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A reverse maximal ergodic theorem

by

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Abstract. A reverse maximal ergodic theorem is proved for a d-parameter discrete semigroup $(T_g\colon g \in Z_+^d)$ of measure preserving transformations on a σ-finite measure space $(X, \mathscr{F}, \mu)$ which is ergodic in the sense that if $E \in \mathscr{F}$ with $E \neq X$ is T_g-invariant for all $g \in Z_+^d$ then $\mu E = 0$ or ∞. A continuous version follows from standard approximation arguments.

1. Introduction. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and $(T_g\colon g \in Z_+^d)$ a d-parameter discrete semigroup of measure preserving transformations on $(X, \mathscr{F}, \mu)$. For $0 \leqslant f \in L_1(\mu) + L_\infty(\mu)$, the *maximal function* f^* is defined by

$$f^*(x) = \sup_{n\geqslant 1} n^{-d} \sum_{g\in V_n} f(T_g x) \quad \text{where} \quad V_n = \{0, \ldots, n-1\}^d.$$

It is then known (cf. [11], [4], [1]) that the maximal inequality holds:

$$\mu\{f^* > a\} \leqslant \frac{1}{B_d a} \int_{\{f > B_d a\}} f\,d\mu \quad \text{for any } a > 0 \tag{1}$$

where B_d is a constant dependent only on the dimension d.

The purpose of this paper is to show that a reverse maximal inequality holds provided that the semigroup $(T_g\colon g \in Z_+^d)$ is *ergodic* in the sense that if $E \in \mathscr{F}$ with $E \neq X$ is T_g-invariant for all $g \in Z_+^d$ then $\mu E = 0$ or ∞. Here it should be noted that N. Dang-Ngoc [2] has shown a similar inequality for an ergodic d-parameter *group* $(T_g\colon g \in Z^d)$ of measure preserving transformations on a probability measure space. However, the maximal function $f^\sim$ he considered is defined by

$$f^\sim(x) = \sup_{n\geqslant 1} (2n-1)^{-d} \sum_{g\in W_n} f(T_g x) \quad \text{where} \quad W_n = \{-n+1, \ldots, n-1\}^d,$$

and he remarked that his argument is not modified if $f^\sim$ is replaced by f^*. Nevertheless, we shall modify his argument to prove our result. For the particular case $(T^n\colon n \in Z_+^1)$ where T is conservative and ergodic in the usual sense, the inequality was already obtained by Derriennic [3] in a slightly different form. We think that Derriennic's argument cannot be applied when the dimension d is greater than one. See also Ornstein [8], Petersen [9], Marcus and Petersen [7], and the author [10].

2. Reverse maximal ergodic theorem.

THEOREM 1. *Let $(T_g: g \in Z_+^d)$ be a d-parameter discrete semigroup of measure preserving transformations on a σ-finite measure space $(X, \mathcal{F}, \mu)$ which is assumed to be ergodic in our sense. Then there exists a constant $C_d > 0$, dependent only on the dimension d, such that*

(i) *if $\mu X = \infty$ then for any $0 \leqslant f \in L_1(\mu)+L_\infty(\mu)$ and any $a > 0$*

$$\mu\{f^* > a\} \geqslant \frac{1}{C_d a} \int_{\{f>C_d a\}} f\,d\mu, \tag{2}$$

(ii) *if $\mu X < \infty$ then for any $0 \leqslant f \in L_1(\mu)$ and any $a > 0$ with $\int f d\mu < a\mu X$, (2) holds.*

To prove the theorem we proceed as in [2]. We need some lemmas. A *quasi-cube* Q of Z^d is a set of the form $Q = \{a_1, \ldots, a_1+b_1\} \times \ldots \times \{a_d, \ldots, a_d+b_d\}$ where $b_i \geqslant 0$ for each i and

$$\sup_{1\leqslant i,j\leqslant d} |b_i - b_j| \leqslant 1.$$

If Q is a quasi-cube of the above form, let $l(Q) = 1 + \sup_{1\leqslant i\leqslant d} b_i$,

$$Q' = \{a_1-b_1, \ldots, a_1\} \times \ldots \times \{a_d-b_d, \ldots, a_d\},$$
$$\bar{Q} = \{a_1-2l(Q), \ldots, a_1+3l(Q)\} \times \ldots \times \{a_d-2l(Q), \ldots, a_d+3l(Q)\}.$$

LEMMA 1. *There exists a constant $C > 0$, dependent only on the dimension d, such that if $Q_1, \ldots, Q_n$ are disjoint quasi-cubes of Z^d then*

$$\sum_{j=1}^{s} |Q'_{k_j}| \geqslant C \sum_{k=1}^{n} |Q_k|$$

for some disjoint $Q'_{k_j} (j = 1, \ldots, s)$, where $|Q|$ denotes the cardinal number of Q.

Proof. Without loss of generality we may assume that $l(Q_1) \geqslant \ldots \geqslant l(Q_n)$. Take $k_1 = 1$. Next take $k_2 = \min\{j: Q'_1 \cap Q'_j = \emptyset\}$, and continue this process. We obtain disjoint Q'_{k_j} $(j = 1, \ldots, s)$ such that

$$\bigcup_{k=1}^{n} Q_k \subset \bigcup_{j=1}^{s} \bar{Q}_{k_j},$$

from which the lemma follows immediately.

LEMMA 2 (cf. [2], Lemma 3). *Let $f \geqslant 0$ be an integrable function on Z^d. Then for any $a > 0$ there exist disjoint quasi-cubes $Q_1, \ldots, Q_n$ such that*

that

(i) *$g \in Z^d \setminus \bigcup_{k=1}^{n} Q_k$ implies $f(g) \leqslant a$,*

(ii) *for each $1 \leqslant k \leqslant n$*

$$a < \frac{1}{|Q_k|} \sum_{g\in Q_k} f(g) \leqslant 3^d a.$$

If f is a non-negative function on Z^d, define

$$Mf(g) = \sup_{n\geqslant 1} \frac{1}{n^d} \sum_{h\in V_n} f(g+h) \quad \text{where} \quad V_n = \{0, \ldots, n-1\}^d.$$

LEMMA 3. *There exists a constant $C' > 0$, dependent only on the dimension d, such that if $f \geqslant 0$ is an integrable function on Z^d, then for any $a > 0$*

$$|\{g: Mf(g) > a\}| \geqslant \frac{1}{C'a} \sum_{\{g: f(g) > C'a\}} f(g).$$

Proof. Let $Q_1, \ldots, Q_n$ be disjoint quasi-cubes in Lemma 2, and let Q'_{k_j} $(j = 1, \ldots, s)$ be as in Lemma 1. Since

$$g \in \bigcup_{j=1}^{s} Q'_{k_j} \quad \text{implies} \quad Mf(g) > a/4^d,$$

we have

$$|\{g: Mf(g) > a/4^d\}| \geqslant \sum_{j=1}^{s} |Q'_{k_j}| \geqslant C \sum_{k=1}^{n} |Q_k|$$
$$\geqslant \frac{C}{3^d a} \sum \{f(g): g \in \bigcup_{k=1}^{n} Q_k\}$$
$$\geqslant \frac{C}{3^d a} \sum_{\{f>a\}} f(g),$$

which establishes the lemma.

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Suppose $\mu X = \infty$, $0 \leqslant f \in L_1(\mu)$ and $a > 0$. It follows from (1) that

$$\mu\{f^* > a\} \leqslant \frac{1}{B_d a} \int_{\{f>B_d a\}} f\,d\mu < \infty. \tag{3}$$

On the other hand, since the semigroup is ergodic in our sense, the pointwise ergodic theorem (cf. [11], [4]) shows that

$$\lim_n n^{-d} \sum_{g \in V_n} f(T_g x) = 0 \text{ a.e. on } X.$$

Hence by (3), given an $\varepsilon > 0$ there exists an integer $N \geqslant 1$ such that

$$\mu\Big\{x\colon N^{-d} \sum_{g \in V_N} f(T_g x) \geqslant a\Big\} < \varepsilon. \tag{4}$$

Put

$$F(g, x) = \begin{cases} 0 & \text{if} \quad g \in Z^d \setminus Z^d_+, \\ f(T_g x) & \text{if} \quad g \in Z^d_+, \end{cases}$$

and

$$F_N(g, x) = 1_{V_N}(g) F(g, x).$$

Define

$$MF(g, x) = \sup_{n \geqslant 1} n^{-d} \sum_{h \in V_n} F(g+h, x)$$

and

$$MF_N(g, x) = \sup_{n \geqslant 1} n^{-d} \sum_{h \in V_n} F_N(g+h, x).$$

Write

$$E = \big\{(g, x) \in \{-N, \ldots, N-1\}^d \times X\colon MF(g, x) > a\big\},$$
$$E_g = \{x\colon (g, x) \in E\} \quad \text{and} \quad E_x = \{g\colon (g, x) \in E\}.$$

Then we observe that

$$\mu(E_g) \leqslant \mu(E_0) = \mu\{f^* > a\} \quad \text{for all } g \in Z^d. \tag{5}$$

In fact, if $g \in Z^d_+$ then

$$E_g \subset \{x\colon MF(g, x) > a\} = T_g^{-1}\{x\colon MF(0, x) > a\}$$
$$= T_g^{-1}\{f^* > a\} = T_g^{-1}(E_0);$$

it follows that $\mu(E_g) = \mu(E_0)$, because T_g is measure preserving. If $g \notin Z^d_+$, let $g' = (a_1', \ldots, a_d')$ where $g = (a_1, \ldots, a_d)$ and $a_i' = \max(0, a_i)$ for each $1 \leqslant i \leqslant d$. Then we have

$$n^{-d} \sum_{h \in V_n} F(g+h, x) \leqslant n^{-d} \sum_{h \in V_n} F(g'+h, x) \leqslant MF(g', x)$$

for each $n \geqslant 1$; thus $MF(g, x) \leqslant MF(g', x)$ and $E_g \subset E_{g'}$. This yields

$\mu(E_g) \leqslant \mu(E_{g'}) = \mu(E_0)$. Write

$$X_\varepsilon = \Big\{x\colon N^{-d} \sum_{g \in V_N} f(T_g x) < a\Big\}.$$

By (4),

$$\mu(X \setminus X_\varepsilon) < \varepsilon. \tag{6}$$

We then deduce that

$$x \in X_\varepsilon \quad \text{implies} \quad \{g\colon MF_N(g, x) > a\} \subset E_x. \tag{7}$$

To see this, let $x \in X_\varepsilon$ and $g \notin \{-N, \ldots, N-1\}^d$. We have

$$n^{-d} \sum_{h \in V_n} F_N(g+h, x) \begin{cases} = 0 & \text{if} \quad n \leqslant N, \\ \leqslant n^{-d} \sum_{h \in V_N} f(T_h x) & \text{if} \quad n > N, \end{cases}$$

and hence $MF_N(g, x) \leqslant a$. This, together with the fact that $MF_N(g, x) \leqslant MF(g, x)$ for all $(g, x) \in Z^d \times X$, proves (7).

We apply (6), (7), Lemma 3 and Fubini's theorem to get:

$$\begin{aligned}
(2N)^d \mu(E_0) &\geqslant \sum \{\mu(E_g)\colon g \in \{-N, \ldots, N-1\}^d\} = \sum_{g \in Z^d} \mu(E_g) \\
&\geqslant \int_{X_\varepsilon} |E_x|\, d\mu(x) \geqslant \int_{X_\varepsilon} |\{g\colon MF_N(g, x) > a\}|\, d\mu(x) \\
&\geqslant \frac{1}{C'a} \int_{X_\varepsilon} \Big(\sum_{\{g: F_N(g,x) > C'a\}} F_N(g, x)\Big) d\mu(x) \\
&= \frac{1}{C'a} \Big(\int_X - \int_{X \setminus X_\varepsilon} \Big) \Big(\sum_{\{g: F_N(g,x) > C'a\}} F_N(g, x)\Big)\, d\mu(x) \\
&= \mathrm{I} - \mathrm{II},
\end{aligned}$$

$$\begin{aligned}
\mathrm{I} &= \frac{1}{C'a} \int_X \Big(\sum_{\{g: F_N(g,x) > C'a\}} F_N(g, x)\Big) d\mu(x) \\
&= \frac{1}{C'a} \sum_{g \in V_N} \int_{\{x: F_N(g,x) > C'a\}} F_N(g, x)\, d\mu(x) \\
&= \frac{1}{C'a} \sum_{g \in V_N} \int_{\{x: f(T_g x) > C'a\}} f(T_g x)\, d\mu(x) \\
&= \frac{N^d}{C'a} \int_{\{f > C'a\}} f\, d\mu,
\end{aligned}$$

$$\mathrm{II} \leqslant \frac{1}{C'a} \int_{X \setminus X_\varepsilon} \Big(\sum_{g \in V_N} F_N(g, x) \Big) \, d\mu(x)$$

$$= \frac{1}{C'a} \sum_{g \in V_N} \int_{X \setminus X_\varepsilon} f(T_g x) \, d\mu(x)$$

$$\leqslant \frac{N_d}{C'a} \eta(\varepsilon) \quad \text{where} \quad \eta(\varepsilon) = \sup \Big\{ \int_{X \setminus X_\varepsilon} f(T_g x) \, d\mu(x) \colon g \in Z_+^d \Big\}.$$

It follows that

$$\mu\{f^* > a\} = \mu(E_0) \geqslant \frac{1}{2^d C'a} \Big(\int_{\{f > C'a\}} f d\mu - \eta(\varepsilon) \Big);$$

and since $\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0$ by an easy approximation argument, (3) holds, with the constant $C_d = 2^d C'$. Applying this and an approximation argument, it is easily seen that if $\mu X = \infty$ then for any $0 \leqslant f \in L_1(\mu) + L_\infty(\mu)$ and any $a > 0$, (3) holds with the constant $C_d = 2^d C'$.

The proof of (ii) is similar and omitted.

As a corollary to the proof of Theorem 1 we have the following continuous version of Theorem 1.

THEOREM 2. *Let $(T_g \colon g \in R_+^d)$ be a d-parameter measurable semiflow of measure preserving transformations on a σ-finite measure space $(X, \mathscr{F}, \mu)$ which is assumed to be ergodic in our sense. For $0 \leqslant f \in L_1(\mu) + L_\infty(\mu)$, define*

$$f'(x) = \sup_{r > 0} r^{-d} \int_{V_r} f(T_g x) \, dg \quad \textit{where} \quad V_r = [0, r]^d.$$

Let C_d be the constant introduced in Theorem 1. Then we have:

(i) *If $\mu X = \infty$ then for any $0 \leqslant f \in L_1(\mu) + L_\infty(\mu)$ and any $a > 0$*

$$\mu\{f' > a\} \geqslant \frac{1}{C_d a} \int_{\{f > C_d a\}} f \, d\mu. \tag{8}$$

(ii) *If $\mu X < \infty$ then for any $0 \leqslant f \in L_1(\mu)$ and any $a > 0$ with $\int f d\mu < a \mu X$, (8) holds.*

Proof. We shall only prove (8) for $0 \leqslant f \in L_1(\mu)$ and $a > 0$ under the assumption $\mu X = \infty$ since the other parts of the theorem are proved similarly as in Theorem 1.

For each $n \geqslant 1$, let

$$f_n(x) = 2^n \int_{V_{2^{-n}}} f(T_g x) \, dg \quad \text{and} \quad f_n^*(x) = \sup_{k \geqslant 1} k^{-d} \sum_{g \in V_{n,k}} f_n(T_g x)$$

where

$$V_{n,k} = \{0, 2^{-n}, \ldots, (k-1) 2^{-n}\}^d.$$

Since the semiflow is ergodic in our sense, the pointwise ergodic theorem shows that

$$\lim_k k^{-d} \sum_{g \in V_{n,k}} f(T_g x) = 0 \text{ a.e. on } X.$$

Thus the proof of Theorem 1 gives

$$\mu\{f_n^* > a\} \geqslant \frac{1}{C_d a} \int_{\{f_n > C_d a\}} f_n \, d\mu.$$

On the other hand, by the definition of f_n^* we have $f_n^* \leqslant f_{n+1}^*$ and $\lim_n f_n^* = f'$ a.e. on X. Further, since f_n converges to f in the norm topology of $L_1(\mu)$, we may assume (if necessary, take a subsequence) that $\lim_n f_n = f$ a.e. on X. Thus

$$\liminf_n 1_{\{f_n > C_d a\}} \cdot f_n \geqslant 1_{\{f > C_d a\}} \cdot f,$$

so by Fatou's lemma,

$$\mu\{f' > a\} = \lim_n \mu\{f_n^* > a\}$$

$$\geqslant \frac{1}{C_d a} \liminf_n \int_{\{f_n > C_d a\}} f_n \, d\mu$$

$$\geqslant \frac{1}{C_d a} \int_{\{f > C_d a\}} f \, d\mu,$$

which completes the proof.

3. Application. Given a constant $w \geqslant 0$, define the subclass $R_w(\mu)$ of $L_1(\mu) + L_\infty(\mu)$ by

$$R_w(\mu) = \Big\{ f \colon \int_{\{|f| > t\}} |f| (\log(|f|/t))^w \, d\mu < \infty \text{ for all } t > 0 \Big\}.$$

Fava [6] proved that $R_w(\mu)$ is a linear space, that $R_{w+1}(\mu) \subset R_w(\mu)$, and that under the assumption $\mu X < \infty$, $f \in R_w(\mu)$ if and only if

$$\int_{\{|f| > 1\}} |f| (\log |f|)^w \, d\mu < \infty.$$

Also Fava [6] proved that the subclass $R_w(\mu)$ is important in pointwise ergodic theory.

Using the maximal inequality (1) and the reverse maximal inequality (2), we can prove the following dominated ergodic theorem.

THEOREM 3. *Let* $(X, \mathscr{F}, \mu)$ *and* $(T_g\colon g \in Z_+^d)$ *be as in Theorem* 1. *Then* $f \in R_w(\mu)$ *if and only if* $f^* \in R_{w+1}(\mu)$.

Proof. By Fubini's theorem we have

$$\int_{\{f^*>t\}} f^*(\log(f^*/t))^w d\mu = \int_{\{f^*>t\}} d\mu(x) \int_t^{f^*(x)} \left([\log(s/t)]^w + tw[\log(s/t)]^{w-1}\right) ds$$
$$= \int_t^{\infty} \left([\log(s/t)]^w + tw[\log(s/t)]^{w-1}\right) \mu\{f^* > s\}\, ds.$$

Thus we may apply (1) together with a well-known argument (see e.g. [5], p. 676) to infer that $f \in R_{w+1}(\mu)$ implies $f^* \in R_w(\mu)$. Similarly, (2) may be applied to infer that $f^* \in R_w(\mu)$ implies $f \in R_{w+1}(\mu)$. The details are omitted. (Cf. [10].)

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Applications of autoreproducing kernel moduli to the study on interpolability and minimality of a class of stationary Hilbertian varieties

by

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Abstract. The autoreproducing kernel modulus $\mathscr{H}(F)$ of an operator-valued spectral measure F is constructed. All of its elements are operator-valued measures. These measures are said (as suggested by the results obtained in [23]) to be Hellinger square integrable relative to F. Then the class of Hilbert space-valued stationary processes $(X_g)_{g\in G}$, having operator valued spectral densities is considered. For such processes, some characterizations of $\mathscr{H}(F)$ are given and compared to that obtained by Makagon in the recent paper [9] on Hellinger square integrable vector measures. From the results on $\mathscr{H}(F)$, the interpolable and minimal processes $(X_g)_{g\in G}$ are then analytically characterized.

Introduction. It is shown in [23] the impossibility for a minimal $\mathscr{S}(U, H)$-valued stationary processes to be of full rank. However this notion can be defined for Banach space-valued stationary processes. So a special class of these processes is considered here.

First it is constructed from a spectral bimeasure a unique autoreproducing kernel Loynes modulus $\mathscr{H}(F)$, all the elements of which are operator-valued measures (cf. Theorem 2). When F is a spectral measure, by analogy with the results obtained in [23], the measures in $\mathscr{H}(F)$ are said to be Hellinger square integrable with respect to (w.r.t.) F.

Then, Hilbert space-valued stationary processes $(X_g)_{g\in G}$, possessing operator-valued spectral densities are considered. The operator time-domains of these processes are proved to have the Radon–Nikodym property w.r.t. F (Theorem 4) and some characterizations of $\mathscr{H}(F)$ are obtained (Theorem 5). Afterwards, analytic conditions for interpolability and minimality studied by [19], [20], [23], [24], [27] are extended to the processes $(X_g)_{g\in G}$, and a criterion for such processes $(X_g)_{g\in G}$ to be minimal of full rank is also given.

We learned quite recently that Makagon in [9] has given a definition and a criterion for vector measures to be Hellinger square integrable w.r.t. a spectral measure. His definition is proved to be equivalent to our Definition 3 (cf. Theorem 3) whereas our criterion (Theorem 5) may be considered as an operator version of Makagon's criterion ([9], Theorem 1.5).