R(D^k)) \times I \to E^n$ with $g \subset \varepsilon(\mathrm{hyp}) \circ S^k \circ \varepsilon(\mathrm{hyp})$. Clearly, $F = (g|(Z \times I))$ is the desired homotopy.

Now for $\varepsilon > 0$, first choose $\alpha$, $0 < \alpha < \varepsilon$, to satisfy Sublemma 3.2; then choose $\beta$, $0 < \beta < \alpha$ to satisfy the conclusion to Lemma 3.2; finally choose $\delta$, $0 < \delta < \beta$ according to the previous paragraph. Suppose $C^k$ is a $(k, \delta^*)$-collapse and $X \subset N_{\delta^*}(R(D^k))$. Let $F^\beta$ be an $(h, \beta)$-homotopy of $X$. Apply Lemma 3.2 with $C^* = C^k$ and $X(3.2) = X$ if $h = k$ or $X(3.2) = F^\beta_{[0,1]}$ if $h < k$. Then $C_X^k = C(3.2)$ and $B_k^X = N(|C_X^k|, E^{n''})$ satisfy (1)–(3) of Lemma 3.3. Properties (4) and (5) follow from our choice of $\alpha$ and the Addendum to Lemma 3.2 respectively.

Case 2. $j \leqslant k-1$. Case 1 began the induction; as in the proof Lemma 2, we verify that Miller's outline works in the relation case.

We assume Lemma 3.3 holds for $(j+1)$ and prove the following.

Claim. *Suppose $\varepsilon' > 0$ is given. There exists $\delta' > 0$ such that if $\bar{C}^k, \ldots, \bar{C}^j$ and $\bar{B}_k, \ldots, \bar{B}_j$ satisfy conditions (1) and (2) of Lemma 3.3 for $\delta^*$, $0 < \delta^* \leqslant \delta'$ and if $Z \subset (\mathrm{bdy}(\bar{B}_{j+1})) \cap (N_{\delta^*}(R(D^j)))$ is a complex of dimension $\leqslant (h-1)$, then there are new sequences $\bar{C}_Z^k, \ldots, \bar{C}_Z^{j+1}$, $\bar{C}^j$ and $\bar{B}_k^Z, \ldots, \bar{B}_{j+1}^Z$, $\bar{B}_j$ satisfying conditions (1) and (2) of Lemma 3.3 with $\delta^*$ replaced by $\varepsilon'$ and such that $|\bar{C}_Z^r| \supset |\bar{C}^r|$ for $j+1 \leqslant r \leqslant k$ and such that there is an $(h, \varepsilon')$-homotopy of $Z$ in $(\mathrm{bdy}(\bar{B}_{j+1}^Z)) \cap N_{\varepsilon'}(R(D^j))$.*

Proof of Claim. We begin with two sets of technical conditions. First, choose $\alpha$, $0 < \alpha < \varepsilon'$, and $\gamma_1 > 0$ so that if $C$ is a $(j+1, \alpha)$-collapse then we have

$(\mathrm{i})_\alpha$ $C(\gamma_1) \cap N_\alpha(R(D^j)) = \emptyset$.

$(\mathrm{ii})_\alpha$ For $h \leqslant j$ we have for $(x, t) \in E^n \times I$

$$((x, t), (\alpha \circ S^{j+1} \circ \alpha) \circ \overline{\gamma}_1)((\alpha \circ S^h \circ \alpha)(x, t), 0) \subset \varepsilon' \circ S^h \circ \varepsilon'.$$

$(\mathrm{iii})_\alpha$ $(\alpha \circ S^{j+1} \circ \alpha) \circ \overline{\gamma}_1(N_\alpha(R(D^j)), 0) \subset N_{\varepsilon'}(R(D^j))$.

Next choose $\beta$, $0 < \beta < \alpha$ and $\gamma_2 > 0$ so that if $C$ is a $((j+1), \beta)$-collapse, then

$(\mathrm{i})_\beta$ $C(\gamma_2) \cap N_\beta(R(D^j)) = \emptyset$.

$(\mathrm{ii})_\beta$ For $1 \leqslant h \leqslant j$ $((x, [0, \gamma_2]), \beta \circ S^{j+1} \circ \beta(x, [0, \gamma_2])) \subset (\alpha \circ S^h \circ \alpha)$.

$(\mathrm{iii})_\beta$ If $Y \subset (N_\beta(R(D^j)) \cap |C|)$ then for $h \leqslant j$

$$\beta \circ S^h \circ \beta(\{C(Y, \gamma_2)\} \times I) \subset N_\alpha(R(D^j)).$$

$(\mathrm{iv})_\beta$ $(\beta \circ S^{j+1} \circ \beta) \circ \overline{\gamma}_2(N_\beta(R(D^j)), 0) \subset N_\alpha(R(D^j))$.

We obtain $\delta'$ by applying induction on Lemma 3.3 with $\varepsilon(\mathrm{ind}) = \beta$ and also requiring $\delta' < (\beta/2)$.

To prove the claim, let $\varrho_t$ denote the composition of first the collapse $\bar{B}_{j+1} \searrow |\bar{C}^{j+1}|$ and then the collapse $|\bar{C}^{j+1}| \searrow \bar{C}^{j+1}(\delta_2)$. If $\varrho \equiv \varrho_1$, then $\varrho(Z) \subset \operatorname{bdy}(\bar{B}_{j+2})$. We apply induction on Lemma 3.3 to $\bar{C}^k, \ldots, \bar{C}^{j+1}$ and $\bar{B}_k, \ldots, \bar{B}_{j+1}$ with $X(\text{ind}) = \varrho(Z)$. We obtain new sequences $\bar{C}^k_{\varrho(Z)}, \ldots, \bar{C}^{j+1}_{\varrho(Z)}$ and $\bar{B}^{\varrho(Z)}_k, \ldots, \bar{B}^{\varrho(Z)}_{j+1}$ satisfying (1)-(5) of Lemma 3.3 for $j+1 \leqslant r \leqslant k$.

Since $h \leqslant j < j+1$ and $\dim(Z) \leqslant j+1$, we have an $(h, \beta)$-homotopy $F\colon \varrho(Z) \times I \to E^n$. Also (i)$_\beta$ and (5) allow $\bar{B}_j$ and $\bar{C}^j$ to be added to the above sequences with the result satisfying (1)-(2) of Lemma 3.3 for $\beta > 0$.

Define a homotopy $G\colon Z \times I \to E^n$ by

$$G(x, t) = \begin{cases} \varrho(x, (t/t_0)) & \text{for } 0 \leqslant t \leqslant t_0, \\ F(x, (t-t_0)/(1-t_0)) & \text{for } t_0 \leqslant t \leqslant 1, \end{cases}$$

For small $t_0$ and $t_0 \leqslant t \leqslant 1$, $G$ is very close to $F$. Also, condition (ii)$_\beta$ implies $((x, [0, \gamma_2]), \delta' \circ \bar{C}^{j+1} \circ 2\delta'(x, [0, \gamma_2])) \subset \alpha \circ S^h \circ \alpha$ and thus,

$$((x, [0, \gamma_2]), \varrho(x, [0, \gamma_2])) \subset \alpha \circ S^h \circ \alpha,$$

taking care of the case $0 \leqslant t \leqslant t_0$. Therefore $G$ is an $(h, \alpha)$-homotopy. Conditions (iii)$_\beta$ and (iv)$_\beta$ imply $\operatorname{image}(G) \subset N_\alpha(R(D^j))$.

We "push" $G$ into $\operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+1})$ by collapsing "past" $G(Z \times I)$ and then using the inverse.

From condition (i)$_\alpha$ we see that

$$\bar{C}^{j+1}_{\varrho(Z)}(\gamma_1) \cap N_\alpha(R(D^j)) = \emptyset.$$

This collapse occurs in $\operatorname{bdy} \bar{B}^{\varrho(Z)}_{j+2}$; let $\Phi_t$ be the isotopy of $\operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+2})$ that Theorem 2.1 associates with the collapse. Note that

$$\Phi_1(\bar{B}^{\varrho(Z)}_{j+1}) = N(\bar{C}^{j+1}_{\varrho(Z)}(\gamma_1), (\operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+2}))'').$$

We can choose our second derived so that $\Phi(\bar{B}^{\varrho(Z)}_{j+1}) \cap N_\alpha(R(D^j)) = \emptyset$. Regular neighborhood theory supplies a PL homeomorphism $p_0$ of $\operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+2})$ which fixes $\bar{C}^{j+1}_{\varrho(Z)}$ and takes $\bar{B}^{\varrho(Z)}_{j+1}$ onto $N(\bar{B}^{\varrho(Z)}_{j+1}, (\operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+2}))'')$. There also is a PL map $p_1\colon N(\bar{B}^{\varrho(Z)}_{j+1}, (\operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+2}))'') \to \bar{B}^{\varrho(Z)}_{j+1}$ that fixes $\bar{B}^{\varrho(Z)}_{j+1}$ and collapses the rest of $N(\bar{B}^{\varrho(Z)}_{j+1}, (\operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+2}))'')$ onto $\operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+1})$. We can assume $(p_1 \circ p_0 | \operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+1})) = \text{identity}$. Define a homotopy $H$ of $Z$ by $H = p_1 \circ \Phi_1^{-1} \circ p_0 \circ G\colon Z \times I \to E^n$.

We have $H_0 = G_0$; also $\Phi_1^{-1} \circ p_0(G(Z \times I)) \subset (N(\bar{B}^{\varrho(Z)}_{j+1}, (\operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+2}))'') - \bar{B}^{\varrho(Z)}_{j+1})$ so that $p_1 \circ \Phi_1^{-1} \circ p_0(G(Z \times I)) \subset \operatorname{bdy}(\bar{B}^{\varrho(Z)}_{j+1})$. Finally, if $x \in (N_\alpha(R(D^j)) \cap |\bar{C}^{j+1}_{(Z)}|)$, we see that $((x, 0), \bar{C}^{j+1}_{\varrho(Z)}(x, [0, \gamma_1])) \subset (\alpha \circ S^{j+1} \circ \alpha) \circ \gamma_1$. Thus, appropriate choices of $p_0$, $p_1$, and $\Phi_t$ allow $p_1 \circ \Phi^{-1} \circ p_0 \subset ((\alpha \circ S^{j+1} \circ \alpha) \circ \bar{\gamma}_1 | E^n \times \{0\})$. Since $G \subset \alpha \circ S^h \circ \alpha$, condition (ii)$_\alpha$ implies $H = p_1 \circ \Phi^{-1} \circ p_0 \circ G \subset \varepsilon' \circ S^h \circ \varepsilon'$. Condition (iii)$_\alpha$ insures $H(Z \times I) \subset N_{\varepsilon'}(R(D^j))$, concluding the proof of the claim.

We verify the hypothesis of Lemma 3.2 for



$\{\mathscr{M}(\delta)\} = \{\operatorname{bdy}(B_{j+1}) \cap N_\delta(R(D^j)) |$ there exist sequences $C^k, \ldots, C^{j+1}$ and $B_k, \ldots, B_{j+1}$ satisfying conditions (1) and (2) of Lemma 3.3 for $\delta > 0\}$.

Given $\varepsilon > 0$, let $\varepsilon'(\text{Claim}) = \varepsilon$ and choose $\delta = \delta'(\text{Claim})$. Suppose $\delta \geqslant \delta^* > 0$ and $Z \subset (\operatorname{bdy}(B_{j+1})) \cap N_{\delta^*}(R(D^j)) = M^* \in \{\mathscr{M}(\delta^*)\}$. Applying the Claim with $C^j = B_j = \emptyset$ added to the sequences of $\mathscr{M}(\delta^*)$, we obtain $C^k_Z, \ldots, C^{j+1}_Z$ and $B^Z_k, \ldots, B^Z_{j+1}$ satisfying (1) and (2) of Lemma 3.3 with $\varepsilon$ replacing $\delta$ and also with $(\operatorname{bdy}(B^Z_{j+1}))$ containing a $(j, \varepsilon)$-homotopy of $Z$. Thus, we are free to use Lemma 3.2.

Given $\varepsilon > 0$, choose $\alpha$, $0 < \alpha < \varepsilon$ according to the conclusion of Lemma 3.2. Then choose $\delta$, $0 < \delta < \alpha$, using the Claim. Suppose $C^k, \ldots, C^j$ and $B_k, \ldots, B_j$ satisfy (1) and (2) of Lemma 3.3 for $\delta^*$, $0 < \delta^* \leqslant \delta$. If $h = j$, let $C^r_x = C^r$ and $B^x_r = B_r$. If $h < j$, then the Claim produces $C^k_X, \ldots, C^{j+1}_X$, $C^j_X = C^j$ and $B^X_k, \ldots, B^X_{j+1}$, $B^X_j = B^j$ satisfying (1) and (2) for $\alpha > 0$ with a $(j, \alpha)$-homotopy $H$ of $X$ in $(\operatorname{bdy}(B^X_{j+1})) \cap N_\alpha(R(D^j))$. We apply Lemma 2 ($\mathscr{M}(\delta)$ as above) with $M^* = (\operatorname{bdy}(B^X_{j+1})) \cap N_\alpha(R(D^j))$, $C^* = C^j$, and $X(3.2) = X$ if $h = j$ or $X(3.2) = H(X \times I)$ if $h < j$. We obtain sequences $C^k_X, \ldots, C^{j+1}_X$ and $B^X_k, \ldots, B^X_{j+1}$ satisfying (1) and (2) for $\varepsilon > 0$. Let $C^j_X = C(3.2)$ and $B^X_j = N(|C^j_X|, (\operatorname{bdy}(B^X_{j+1}))'')$.

The resulting sequences satisfy (1)–(5) of Lemma 3.3 for $\varepsilon > 0$

LEMMA 3.4 ([Mi[3], Lemma 12]) ($1 \leqslant j \leqslant k$). *For each $\varepsilon > 0$ there are sequences $C^k, \ldots, C^j$ and $B_k, \ldots, B_j$ such that for $j \leqslant r \leqslant k$*

1. *$C^r$ is an $(r, \varepsilon)$-collapse in $N_\varepsilon(R(D^r)) \cap (\operatorname{bdy}(B_{r+1}))$.*
2. *$B_k = N(|C^k|, (E^n)'')$ and $B^r(|C^r|, (\operatorname{bdy}(B_{r+1}))'')$.*
3. *There is a PL map $g_{r-1}\colon D^{r-1} \to |C^r|$ such that $g^{r-1} \subset \varepsilon \circ (RD^{r-1}) \circ \varepsilon$,*

Proof. The proof is generalized directly from that of Miller. The proof is by downward induction on $j$.

Case 1. $j = k$. Suppose $\varepsilon > 0$ is given. Let $\varepsilon(3.3) = \varepsilon$ and choose $\delta = \delta(3.3)$. Let $g_{k-1}\colon D^{k-1} \to N_\delta(R(D^k))$ be a PL map with $g_{k-1} \subset \delta \circ (R|D^{k-1}) \circ \delta$ chosen by the Approximation Theorem. Apply Lemma 3.3 with $j = k$, $C^k(3.3) = \emptyset$ and $X(3.3) = g_{k-1}(D^{k-1})$. Then set $C^k(3.4) = C^k_X(3.3)$ and $B_k(3.4) = B^X_k(3.3)$.

Case 2. $j < k$. Choose $\alpha$, $0 < \alpha < \varepsilon$, according to Lemma 3.3; then choose $\beta$, $0 < \beta < \alpha$, so that if $f\colon D^{j-1} \to E^n$ is contained in $\beta \circ (R|D^j) \circ \beta$, then $f \subset \alpha \circ (R|D^{j-1}) \circ \alpha$. Finally choose $\delta$, $0 < \delta < \beta$, in the manner as $\alpha$ was chosen in Lemma 3.3. Induction on Lemma 3.4 yields $C^k, \ldots, C^{j+1}$ and $B_k, \ldots, B_{j+1}$ satisfying (1)-(3) above for $\delta > 0$. Let $g_m\colon D^j \to |C^{j+1}|$ be the PL map. As in Lemma 3.3 we push $g_j(D^j)$ into $\operatorname{bdy}(B_{j+1})$ via a PL map $p$, constructed using the inverse of the collapse of $|C^{j+1}|$ past $g_j(D^j)$. Since $p \circ g_j \subset \beta \circ (R|D^j) \circ \beta$, we have $p \circ g | D^{j-1} \subset \alpha \circ (R|D^{j-1}) \circ \alpha$. Apply Lemma 3.3 to the above sequences with $C^j = B_j = \emptyset$ added and $X(3.3) = p \circ g_{j-1}(D^{j-1})$. The result satisfies Lemma 3.4.

LEMMA 3.5. ([Mi³, Lemma 13]). *For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $C^k, \ldots, C^1$ and $B_k, \ldots, B_1$ are complexes satisfying conditions* (1)-(3) *of Lemma* 3.4 *with $\varepsilon$ replaced by $\delta$, then for each $j$, $1 \leqslant j \leqslant k$, there is a* PL *embedding $p_j\colon D^j \to B_j$ such that $p_j \subset \varepsilon\circ(R|D^j)\circ\varepsilon$.*

The proof is by upward induction on $j$. For $j = 3$, we have a PL map $g_2\colon D^2 \to |C^3|$. Without loss, we can assume $g_2\colon D \to \operatorname{bdy}(B_3)$ (see proof of Lemma 3.4) and $g_2$ is a PL embedding $(\dim(\operatorname{bdy}(B_3)) \geqslant 5)$. An application of Lemma 3.1 yields a PL embedding $p_3\colon D^3 \to B_3$.

For $j > 3$, choose $\alpha$, $0 < \alpha < \varepsilon$, using Lemma 3.1; then choose $\delta$, $0 < \delta < \alpha$ by induction on Lemma 5. We obtain a PL embedding $p_{j-1}\colon D^{j-1} \to B_{j-1}$ with $p_{j-1} \subset \alpha\circ(R|D^{j-1})\circ\alpha$. Apply Lemma 3.1 with $C(3.1) = C^j$ and $M(3.1) = \operatorname{bdy}(B_{j+1})$ to obtain $p_j\colon D^j \to B_j = N\left(|C^j|, (\operatorname{bdy}(B_{j+1}))''\right)$ with $p_j \subset \varepsilon\circ(R|D^j)\circ\varepsilon$.

The case $j = k$ finishes Theorem 3.0.

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