Cohomology groups of sheaves $\mathscr{S}\mathscr{E}L$ and splitness of Dolbeault complexes of sheaves $J_{V,\xi}$

by

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Abstract. It is shown that if X is an increasing union of Stein open sets and $\mathscr{S}$ a coherent analytic sheaf on X, then $H^q(X, \mathscr{S}\mathscr{E}L) = 0$ for every $q \geqslant 2$ and for every Hausdorff complete locally convex space L and $H^1(X, \mathscr{S}\mathscr{E}F') \neq 0$ for every Fréchet space F which does not admit a continuous norm. We prove also that on such manifolds Dolbeault complexes of sheaves $J_{V,\xi}$ split only at positive dimensions, where V is a closed submanifold of X, ξ a holomorphic Banach bundles over X and $J_{V,\xi}$ is the sheaf of germs of holomorphic sections of ξ on X vanishing on V.

Introduction. Let X be a topological space, $\mathscr{S}$ a sheaf of Hausdorff complete locally convex spaces and L a Hausdorff complete locally convex space. By $\mathscr{S}\mathscr{E}L$ we denote the sheaf on X given by the formula

$$U \mapsto \mathscr{S}(U)\mathscr{E}L \quad \text{for all open sets } U \text{ in } X,$$

where $\mathscr{S}(U)\mathscr{E}L = \mathrm{HOM}_{\mathscr{E}}\big(L'_c, \mathscr{S}(U)\big)$ is the $\mathscr{E}$-product of $\mathscr{S}(U)$ and L [3] (L'_c denotes the vector space of continuous linear functionals on L equipped with the compact-open topology and $\mathrm{HOM}_{\mathscr{E}}\big(L'_c, \mathscr{S}(U)\big)$ the space of continuous linear maps from L'_c into $\mathscr{S}(U)$ equipped with the topology of uniform convergence on equicontinuous subsets of L').

If X is an analytic space, $\mathscr{S}$ a coherent analytic sheaf on X and L a Fréchet space, the cohomology groups of sheaves $\mathscr{S}\mathscr{E}L$ are investigated by several authors [2], [3]. If L is a Hausdorff complete locally convex space, the cohomology groups $\mathscr{S}\mathscr{E}L$ are investigated in [11]. The aim of this paper is to continue study the cohomology groups of $\mathscr{S}\mathscr{E}L$. The obtained results are applied to study the splitness of Dolbeault complexes of the sheaves $J_{V,\xi}$.

In §1 we study the groups $H^q(X, \mathscr{S}\mathscr{E}L)$, where X either is an increasing union of Stein open sets or has a Stein morphism. The main results of this section are Theorems 1.2 and 1.6. We prove that if X is an increasing union of Stein open sets, then $H^q(X, \mathscr{S}\mathscr{E}L) = 0$ for every $q \geqslant 2$ and for every Hausdorff complete locally convex space L and $H^1(X, \mathscr{S}\mathscr{E}F') \neq 0$ for every Fréchet space F which does not admit a continuous norm.

Section 2 is devoted to prove this statement for the sheaves $J_{V,\xi}\mathscr{E}L$.

The cohomology groups of the sheaves $\mathscr{M}\mathscr{E}L$ of meromorphic functions on Riemann surfaces with values in Hausdorff complete locally convex space L are investigated in § 3. We prove that $H^1(X, \mathscr{M}\mathscr{E}L) = 0$ for every Hausdorff complete locally convex spaces L if and only if X is compact. The results of § 2 are applied to give the splitness of Dobeault complexes of the sheaves $J_{V,\xi}$ on complex manifolds which are increasing unions of Stein open sets only at positive dimensions.

§ 1. Cohomology groups of the sheaves $\mathscr{S}\mathscr{E}L$. Let $\mathscr{S}$ be a sheaf of Hausdorff complete locally convex spaces on a paracompact space X. Then the groups

$$H^q(X, \mathscr{S}) \overset{\text{def}}{=} \lim_{\to} H^q(\mathscr{U}, \mathscr{S})$$

are equipped with the inductive topology (where $\mathscr{U}$ is an open covering of X and

$$H^q(\mathscr{U}, \mathscr{S}) \overset{\text{def}}{=} \operatorname{Ker} \delta^q / \operatorname{Im} \delta^{q-1}$$

if

$$\delta^q = \delta^q(\mathscr{U}, \mathscr{S}): C_q(\mathscr{U}, \mathscr{S}) \to C^{q-1}(\mathscr{U}, \mathscr{S})$$

are the coboundary maps and $\delta^q(\mathscr{U}, \mathscr{S})$ are endoved with the product topology).

1.1. PROPOSITION. *Let X be an analytic space which is an increasing union of Stein open sets, $\mathscr{S}$ a coherent analytic sheaf on X and L a Hausdorff complete locally convex space. Then*

$$[H^q(X, \mathscr{S}\mathscr{E}L)]_s = 0 \quad \text{for every } q \geqslant 1.$$

Here for each locally convex space L by L_s we denote the space $L/\{\bar{0}\}$.

Proof. (a) We write $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}$ is an increasing sequence of Stein open sets. Let $\mathscr{U}_n = \{X_1, \ldots, X_n\}$, $\mathscr{U} = \{X_1, \ldots, X_n, \ldots\}$. Then for each $U \in \mathscr{U}(L)$, where $\mathscr{U}(L)$ denotes the set of all balanced convex neighbourhoods of zero in L, and for all $n, q \geqslant 1$ we have [3]

$$\operatorname{Im} \delta^{q-1}(\mathscr{U}_n, \mathscr{S}\mathscr{E}L(U)) = \operatorname{Ker} \delta^q(\mathscr{U}_n^{\bar{3}}, \mathscr{S}\mathscr{E}L(U)).$$

$L(U)$ denotes the completion of $L/\varrho_U \overset{\text{def}}{=} L/\varrho_U^{-1}(0)$ equipped with the norm ϱ_U generated by U. This implies that

(1.1) $\operatorname{Im} \delta^{q-1}(\mathscr{U}, \mathscr{S}\mathscr{E}L(U))$ is dense in $\operatorname{Ker} \delta^q(\mathscr{U}, \mathscr{S}\mathscr{E}L(U))$.

(b) Given $\alpha \in H^q(X, \mathscr{S}\mathscr{E}L)$, $q \geqslant 1$. Take a representative $\tilde{\alpha} \in \operatorname{Ker} \delta^q(\mathscr{U}_1, \mathscr{S}\mathscr{E}L)$ of α, where $\mathscr{U}_1 < \mathscr{U}$ is a Stein open covering of X. It suffices to show that

(1.2) $\pi(U)\tilde{\alpha} \in \overline{\operatorname{Im} \delta^{q-1}(\mathscr{U}_1, \mathscr{S}\mathscr{E}L(U))}$ for $U \in \mathscr{U}(L)$,

where $\pi(U)$ is the canonical map of L into $L(U)$. Since $\mathscr{U}$ and $\mathscr{U}_1$ are Leray coverings of X for the sheaf $\mathscr{S}\mathscr{E}L(U)$, we find $\tilde{\alpha}_1 \in \operatorname{Ker} \delta^q(\mathscr{U}, \mathscr{S}\mathscr{E}L(U))$ such that

(1.3) $\pi(U)\tilde{\alpha} - \tilde{\alpha}_1|\mathscr{U}_1 \in \operatorname{Im} \delta^{q-1}(\mathscr{U}_1, \mathscr{S}\mathscr{E}L(U))$.

From (1.1) and (1.3) we infer that (1.2) holds.

1.2. THEOREM. *Let X be an analytic space which is an increasing union of Stein open sets. Then*

(i) *$H^q(X, \mathscr{S}\mathscr{E}L) = 0$ for every $q \geqslant 2$ and every Hausdorff complete locally convex space L;*

(ii) *$H^1(X, \mathscr{S}\mathscr{E}F') \neq 0$ for every Fréchet space F which does not admit a continuous norm and for every coherent analytic sheaf $\mathscr{S}$ on X with $\dim \operatorname{supp} \mathscr{S} > 0$;*

(iii) *For every $n \geqslant 2$ there exists a coherent analytic subsheaf $\mathscr{S}$ of the sheaf $\mathcal{O}$ on C^n such that*

$$H^1(C^n, \mathscr{S}\mathscr{E}F') \neq 0$$

for some Fréchet space which has a continuous norm.

We need the following

1.3. LEMMA. *Let X be an irreducible analytic space of dimension 1 and $\mathscr{S}$ a torsion free coherent analytic sheaf on X. Then $H^0(X, \mathscr{S})$ has a continuous norm.*

Proof. Let K be a relatively compact open subset of $R(X)$, $K \neq \varnothing$, where $R(X)$ denotes the regular part of X. It suffices to show that if $\sigma \in H^0(X, \mathscr{S})$, $\sigma|K = 0$, then $\sigma \equiv 0$.

Since $\mathcal{O}_z$ is a domain of principal ideals for $z \in R(X)$, it follows that $\mathscr{S}|R(X)$ is locally free. Hence, by the connectedness of $R(X)$ we infer that $\sigma|R(X) = 0$. Let $z \notin R(X)$ and (U, π, Δ) be an analytic covering on some neighbourhood U of z, where $\Delta = \{z \in C: |z| < 1\}$. Since π is proper, by the Grauert theorem [4] it follows that $\pi_*\mathscr{S}$ is coherent analytic on Δ. Obviously $\pi_*\mathscr{S}$ is torsion free. This implies that $\pi_*\mathscr{S}$ is locally free on Δ. Combining this with the relations $\sigma|U \in H^0(U, \mathscr{S}) = H^0(\Delta, \pi_*\mathscr{S})$, $\sigma|R(U) = 0$ we infer that $\sigma = 0$ on U. Hence $\sigma \equiv 0$.

The lemma is proved.

Proof of Theorem 1.2. (i): For X being Stein (i) has been proved in [11], Theorem 1.2. Let us note that the proof in [11] gives also a proof

of (i) if we write $X = \bigcup_{n=1}^{\infty} W_n$, where W_n are relatively compact open sets in X_n, such that

$$W_n \subset \hat{W}_n = \mathcal{O}(X_n) - \text{hull}(W_n) \subset W_{n+1},$$

where $\{X_n\}$ is an increasing sequence of Stein open sets in X.

(ii): Assuming the contrary, we have

(1.4) $$H^1(X, \mathscr{S}\mathscr{E}C^{\infty\prime}) = 0$$

since by a theorem of Bessaga and Pełczyński F contains a complemented subspace isomorphic to C^{∞} [1].

(a) We first assume supp $\mathscr{S}$ contains an irreducible subvariety V of dimension 1. Let $\mathscr{S}_1 = \mathscr{S}/J_V\mathscr{S}$, where J_V denotes the ideal sheaf associated with V. By (i) and by the exactness of the cohomology sequence associated with the exact sequence

(1.5) $$0 \to J_V\mathscr{S}\mathscr{E}C^{\infty\prime} \to \mathscr{S}\mathscr{E}C^{\infty\prime} \to \mathscr{S}_1\mathscr{E}C^{\infty\prime} \to 0$$

it follows that

(1.6) $$H^1(V, \mathscr{S}_1\mathscr{E}C^{\infty\prime}) = 0.$$

The exactness of sequence (1.5) follows from a result of Bungart [3], Corollary 17.2.

Let $T(\mathscr{S}_1)$ denote the torsion subsheaf of $\mathscr{S}_1$. Since for every $z \in R(V)$, $\mathcal{O}_z$ is a domain of principal ideals, it follows that $\mathscr{S}_2 = \mathscr{S}_{1/T(\mathscr{S}_1)}$ is non-zero torsion free on V. Note that $\mathscr{S}_2$ satisfies (1.6).

By $R(\mathscr{S}_2)$ we denote the set consisting of points at which $\mathscr{S}_2$ is locally free on supp $\mathscr{S}_2 = V$. It is known [4], Corollary 2.13, that $R(\mathscr{S}_2)$ is open and dense in V. Since V is non-compact and dim $V = 1$, V is Stein [7]. Thus there exists $\sigma \in H^0(V, \mathscr{S}_2)$ such that $\sigma \neq 0$. Take a discrete sequence $\{z_j\} \subset R(\mathscr{S}_2)$ in V such that $\sigma(z_j) \neq 0$ for every $j \geq 1$. Select a holomorphic function f on V and an open covering $\mathscr{U} = \{U_j\}_{j=0}^{\infty}$ of V such that

$$f(z_j) = 0, \quad z_j \in U_j, \quad U_i \cap U_j = \varnothing \quad \text{for all } i,j \geq 1, i \neq j$$

and

$$f^{-1}(0) \cap U_0 = \varnothing.$$

Put

$$f_{ij} = 0 \quad \text{for} \quad i,j \geq 1 \quad \text{and} \quad f_{0j} = -f_{j0} = \sigma/f \otimes e_j \quad \text{for} \quad j \geq 0,$$

where $e_0 = 0$, $e_j = \underbrace{(0, \ldots, 0, 1)}_{j} \in C^{\infty\prime}$ for every $j \geq 1$. Obviously

$$(f_{ij}) \in \text{Ker } \delta^1(\mathscr{U}, \mathscr{S}_2 \, \mathscr{E}C^{\infty\prime}).$$

Since $H^1(V, \mathscr{S}_2\mathscr{E}C^{\infty\prime}) = 0$, without loss of generality we can assume that there exists $(f_j) \in C^0(\mathscr{U}\mathscr{E}C^{\infty\prime})$ such that $\delta^0(f_j) = (f_{ij})$. Thus we have

$$\sigma \otimes e_i + ff_i = \sigma \otimes e_j + ff_j \quad \text{on} \quad U_i \cap U_j, i,j \geq 0.$$

It follows that the formula

$$\tilde{\sigma} = \sigma \otimes e_j + ff_j \quad \text{on} \quad U_j, \quad j \geq 0$$

defines an element $\tilde{\sigma} \in H^0(V, \mathscr{S}_2 \, \mathscr{E}C^{\infty\prime}) = H^0(V, \mathscr{S}_2)\mathscr{E}C^{\infty\prime}$ such that

$$\tilde{\sigma}(z_j) = \sigma(z_j) \otimes e_j \quad \text{for every } j \geq 0.$$

Since $H^0(V, \mathscr{S}_2)$ has a continuous norm, it follows that

(1.7) $$\tilde{\sigma} = \sum_{j=1}^{n} \gamma_j \otimes e_j \quad \text{for some } n,$$

where $\gamma_j \in H^0(V, \mathscr{S}_2)$ for every $j \geq 1$.

By (1.7) we have

$$\tilde{\sigma}(z_{n+1}) \otimes e_{n+1} = \sum_{j=1}^{n} \gamma_j(z_{n+1}) \otimes e_j.$$

This is imposible since $\sigma(z_{n+1}) \neq 0$. Hence the case where supp $\mathscr{S}$ contains an irreducible subvariety of dimension 1 is proved.

(b) In general it suffices to show that supp $\mathscr{S}$ contains an irreducible subvariety of dimension 1. First we show that $H^0(X, \mathscr{S}) \neq 0$. Let

$$z_0 \in \tilde{R}(\mathscr{S}) \overset{\text{def}}{=} R\big((\mathscr{S}/J_{\text{supp }\mathscr{S}})\mathscr{S}|\text{supp }\mathscr{S}\big).$$

Consider the exact sequence

$$H^0(X, \mathscr{S}) \overset{\eta}{\to} H^0(X, \mathscr{S}/J_{z_0}\mathscr{S}) \to H^1(X, J_{z_0}\mathscr{S}) \to 0.$$

Since

$$\dim H^0(X, \mathscr{S}/J_{z_0}\mathscr{S}) = \dim H^0(\{z_0\}, \mathscr{S}/J_{z_0}\mathscr{S}) < \infty,$$

it follows that $\dim H^1(X, J_{z_0}\mathscr{S}) < \infty$. By Proposition 1.1 we infer that $H^1(X, J_{z_0}\mathscr{S}) = 0$. This implies that the map η is surjective. Thus $H^0(X, \mathscr{S}) \neq 0$.

By considering the sheaf $\mathscr{S}/J_V\mathscr{S}$, where V is an irreducible subvariety of supp $\mathscr{S}$ of dimension > 0 and by the proof in (a) we can assume that supp $\mathscr{S}$ is irreducible. Take $\sigma \in H^0(X, \mathscr{S})$, $\sigma|\tilde{R}(\mathscr{S}) \neq 0$. Consider the sheaf $\mathscr{S}_1 = \mathscr{S}/\sigma\mathcal{O}$. Then $\mathscr{S}_1$ satisfies (1.4). Thus we can assume that supp $\mathscr{S}_1$ is irreducible and supp $\mathscr{S} = \text{supp }\mathscr{S}_1$. Since $\sigma\mathcal{O} \neq 0$, this implies that

$$\dim \mathscr{S}_z/m_z\mathscr{S}_z > \dim \mathscr{S}_{1z}/m_z\mathscr{S}_{1z}$$

for all $z \in \check{R}_1(\mathscr{S}) \cap R(\check{\mathscr{S}})$, $\sigma(z) \neq 0$, where m_z denotes the maximal ideal in $(\mathcal{O}_{\mathrm{supp}\,\mathscr{S}})_z$. Note that $\check{R}(\mathscr{S}) \cap \check{R}(\mathscr{S}_1)$ is dense in $\mathrm{supp}\,\mathscr{S}$.

Continuing this process we get a coherent analytic sheaf $\mathscr{S}_{k_1}$ on X such that $\mathscr{S}_{k_1}$ satisfies (1.4), $\mathrm{supp}\,\mathscr{S}_{k_1}$ is irreducible and

$$\dim \mathscr{S}_{k_1 z}/m_z \mathscr{S}_{k_1 z} = 1$$

for every z belonging to a dense open subset of $\mathrm{supp}\,\mathscr{S}$.

Let $\beta \in H^0(X, \mathscr{S}_{k_1})$, $\beta | \check{R}(\mathscr{S}_{k_1}) \neq 0$. Consider the sheaf $\mathscr{S}^1 = \mathscr{S}k_1/\beta\mathcal{O}$. Then $\mathscr{S}^1 \neq 0$, $\mathscr{S}^1$ satisfies (1.4) and $\mathrm{supp}\,\mathscr{S}^1$ contains an irreducible subvariety X_1 of dimension $< \dim X$. Using the above argument to $\mathscr{S}^1/J_{X_1}\mathscr{S}^1$ we get an irreducible subvariety X_2 of X_1 of dimension $< \dim X_1$. Continuing this process we infer that $\mathrm{supp}\,\mathscr{S}$ contains an irreducible subvariety of dimension 1.

(iii): Let $\Delta = \{z \in C : |z| < 1\}$. Select a holomorphic function f on Δ such that f is not locally bounded at each point of $\partial \Delta$. Then the map $\gamma : \Delta \to C^2$ given by $\gamma(z) = (z, f(z))$ is a proper embedding of Δ into $C^2 \subset C^n$. Let J denote the ideal sheaf associated with $\gamma(\Delta)$. Since it does not exist a continuous linear extension map from $\mathcal{O}(\gamma(\Delta))$ into $\mathcal{O}(C^n)$ ([9], Proposition 5.3), by the exactness of the cohomology sequence associated with the exact sequence

$$0 \to J\mathscr{E}F' \to \mathcal{O}\mathscr{E}F' \to \mathcal{O}/J\mathscr{E}F' \to 0,$$

where $F = \mathcal{O}(\gamma(\Delta))$, it follows that $H^1(C^n, J\mathscr{E}F') = 0$. The theorem is proved.

Let X be an analytic space. We say that X has a *Stein morphism* if there exists a holomorphic map π from X into a Stein space W such that $\pi^{-1}(U)$ is Stein for every U belonging to some Stein open covering $\mathscr{U}$ of W.

Since $\pi(\partial\pi^{-1}(V)) \subset \partial V$ for every open subset V of W, it is easy to check that $\pi^{-1}(V)$ is Stein for every Stein open set V in W contained in some $U \in \mathscr{U}$. Thus by the covering lemma of Stehlé [18] there exists a Stein open covering $\{V_j\}$ of W such that

(S₁) $\Omega_j = \bigcup_{i \leqslant j} V_i$ are Stein;

(S₂) $\Omega_j \cap V_{j+1}$ are Runger in V_{j+1};

(S₃) $\tilde{\mathscr{U}} = \{X'_j = \pi^{-1}(V_j)\}$ is a Stein open covering of X.

Since X'_{j+1} is Stein, by (S₂) it is easy to check that $X_j \cap X'_{j+1}$ is Runger in X'_{j+1}, $X_j = \pi^{-1}(\Omega_j)$. Take a Stein open covering $\tilde{\mathscr{U}}_1$ of X such that $\tilde{\mathscr{U}}_1 < \tilde{\mathscr{U}}$, $\tilde{\mathscr{U}}_1$ forms a basis of open sets in X and

$$\tilde{\mathscr{U}}_1 | X_{j+1} = \tilde{\mathscr{U}}_1 | X_j \cup \tilde{\mathscr{U}}_1 | X'_{j+1},$$

where for every open subset G of X we write

$$\tilde{\mathscr{U}}_1 | G' = \{U \in \mathscr{U}_1 : U \subset G\}.$$

Then by an argument as in [8], Theorems 1 and 2, we get the following

1.4. PROPOSITION. *Let X be an analytic space having a Stein morphism and $\mathscr{S}$ a Fréchet sheaf on X satisfying the following conditions:*

(i) $H^q(U, \mathscr{S}) = 0$ *for every Stein open subset of X and for every $q \geqslant 1$;*

(ii) *The restriction map* $H^0(U, \mathscr{S}) \to H^0(V, \mathscr{S})$ *has dense image for every Runger domain V in any Stein open subset U of X.*

Then

(i) $H^q(X, \mathscr{S}) = 0$ *for every $q \geqslant 2$;*

(ii) *The coboundary map*

$$\delta^0 : C^0(\tilde{\mathscr{U}}_1, \mathscr{S}) \to \mathrm{Ker}\,\delta^1(\tilde{\mathscr{U}}_1, \mathscr{S})$$

has dense image.

Since for every coherent analytic sheaf $\mathscr{S}$ on X and for every Fréchet space F the sheaf $\mathscr{S}\mathscr{E}F$ satisfies the conditions in Proposition 1.4, by an argument as in Proposition 1.1 we get the following

1.5. PROPOSITION. *Let X be an analytic space having a Stein morphism, $\mathscr{S}$ a coherent analytic sheaf on X and L a Hausdorff complete locally convex space. Then*

$$[H^q(X, \mathscr{S}\mathscr{E}L)]_s = 0 \quad \text{for every } q \geqslant 1.$$

1.6. THEOREM. *Let X be an analytic space having a Stein morphism and $\mathscr{S}$ a coherent analytic sheaf on X. Then*

(i) $H^q(X, \mathscr{S}\mathscr{E}L) = 0$ *for every $q \geqslant 3$ and for every Hausdorff complete locally convex space L;*

(ii) *If $H^0(X, \mathscr{S})$ has a continuous norm, then*

$$H^1(X, \mathscr{S}\mathscr{E}F') \neq 0$$

for every Fréchet space F which does not admit a continuous norm.

Proof. (i): By Theorem 1.2 (i) we have $H^q(Z, \mathscr{S}\mathscr{E}L) = 0$ for every $q \geqslant 2$ and for every Stein open subset Z of X. Thus by induction on j and by considering the Mayer–Vietoris sequence of pairs (X_j, X'_{j+1}) we infer that

$$(1.8) \qquad H^q(X_j, \mathscr{S}\mathscr{E}L) = 0 \quad \text{for every } q \geqslant 3 \text{ and } j \geqslant 1.$$

Take a flabby resolution

$$0 \to \mathscr{S}\mathscr{E}L \to J_0 \xrightarrow{d_0} J_1 \xrightarrow{d_1} \cdots$$

of $\mathscr{S}\mathscr{E}L$. We prove that for every $j \geqslant 1$ and $q \geqslant 3$ the restriction map

$$\mathrm{Ker}\,\hat{d}_{q-1} | X_{j+1} \to \mathrm{Ker}\,\hat{d}_{q-1} | X_j$$

is surjective.

Given $\alpha \in \operatorname{Ker} \hat{d}_{q-1}|X_j$. Since $X_j \cap X'_{j+1}$ is Stein and $q-1 \geqslant 2$, we find $\eta \in J_{q-2}(X_j \cap X'_{j+1})$ such that

$$\hat{d}_{q-2}|X_j \cap X'_{j+1}\eta = \alpha|X_j \cap X'_{j+1}.$$

Let $\tilde{\eta}$ be an extension of η on X'_{j+1}. Setting

$$\tilde{\alpha}|X_j = \alpha \quad \text{and} \quad \tilde{\alpha}|X'_{j+1} = \hat{d}_{q-2}|X'_{j+1}\tilde{\eta},$$

we get an element $\tilde{\alpha} \in \operatorname{Ker} \hat{d}_{q-1}|X_{j+1}$ extending α. Now let $\alpha \in \operatorname{Ker} \hat{d}_q$, $q \geqslant 3$. Take $\beta_1 \in J_{q-1}(X_1)$ such that

$$\hat{d}_{q-1}|X_1\beta_1 = \alpha|X_1.$$

By (1.8) there exists $\beta'_2 \in J_{q-1}(X_2)$ such that

$$\hat{d}_{q-1}|X_2\beta'_2 = \alpha|X_2.$$

Since $(\beta'_2 - \beta_1)|X_1 \in \operatorname{Ker} \hat{d}_{q-1}|X_1$, we find $\beta''_2 \in \operatorname{Ker} \hat{d}_{q-1}|X_2$ extending $(\beta'_2 - \beta_1)|X_1$. Put $\beta_2 = \beta'_2 - \beta''_2$. Then

$$\beta_2 \in J_{q-1}(X_2) \quad \text{and} \quad \beta_2|X_1 = \beta_1, \quad \hat{d}_{q-1}|X_2\beta_2 = \alpha|X_2.$$

Continuing this process we get elements $\beta_n \in J_{q-1}(X_n)$ such that

$$\beta_n|X_{n-1} = \beta_{n-1} \quad \text{and} \quad \hat{d}_{q-1}|X_n\beta_n = \alpha|X_n \quad \text{for every } n \geqslant 2.$$

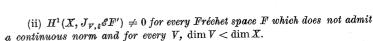Thus the formula $\beta|X_n = \beta_n$ for every $n \geqslant 1$ defines an element $\beta \in J_{q-1}(X)$ such that $\hat{d}_{q-1}\beta = \alpha$. Hence (i) is proved.

(ii): By the proof of Theorem 1.2 it suffices to show that if $H^1(X, \mathscr{S}\mathscr{E}C^{\infty\prime}) = 0$, then

$$H^0(X, \mathscr{S}) \neq 0 \quad \text{and} \quad \dim \mathscr{O}(X) > 1.$$

Obviously $\dim \mathscr{O}(X) > 1$ since X has a Stein morphism. That $H^0(X, \mathscr{S}) \neq 0$ follows from Proposition 1.4 and from the proof of Theorem 1.2 (ii). The theorem is proved.

§2. Cohomology groups of sheaves $J_{V,\xi}\mathscr{E}L$.

Let X be an analytic space and ξ a holomorphic Banach bundle over X. Let V be a subvariety of X. By $J_{V,\xi}$ we denote the Fréchet sheaf of germs of holomorphic sections of ξ on X vanishing on V. In this section we investigate the cohomology groups of the sheaves $J_{V,\xi}\mathscr{E}L$, where L is a Hausdorff complete locally convex space.

2.1. THEOREM. *Let X, V and ξ be as above. Let X be an increasing union of Stein open sets. Then*

(i) $H^q(X, J_{V,\xi}\mathscr{E}L) = 0$ *for every $q \geqslant 2$ and for every Hausdorff complete locally convex space L;*

(ii) $H^1(X, J_{V,\xi}\mathscr{E}F') \neq 0$ *for every Fréchet space F which does not admit a continuous norm and for every V, $\dim V < \dim X$.*

We need the following

2.2. LEMMA [14]. *Let*

$$0 \to \{G_n, \beta_n^m\} \overset{\{f_n\}}{\to} \{F_n, \omega_n^m\} \overset{\{g_n\}}{\to} \{E_n, \alpha_n^m\} \to 0$$

be a complex of projective systems of Fréchet spaces and let

$$\operatorname{Ker} f_n = 0, \quad \operatorname{Im} f_n = \operatorname{Ker} g_n \quad \text{and} \quad \operatorname{Im} g_n = E_n \quad \text{for all } n \geqslant 1.$$

Then the map $\varprojlim g_n$ is surjective if and only if for every n_0 there exists $n(n_0) \geqslant n_0$ such that

(2.1) $\operatorname{Im} \beta_n^{n_0}$ *is dense in* $\operatorname{Im} \beta_{n(n_0)}^{n_0}$ *for every* $n \geqslant n(n_0)$.

Proof of Theorem 2.1. (i): We write $X = \bigcup_{n=1}^{\infty} W_n$ as in the proof of Theorem 1.2 (i). Since every holomorphic Banach bundle over a Stein space is complemented in some trivial Banach bundle [19], Theorem 3.9, by a theorem of Bungart [3], Theorem B*, it follows that

(2.2) $$H^q(\hat{W}_n, J_{V,\xi}\mathscr{E}L) = 0 \quad \text{for every } q, n \geqslant 1.$$

From (2.2) by an argument as in the proof of Theorem 1.2 (i) we infer that $H^q(X, J_{V,\xi}\mathscr{E}L) = 0$ for every $q \geqslant 2$.

(ii): For a contradiction we have $H^1(X, J_{V,\xi}\mathscr{E}C^{\infty\prime}) = 0$. Let W be an irreducible subvariety of X such that $W \cap X_n \setminus V$ is infinite for sufficiently large n, let

$$\gamma: \mathscr{O} \to \mathscr{O}/J_W \quad \text{and} \quad \gamma_\xi: \mathscr{O}_\xi \to \mathscr{O}_\xi/J_{W,\xi}, \quad \mathscr{O}_\xi = J_{\varnothing,\xi}$$

be the canonical maps. Put $\mathscr{S} = \gamma(J_V)$, $\mathscr{S}_\xi = \gamma(J_{V,\xi})$. Then $\mathscr{S}$ is coherent analytic sheaf and $\operatorname{supp} \mathscr{S} \cap X_n$ is infinite for sufficiently large n. It is easy to check that if $\xi|U \cong B$, where B denotes the trivial bundle over U with fiber B, then

(2.3) $$\operatorname{Ker} \gamma_\xi^0|U \cong \operatorname{Ker} \gamma^0|U\mathscr{E}B, \quad \mathscr{S}_\xi|U \cong \mathscr{S}|U\mathscr{E}B$$

and

(2.4) $$\operatorname{Ker} \gamma_{\xi \oplus \eta}^0 \cong \operatorname{Ker} \gamma_\xi^0 \oplus \operatorname{Ker} \gamma_\eta^0,$$

where $\gamma^0 = \gamma|J_V$, $\gamma_\xi^0 = \gamma_\xi|J_{V,\xi}$.

From (2.3) by the associability of the $\mathscr{E}$-product it follows that the sequence

(2.5) $$0 \to \operatorname{Ker} \gamma_\xi^0 \mathscr{E}C^{\infty\prime} \to J_{V,\xi}\mathscr{E}C^{\infty\prime} \to \mathscr{S}_\xi \mathscr{E}C^{\infty\prime} \to 0$$

is exact. Since every holomorphic Banach bundle over a Stein space is

isomorphic to a complemented subbundle of a trivial bundle, from (2.4) it follows that

$$(2.6) \qquad H^q(\hat{W}_n, \operatorname{Ker} \gamma_\xi^0 \mathscr{E} C^{\infty\prime}) = 0 \qquad \text{for every } q, n \geqslant 1$$

and

$$(2.7) \qquad H^q(Z, \mathscr{S}_\xi) = 0$$

for every $q \geqslant 1$ and for every Stein open subset Z of X. By (2.6) as in (i) we have

$$(2.8) \qquad H^q(X, \operatorname{Ker} \gamma_\xi^0 \mathscr{E} C^{\infty\prime}) = 0$$

for every $q \geqslant 2$. From (2.8) by the exactness of sequence (2.5) we infer that

$$(2.9) \qquad H^1(X, \mathscr{S}_\xi \mathscr{E} C^{\infty\prime}) = H^1(X, \mathscr{S}_\xi) = 0.$$

Consider the complex of projective system of Fréchet spaces

$$0 \to \{\mathscr{S}_\xi(X_n)\} \to \{C_0(\mathscr{U}_n, \mathscr{S}_\xi)\} \xrightarrow{\{\delta_n^0\}} \{\operatorname{Ker}\delta^1(\mathscr{U}_n, \mathscr{S}_\xi)\} \to 0,$$

where $\mathscr{U} = \{X_n\}, \mathscr{U}_n = \{X_1, \ldots, X_n\}$. By (2.7) δ_n^0 are surjective and $\mathscr{U}$ is a Leray covering of X for $\mathscr{S}_\xi$. By (2.9) the map $\varprojlim \delta_n^0$ is surjective. Hence the restriction maps

$$R_{\xi n}^m : \mathscr{S}_\xi(X_n) \to \mathscr{S}_\xi(X_n)$$

satisfy (2.1). We prove that the restriction maps $R_n^m : \mathscr{S}(X_n) \to \mathscr{S}(X_m)$ satisfy also (2.1).

(a) First assume that ξ is infinite dimensional. Given n_0. Take $n(n_0) \geqslant n_0$ such that (2.1) holds for $R_{\xi n}^m$. Let $n \geqslant n(n_0)$. By a theorem of Zajdenberg–Krejn–Kusment–Pankov [19], Theorem 3.13, there exists $\sigma \in \mathscr{O}_\xi(X_n)$ such that $\sigma(z) \neq 0$ for every $z \in X_n$. Let η denote the subbundle of $\xi|X_n$ spanned by $\sigma(X_n)$ and θ the isomorphism of C onto η given by

$$\theta(z, \lambda) = \lambda \sigma(z) \qquad \text{for all } (z, \lambda) \in C.$$

By (2.1) and since η is complemented in $\xi|X_n$ ([19], Theorem 3.11), it follows that $\operatorname{Im} R_n^{n_0}$ is dense in $\operatorname{Im} R_n^{n(n_0)}$ for every $n \geqslant n(n_0)$.

(b) Now assume that ξ is finite dimensional. Consider the infinite dimensional bundle $\xi \mathscr{E} B$, where B is a Banach space $\dim B = \infty$. Then by (2.9) and by the nuclearity of the space $C^0(\mathscr{U}, \mathscr{S}_\xi)$ we infer that $H^1(X, \mathscr{S}_{\xi \mathscr{E} B}) = 0$. Combining this with (a) it follows that the maps R_n^m satisfy (2.1). Since X_n is Stein, and $\operatorname{supp}\mathscr{S} \cap X_n$ is infinite for sufficiently large n, we infer that $\dim \mathscr{S}(X_n) = \infty$ for sufficiently large ([12], Lemma 2.3). Hence $\dim \mathscr{S}(X) = \infty$ since the maps R_n^m satisfy (2.1). On the other

hand, since

$$\mathscr{S}(X_n)\mathscr{O}_\xi(W \cap X_n) \subset \mathscr{S}_\xi(X_n),$$

it follows that $\mathscr{S}_\xi(X_n) \neq 0$ for sufficiently large n. Hence $\mathscr{S}_\xi(X) \neq 0$ since $R_{\xi n}^m$ satisfy (2.1). Note that $\mathscr{S}_\xi(X) \hookrightarrow \mathscr{O}_\xi(W)$ and hence $\mathscr{S}_\xi(X)$ has a continuous norm. Let $f \in \mathscr{S}(X)$ and $f \neq$ constant. By the irreducibility of W we can assume that f is bounded since if $a = \sup|f(z)| < \infty$, we replace f by the function $(r - f(z))^{-1}$, where $|r| = a$ and $r = \lim f(z_n)$ for some sequence $\{z_n\} \subset X$. Let $\sigma \in \mathscr{S}_\xi(X)$, $\sigma \neq 0$. Take a sequence $\{z_j\}_{j=1}^\infty$ in X such that $\sigma(z_j) \neq 0$ for every $j \geqslant 1$, $|f(z_j)| \to \infty$ and $f(z_j) \neq f(z_k)$ for every $j \neq k$. Let $\varphi \in \mathscr{O}(C)$ such that $\varphi^{-1}(0) = \{f(z_j)\}_{j=1}^\infty$. Since $\{f(z_j)\}_{j=1}^\infty$ is discrete, there exists an open covering $\mathscr{U}_1 = \{U_j\}_{j=0}^\infty$ of X such that

$$z_j \in U_j, \qquad U_i \cap U_j = \varnothing \qquad \text{for every } i, j \geqslant 1, \ i \neq j$$

and

$$g^{-1}(0) \cap U_0 = \varnothing, \qquad \text{where} \qquad g = \varphi(f).$$

Since $\mathscr{S}_\xi(X)$ has a continuous norm, using the proof of Theorem 1.2 (ii) to $\mathscr{S}_\xi$, $\mathscr{U}_1$, σ, and g we complete the proof. Theorem is proved.

2.3. THEOREM. *Let X have a Stein morphism and V, ξ be as in Theorem 2.1. Then*

(i) $H^q(X, J_{V,\xi}\mathscr{E}L) = 0$ *for every $q \geqslant 3$ and for every Hausdorff complete locally convex space L;*

(ii) *If X is irreducible, then $H^1(X, J_{V,\xi}\mathscr{E}F') \neq 0$ for every Fréchet space F which does not admit a continuous norm.*

Proof. (i) follows from Theorem 2.1 (i) and from the proof of Theorem 1.6 (i).

(ii): Since X is irreducible, $J_{V,\xi}(X)$ has a continuous norm. Hence (ii) follows from the proof of Theorem 1.6 (ii) and from the following

2.4. PROPOSITION. *Let X, V and ξ be as in Theorem 2.3. Let $V \neq X$, $\xi \neq 0$ and $H^1(X, J_{V,\xi}) = 0$. Then $J_{V,\xi}(X) \neq 0$.*

Proof. For a contradiction we get a commutative and exact diagram

$$
\begin{array}{ccccccccc}
& & 0 & \to & \xi_{z_0} & & \to H^1(X, \mathscr{S}') \to 0 \\
& & & & \downarrow & & \downarrow \\
0 \to & C^0(\tilde{\mathscr{U}}_1, \mathscr{S}') \to & & C^0(\tilde{\mathscr{U}}_1, \mathscr{S}) \to & C^0(\tilde{\mathscr{U}}_1, \mathscr{S}'') \to 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 \to \operatorname{Ker}\delta^1(\tilde{\mathscr{U}}_1, \mathscr{S}') \to & \operatorname{Ker}\delta^1(\tilde{\mathscr{U}}_1, \mathscr{S}) \to & & \operatorname{Ker}\delta^1(\tilde{\mathscr{U}}_1, \mathscr{S}'') \\
& & & \downarrow \\
& & & 0
\end{array}
$$

whe~ s the open covering of X as in Proposition 1.4, $z_0 \notin V$, $\mathscr{S} = J_{V,\xi}$,

$\mathscr{S}' = J_{VU\{z_0\},\xi}$, $\mathscr{S}'' = \mathscr{S}/\mathscr{S}'$. Whence, we infer that $\operatorname{Im}\delta^0(\tilde{\mathscr{U}}_1, \mathscr{S}')$ is closed in $\operatorname{Ker}\delta^1(\tilde{\mathscr{U}}_1, \mathscr{S}')$ and hence by Proposition 1.4 we have

$$H^1(X, \mathscr{S}') = \operatorname{Ker}\delta^1(\tilde{\mathscr{U}}_1, \mathscr{S}')/\operatorname{Im}\delta^0(\tilde{\mathscr{U}}_1, \mathscr{S}') = 0.$$

This implies that $\xi_{z_0} = 0$ which contradics $\xi \neq 0$. The proposition is proved.

§3. Cohomology groups of sheaves $\mathscr{M}\mathscr{E}L$. Let X be a complex manifold and L a Hausdorff complete locally convex space. A holomorphic function f on a dense open subset G of X with values in L is called *meromorphic on X* if for each $z \in X$ there exist a neighbourhood U of z in X and a non-zero holomorphic function σ on U such that σf can be extended to a holomorphic function on U.

By $\mathscr{M}\mathscr{E}L$ we denote the sheaf of germs of meromorphic functions on X with values in L. In this section we investigate the cohomology groups of the sheaves $\mathscr{M}\mathscr{E}L$ on Riemann surfaces. We prove the following

3.1. THEOREM. *Let X be a Riemann surface. Then the following conditions are equivalent:*

(i) *X is compact;*

(ii) *$H^1(X, \mathscr{M}\mathscr{E}L) = 0$ for every Hausdorff complete lovally convex space L;*

(iii) *$H^1(X, \mathscr{M}\mathscr{E}F') = 0$ for every Fréchet space F which does not admit a continuous norm;*

(iv) *For every closed subspace E of every Fréchet space F the canonical map $H^0(X, \mathscr{M}\mathscr{E}F') \to H^0(X, \mathscr{M}\mathscr{E}E')$ induced by the restriction map $F' \to E'$ is surjective.*

Let $f \in H^0(U, \mathscr{M}\mathscr{E}L)$, where U is an open subset of X. Then the pole

$$P(f) = \{z \in X : f_z \notin (\mathcal{O}\mathscr{E}L)_z\}$$

is discrete in U [13], Theorem 1.1. Hence for every $z_0 \in U$ there exists a Laurent expansion

$$f(z) = \sum_{j=k}^{\infty} a_j z^j, \qquad a_k \neq 0$$

of f in some neighbourhood of z_0, where z is a local coordinate in a neighbourhood of z_0. It is easy to see that the number k is independent of choice of the local coordinate z. Hence we can put $\operatorname{ord}_{z_0}(f) = k$.

Let D be a divizor on X. By $\mathcal{O}_D^L$ we denote the subsheaf of the sheaf $\mathscr{M}\mathscr{E}L$ given by the formula

$$U \mapsto \{f \in H^0(U, \mathscr{M}\mathscr{E}L) : \operatorname{ord}_z(f) \geqslant -D(z) \text{ for every } z \in U\}.$$

We write $\mathcal{O}_D^C = \mathcal{O}_D$. Note that $\mathcal{O}_D$ is locally free and for every Stein open set $U \subset X$ we have

$$(3.1) \qquad f \in \mathcal{O}_D^L(U) \Leftrightarrow \sigma f \in (\mathcal{O}\mathscr{E}L)(U) = \operatorname{HOM}\big(L_c', \mathcal{O}(U)\big)$$
$$\Leftrightarrow f \in \operatorname{HOM}_{\mathscr{E}}\big(L_c', \mathcal{O}_D(U)\big) = (\mathcal{O}_D\mathscr{E}L)(U),$$

where $\sigma \in \mathscr{M}(U)$, $\operatorname{ord}_z\sigma = D(z)$ for $z \in U$. From (3.1) we infer that $\mathcal{O}_D^L = \mathcal{O}_D\mathscr{E}L$.

Proof of Theorem 3.1. (i)$\Rightarrow$(ii): (a) We first prove that $H^1(X, \mathcal{O}_D\mathscr{E}L) = 0$ for every divizor D on X, $\deg D \geqslant 2g-2$, $g = \dim H^1(X, \mathcal{O}_D)$. It is known [5] that $H^1(X, \mathcal{O}_D) = 0$. Hence by the compactness of X and by Theorem 1.2 (ii) in [11], we infer that $H^1(X, \mathcal{O}_D\mathscr{E}L) = 0$.

(b) In general, let $\eta \in H^1(X, \mathcal{O}_D\mathscr{E}L)$ and let $\mathscr{U}$ be a finite open covering of X such that η is image of a cochain $(f_{ij}) \in \operatorname{Ker}\delta^1(\mathscr{U}, \mathscr{M}\mathscr{E}L)$. Since $\mathscr{U}$ is finite, there exists a divizor D on X such that

$$\deg D \geqslant 2g-2 \qquad \text{and} \qquad (f_{ij}) \in \operatorname{Ker}\delta^1(\mathscr{U}, \mathcal{O}_D\mathscr{E}L).$$

By (a) the cochain (f_{ij}) can be written in the form

$$f_{ij} = f_i - f_j, \qquad \text{where} \qquad f_j \in H^0(U_j, \mathcal{O}_D\mathscr{E}L).$$

Hence $(f_{ij}) = \delta^0(f_j)$ and $\eta = 0$.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i): Since $C^{\infty\prime}$ does not admit a continuous norm, it suffices to prove that if X is not compact, then $H^1(X, \mathscr{M}\mathscr{E}C^{\infty\prime}) \neq 0$. Select an open covering $\mathscr{U} = \{U_j\}_{j=0}^{\infty}$ of X such that $\overline{U}_i \cap \overline{U}_j = \emptyset$ for every $i, j \geqslant 1$, $i \neq j$, $U_0 \cap U_j$ is isomorphic to a bounded domain in C for every $j \geqslant 1$ and U_0 is connected. Obviously such a covering exists. For each $j \geqslant 1$ we find an $\sigma_j \in \mathcal{O}(U_0 \cap U_j)$ which is not locally bounded at every point of $\partial(U_0 \cap U_j)$. Put

$$m_{ij} = 0 \quad \text{for} \quad i, j \geqslant 1 \quad \text{and} \quad m_{0j} = -m_{j0} = \sigma_j e_j \quad \text{for} \quad j \geqslant 0.$$

Then $(m_{ij}) \in \operatorname{Ker}\delta^1(\mathscr{U}, \mathscr{M}\mathscr{E}C^{\infty\prime})$. We prove that $m \neq 0$ in $H^1(X, \mathscr{M}\mathscr{E}C^{\infty\prime})$. For a contradiction there exists an open covering $\mathscr{U}' = \{V_j\}_{j \in Z^+}$, $Z^+ = \{0, 1, \ldots\}$ of X such that

$$(3.2) \qquad \begin{aligned} V_j &\subset U_{a(j)} \quad \text{for} \quad j \in Z^+, \\ V_j &\text{ is connected} \quad \text{for} \quad j \in Z^+, \\ m &= 0 \quad \text{in} \quad H^1(\mathscr{U}', \mathscr{M}\mathscr{E}C^{\infty\prime}). \end{aligned}$$

For each $z \in U_0$ take an open neighbourhood of z $V_z \subset U_0 \cap V_{\beta(z)}$ for some $\beta(z) \in Z^+$. Then $\mathscr{U}'' = \{V_j, V_z\} < \mathscr{U}' < \mathscr{U}$ and $m = 0$ in $H^1(\mathscr{U}'', \mathscr{M}\mathscr{E}C^{\infty\prime})$.

Thus there exist $m_j \in H^0(V_j, \mathcal{M}\mathcal{E}C^\infty{}')$ and $m_z \in H^0(V_z, \mathcal{M}\mathcal{E}C^\infty{}')$ such that

$$
(3.3) \qquad
\begin{aligned}
m_i - m_j &= m_{a(i)a(j)}, \\
m_z - m_j &= m_{0a(j)} = \sigma_{a(j)}e_{a(j)}, \\
m_z - m_y &= m_{00} = 0.
\end{aligned}
$$

From (3.3) it follows that there exists $\tilde{m}_0 \in H^0(U_0, \mathcal{M}\mathcal{E}C^\infty{}')$ such that

$$(3.4) \qquad \tilde{m}_0 - m_j = \sigma_{a(j)}e_{a(j}) \qquad \text{for} \quad j \in Z^+.$$

By the connectedness of U_0 and of V_j it follows that

$$(3.5) \qquad \tilde{m}_0 = \sum_{k=1}^{n_0} m_k^0 e_k, \qquad m_k^0 \text{ are meromorphic on } U_0,$$

$$(3.6) \qquad m_j = \sum_{k=1}^{n_j} m_k^j e_k, \qquad m_k^j \text{ are meromorphic on } V_j.$$

Let $j_0 \in Z^+$ such that $a(j_0) > n_0$. Since

$$\partial U_0 \cap U_{a(j_0)} \subseteq \bigcup_{a(j)=a(j_0)} V_j,$$

there exists $j_1 \in Z^+$ such that $V_{j_1} \subset U_{a(j_0)}$, $V_{j_1} \cap U_0 = \varnothing$ and $V_1 \nsubseteq U_0$. From (3.4), (3.5) and (3.6) we infer that $m_{a(j_1)}^{j_1} = \sigma a(j_1)$ on $U_0 \cap V_{j_1}$. Since V_{j_1} is connected, $V_{j_1} \nsubseteq U_0$ and $U_0 \cap V_{j_1} \neq \varnothing$, it follows that $\partial(U_0 \cap U_{a(j_1)}) \cap \cap V_{j_1}$ is not discrete. On the other hand, since $P(m_{a(j_1)}^{j_1})$ is discrete, we infer that $\sigma_{a(j_1)}$ is locally bounded at some $z \in \partial(U_0 \cap U_{a(j_1)})$. This contradics the choice of $\sigma_{a(j_1)}$.

(ii) $\Rightarrow$ (iv): Since F is Fréchet, it is easy to check that the sequence

$$0 \to \mathcal{O}\mathcal{E}G' \to \mathcal{O}\mathcal{E}F' \to \mathcal{O}\mathcal{E}E' \to 0,$$

where $G = F/E$, is exact. Hence the sequence

$$(3.7) \qquad 0 \to \mathcal{M}\mathcal{E}G' \to \mathcal{M}\mathcal{E}F' \to \mathcal{M}\mathcal{E}E' \to 0$$

is also exact. Thus by the exactness of the cohomology associated with the exact sequence (3.7) it follows that the map

$$H^0(X, \mathcal{M}\mathcal{E}F') \to H^0(X, \mathcal{M}\mathcal{E}E')$$

is surjective.

(iv) $\Rightarrow$ (i): We write $X = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}$ is an increasing sequence of compact subsets of X. Let

$$\theta: \mathcal{O}(X) \to J = \prod_{n=1}^{\infty} [\mathcal{O}(\hat{X})/\varrho_{K_n}],$$

---

where ϱ_{K_n} is the norm defined by K_n, be the canonical embedding. Applying (iv) to the holomorphic map

$$Ev: X \to \mathcal{O}(X)': [(Ev)z]\sigma = \sigma(z)$$

we get a meromorphic map

$$f: X \to J' = \bigoplus_{n=1}^{\infty} \widehat{[\mathcal{O}(X)/\varrho_{K_n}]}'$$

such that

$$(3.8) \qquad [\theta'fz]\sigma == \sigma(z) \qquad \text{for} \quad z \in P(f) \text{ and } \sigma \in \mathcal{O}(X).$$

By the connectedness of X it follows that

$$(3.9) \qquad f \in H^0(X, \mathcal{M}\mathcal{E} \bigoplus_{n=1}^{m} [\mathcal{O}(X)/\varrho_{K_n}]' \qquad \text{for some } m.$$

Since $X \setminus P(f)$ is dense in X, from (3.8) and (3.9) it is easy to see that $X = \mathcal{O}(X) - \text{hull}(K_m)$. Combining this with the fact that every non-compact Riemann surface is Stein we infer that X is compact. The theorem is proved.

3.2. THEOREM. *Let X be a Riemann surface and F a Fréchet space. Then $H^1(X, \mathcal{M}\mathcal{E}F) = 0$.*

Proof. By Theorem 3.1 it suffices to consider the case, where X is non-compact and hence X is Stein [7]. Thus $H^1(X, \mathcal{O}\mathcal{E}F) = 0$ [3]. Whence we infer that

$$(3.10) \qquad H^1(X, \mathcal{M}\mathcal{E}F) \cong H^1(X, \mathcal{M}\mathcal{E}F/\mathcal{O}\mathcal{E}F).$$

Since $\mathrm{supp}\,\sigma$ is discrete for every $\sigma \in H^0(U, \mathcal{M}\mathcal{E}F/\mathcal{O}\mathcal{E}F)$, it follows that the sheaf $\mathcal{M}\mathcal{E}F/\mathcal{O}\mathcal{E}F$ is soft. Hence $H^1(X, \mathcal{M}\mathcal{E}F/\mathcal{O}\mathcal{E}F) = 0$. Whence, by (3.10) we get $H^1(X, \mathcal{M}\mathcal{E}F) = 0$. The theorem is proved.

§ 4. The splitness of Dolbeault complexes of sheaves $J_{V,\xi}$. Let X be a complex manifold having a countable topology, V a closed submanifold of X and ξ a holomorphic Banach bundle over X. For each $q \geqslant 0$ by Ω_ξ^q we denote the sheaf of germs of C^∞-forms of bidegree $(0, q)$ on X with values in ξ. Let $e: V \hookrightarrow X$ denote the canonical embedding and $e^*: \Omega_\xi^q \to \Omega_{\xi_V}^q$, where $\xi_V = \xi|V$ is the map induced by e. Let $\Omega_{V,\xi}^q$ denote the sheaf on X given by the formula

$$U \mapsto \{\sigma \in \Omega_\xi^q(U): e^*\sigma = 0\}$$

for all open sets $U \subset X$. Since the maps $\bar{\partial}_{\xi_V}^q$ commute with e^*, we can consider the complex

$$(4.1) \qquad 0 \to J_{V,\xi} \to \Omega_{V,\xi}^0 \xrightarrow{\bar{\partial}_{V,\xi}^0} \Omega_{V,\xi}^1 \xrightarrow{\bar{\partial}_{V,\xi}^1} \cdots,$$

where

$$\bar{\partial}^q_{V,\xi} = \bar{\partial}^q_\xi | \Omega^q_{V,\xi}.$$

The complex

$$D(J_{V,\xi}): \quad 0 \to J_{V,\xi}(X) \to \Omega^0_{V,\xi}(X) \xrightarrow{\hat{\bar{\partial}}^0_{V,\xi}} \Omega^1_{V,\xi}(X) \xrightarrow{\hat{\bar{\partial}}^1_{V,\xi}} \dots$$

of global sections of (4.1) is called the *Dolbeault complex* of $J_{V,\xi}$ on X. We say that $D(J_{V,\xi})$ *splits at* q if there exists a continuous linear map

$$\gamma_q: \operatorname{Im} \hat{\bar{\partial}}^q_{V,\xi} \to \Omega^q_{V,\xi}(X)$$

such that $\hat{\bar{\partial}}^q_{V,\xi}\gamma_q = \mathrm{id}$.

In this section we prove the following

4.1. THEOREM. *Let X be an increasing union of Stein open sets and V, ξ be as above. Then $D(J_{V,\xi})$ splits only at $q > 0$.*

4.2. THEOREM. *Let X have a Stein morphism and V, ξ be as in Theorem 4.1. Then*

(i) *$D(J_{V,\xi})$ splits at $q > 1$;*

(ii) *$D(J_{V,\xi})$ does not split at 0.*

Proof of Theorem 4.1. (a) First we show that the sequence

$$(4.2) \qquad 0 \to J_{V,\xi}\mathscr{E}L \to \Omega^0_{V,\xi}\mathscr{E}L \to \Omega^1_{V,\xi}\mathscr{E}L \to \dots,$$

which is obtained by tensoring sequence (4.1) with L, is exact, where L is a Hausdorff arbitrary complete locally convex space. Let $\sigma \in (\operatorname{Ker} \bar{\partial}^q_{V,\xi}\mathscr{E}\mathrm{id})_{z_0}$. We can assume that $X = \varDelta^n$, $V = \varDelta^r \times 0$ and $z_0 = 0$. In [11], Lemma 1.11, we have proved that $\sigma = \bar{\partial}^{q-1}_{V,\xi}\beta$ for some $\beta \in (\Omega^{q-1}_{V,\xi}\mathscr{E}L)_0$. Put $\tilde{\beta} = \beta - \pi^* e^* \beta$, where $\pi: \varDelta^n \to \varDelta^r \times 0$ denotes the canonical projection. Then

$$e^*\tilde{\beta} = e^*\beta - e^*\pi^* e^*\beta = e^*\beta - e^*\beta = 0, \quad \bar{\partial}^{q-1}_\xi \tilde{\beta} = \sigma.$$

(b) By the exactness of (4.2) and since the sheaves $\Omega^q_{V,\xi}\mathscr{E}\mathscr{L}$ are fine, we get

$$(4.3) \qquad H^p(X, J_{V,\xi}\mathscr{E}L) = \operatorname{Ker} \hat{\bar{\partial}}^p_{V,\xi}\mathscr{E}L / \operatorname{Im}(\hat{\bar{\partial}}^{p-1}_{V,\xi}\mathscr{E}\mathrm{id})$$

for every $p \geqslant 1$. Let $q > 0$. Using (4.3) to $L = [\operatorname{Im} \hat{\bar{\partial}}^q_{V,\xi}]'_c$, by the relation $L'_c = \operatorname{Im} \hat{\bar{\partial}}^q_{V,\xi}$ and by Theorem 2.1 (i) it follows that the map

$$\operatorname{HOM}_{\mathscr{E}}\big(\operatorname{Im} \hat{\bar{\partial}}^q_{V,\xi}, \Omega^q_{V,\xi}(X)\big) \to \operatorname{HOM}_{\mathscr{E}}(\operatorname{Im} \hat{\bar{\partial}}^q_{V,\xi}, \operatorname{Im} \hat{\bar{\partial}}^q_{V,\xi})$$

induced by $\hat{\bar{\partial}}^q_{V,\xi}$ is surjective. This gives the splitness of $D(J_{V,\xi})$ at q.

Assume that $q = 0$. Using (4.3) to $L = C^{\infty\prime}$, $p = 1$, by Theorem 2.1 (ii) it follows that $D(J_{V,\xi})$ does not split at 0. By Theorem 2.3 the proof of Theorem 4.2 is similar as Theorem 4.1.

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