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Structure of spaces of holomorphic functions on infinite dimensional polydiscs

by

REINHOLD MEISE (Düsseldorf)
and
DIETMAR VOGT (Wuppertal)

Abstract. Let $\Lambda_1(\alpha)$ be a nuclear finite type power series space. We characterize the open polydiscs $\boldsymbol{D}_a$ in $\Lambda_1(\alpha)'_b$ for which the space $(H(\boldsymbol{D}_a), \tau_0)$ of all holomorphic functions on $\boldsymbol{D}_a$ under the compact-open topology τ_0 is a power series space. This characterization implies the existence of open polydiscs $\boldsymbol{D}_a$ and $\boldsymbol{D}_b$ with finite radii for which $(H(\boldsymbol{D}_a), \tau_0)$ and $(H(\boldsymbol{D}_b), \tau_0)$ are not isomorphic. Furthermore we give sufficient conditions on nuclear Fréchet spaces $\Lambda(P)$ and on $a \in \Lambda(P)$ implying that for the open polydisc $\boldsymbol{D}_a$ in $\Lambda(P)'_b$ the space $(H(\boldsymbol{D}_a), \tau_0)$ is isomorphic to a subspace (resp. a quotient) of a power series space.

Preface. Let E be a nuclear Fréchet space and let U be an open subset of E'_b. Then the space $(H(U), \tau_0)$ of all holomorphic functions on U, endowed with the compact-open topology τ_0 is a nuclear Fréchet space by the theorem of Boland [7] and Waelbroeck [27]. If E is, moreover, a Köthe sequence space $\Lambda(P)$, then Boland and Dineen [8] have shown that the monomials in the coordinate functions form an absolute basis of $(H(U), \tau_0)$ for all open polydiscs U in $\Lambda(P)'_b$, where for $a \in \Lambda(P)$, $a \geqslant 0$ (for convenience), the sets $\boldsymbol{D}_a := \{y \in \Lambda(P)' \mid \sup_{j \in \boldsymbol{N}} |y_j| a_j < 1\}$ are called open polydiscs. From this result Börgens, Meise and Vogt [3] have derived that $H(\Lambda_\infty(\alpha)'_b, \tau_0)$ is isomorphic to $\Lambda_\infty(\beta(\alpha))$ and that in $\Lambda_1(\alpha)'_b$ there exists an open polydisc $\boldsymbol{D}$ with $(H(\boldsymbol{D}), \tau_0) \simeq \Lambda_1(\beta(\alpha))$. Furthermore, they have determined $\beta(\alpha)$ up to equivalence for many interesting sequences α.

The main result of the present article is the following: Let $\Lambda_1(\alpha)$ be nuclear and take $a \in \Lambda_1(\alpha)$, $a \geqslant 0$. Then $(H(\boldsymbol{D}_a), \tau_0)$ is isomorphic to $\Lambda_1(\beta(\alpha))$ if and only if $(H(\boldsymbol{D}_a), \tau_0)$ is isomorphic to a quotient space of a finite type power series space, and this happens if and only if $a > 0$ and $1/a \in \Lambda_1(\alpha)$ which is equivalent to $a > 0$ and $\lim_{j\to\infty} (1/\alpha_j) \ln a_j = 0$. This implies in particular that there exist $a, b \in \Lambda_1(\alpha)$, $a > 0$, $b > 0$ such that $(H(\boldsymbol{D}_a), \tau_0)$ and $(H(\boldsymbol{D}_b), \tau_0)$ are not isomorphic as locally convex spaces; a phenomenon which does not occur in the finite dimensional situation.

The proof of this result is based on a linear topological invariant $(\overline{\Omega})$ which has been used by Wagner [28], to characterize the quotient spaces with a basis of stable power series spaces of finite type.

Knowing this, it was natural to consider also the other linear topological invariants from the structure theory of nuclear Fréchet spaces which have been used to characterize the subspaces and quotients of stable nuclear power series spaces. These characterizations are given in terms of the Λ_N (resp. the Λ_1)-nuclearity and of one of the invariants (DN), $(\underline{\mathrm{DN}})$, (Ω) and $(\overline{\Omega})$ (see Dubinsky [11], Vogt [21]–[24], Vogt and Wagner [25], [26] and Wagner [28]). Since optimal theorems on the $\Lambda(\gamma)$-nuclearity of spaces of type $(H(U), \tau_0)$ have been given by Börgens, Meise and Vogt [4], and since $\beta(\alpha)$ is always stable, the characterization theorems can be applied to $(H(\boldsymbol{D}_a), \tau_0)$ as soon as this space has one of these invariants. In this direction we prove the following: If $\Lambda(P)$ is a quotient of $\Lambda_1(\alpha)$, then $(H(\boldsymbol{D}_a), \tau_0)$ is isomorphic to a quotient space of $\Lambda_1(\beta(\alpha))$ if and only if a satisfies a certain condition. If $\Lambda(P)$ is a quotient of $\Lambda_\infty(\alpha)$ (resp. a subspace of $\Lambda_1(\alpha)$), then $(H(\boldsymbol{D}_a), \tau_0)$ is isomorphic to a quotient of $\Lambda_\infty(\beta(\alpha))$ (resp. a subspace of $\Lambda_1(\beta(\alpha))$ for all open polydiscs $\boldsymbol{D}_a$ in $\Lambda(P)'_b$. The proofs rely on the fact that for Köthe spaces $\Lambda(Q)$ the invariants can be expressed in terms of the Köthe set Q. Hence the basis theorem of Boland and Dineen, mentioned above, gives the opportunity to check the invariants by direct calculation. Furthermore we apply an extension theorem of Boland [5], a duality theorem of Boland and Dineen [8] and the isomorphism $H(s'_b) \simeq s$ proved by Börgens, Meise and Vogt [3].

Concluding, we briefly indicate the content of the four sections of this article. In the first section we recall some definitions and results and fix the notation. In the second one we state the basis theorem of Boland and Dineen in an appropriate form and provide some lemmas. These are applied in Section 3 to obtain our main result as well as the invariants related with finite type power series spaces. The same topic for infinite type power series spaces is treated in Section 4.

1. Preliminaries. In this section we introduce some notation and conventions used throughout the whole article. We also mention some results which we shall use in the subsequent sections without further references.

Notation. The symbols $\boldsymbol{C}, \boldsymbol{R}, \boldsymbol{R}_+, \boldsymbol{N}$ and $\boldsymbol{N}_0$ denote the sets of complex, real, non-negative real, natural and non-negative integer numbers, respectively. All locally convex (l.c.) vector spaces E are assumed to be complex vector spaces and Hausdorff. For an l.c. vector space E, the strong dual is denoted by E'_b. An l.c. space E is called a *subspace* (*quotient space*) of an l.c. space F if E is isomorphic to a topological subspace (quotient space) of F. The space $\boldsymbol{C}^{\boldsymbol{N}}$ of all complex sequences is denoted by ω, while $\varphi \subset \omega$ denotes the space of all complex sequences which are eventually zero. For $a \in \omega$ the set

$$N_a := \{x \in \omega \mid |x_n| \leqslant |a_n| \text{ for all } n \in \boldsymbol{N}\}$$

is called the *normal hull* of a. For $a, b \in \omega$ we denote by a/b the sequence c which is defined by $c_n = a_n/b_n$ if $b_n \neq 0$ and $c_n = 0$ if $b_n = 0$. Results from the theory of l.c. spaces which we use without any reference can be found in the books of Köthe [13], Pietsch [16] or Schaefer [20].

1.1. *Köthe sequence spaces.* We use the Köthe sequence spaces $\Lambda(P)$ as they are defined in Pietsch [16], 6.1.1. Then the *Grothendieck–Pietsch criterion* (see Pietsch [16], 6.1.2) tells that the nuclearity of $\Lambda(P)$ is characterized by the property: For any $p \in P$ there exist $q \in P$ and $c \in l^1$ such that $p \leqslant c \cdot q$. If $\Lambda(P)$ is nuclear, then a subset B of $\Lambda(P)$ is bounded iff B is contained in the normal hull of some $b \in \Lambda(P)$. If $\Lambda(P)$ is nuclear and reflexive, then a subset K of $\Lambda(P)'_b$ is relatively compact iff K is contained in the normal hull of some $c \in \Lambda(P)'$, where we identify $\Lambda(P)'$ with the linear span of $\bigcup_{p \in P} N_p$ in ω. This description of the relatively compact subsets of $\Lambda(P)'_b$ also holds if $\Lambda(P)$ is a Mackey space for which $\Lambda(P)'_b$ is complete (see Börgens, Meise and Vogt [3], 1.1).

1.2. *Power series spaces.* Let α be an increasing sequence of positive real numbers with $\lim_{n\to\infty} \alpha_n = \infty$ (α will be called an *exponent sequence*). For $0 < R \leqslant \infty$ we define the *power series space*

$$\Lambda_R(\alpha) := \Big\{x \in \omega \mid \pi_r(x) := \sum_{n=1}^{\infty} |x_n| r^{\alpha_n} < \infty \text{ for any } 0 < r < R\Big\},$$

which is given the l.c. topology induced by the semi-norms $\{\pi_r \mid 0 < r < R\}$. Obviously $\Lambda_R(\alpha)$ is a Fréchet space.

$\Lambda_R(\alpha)$ is called *power series space of finite type* if $R < \infty$ and *of infinite type* if $R = \infty$. For α fixed and $0 < R < \infty$ all the spaces $\Lambda_R(\alpha)$ are isomorphic. A power series space of infinite type cannot be isomorphic to a power series space of finite type.

For $R = 1$ and $R = \infty$ the identity $\Lambda_R(\alpha) = \Lambda_R(\bar{\alpha})$ holds iff α and $\bar{\alpha}$ are *equivalent* in the following sense: There exists $D \geqslant 1$ such that

$$\frac{1}{D}\alpha_n \leqslant \bar{\alpha}_n \leqslant D\alpha_n \quad \text{for any } n \in \boldsymbol{N}.$$

The nuclearity of $\Lambda_\infty(\alpha)$ is equivalent to $\sup_{n\in \boldsymbol{N}} (\ln(n+1)/\alpha_n) < \infty$, while the nuclearity of $\Lambda_1(\alpha)$ is equivalent to $\lim_{n\to\infty} (\ln(n+1)/\alpha_n) = 0$.

An exponent sequence α is called *stable* if $\sup_{n\in N}(\alpha_{2n}/\alpha_n) < \infty$, which is equivalent to the isomorphism $\Lambda_\infty(\alpha)\times\Lambda_\infty(\alpha) \simeq \Lambda_\infty(\alpha)$ (see Dubinsky and Ramanujan [12], 2.10).

If $\Lambda_1(\alpha)$ is nuclear (resp. if $\Lambda_\infty(\alpha)$ is nuclear) then one can define the class of $\Lambda_1(\alpha)$ (resp. $\Lambda_N(\alpha)$; resp. $\Lambda_\infty(\alpha)$)-nuclear l.c. spaces, as it was done by Dubinsky and Ramanujan [12], Robinson [19] and Ramanujan and Terzioglu [18]. Since the definitions of these classes are a bit involved, we do not give them but refer to the articles cited above. A brief introduction which suffices for our purposes, is given in Börgens, Meise and Vogt [4], 1.4.

For stable nuclear power series spaces $\Lambda_R(\alpha)$, $R = 1, \infty$, the subspaces and quotient spaces have been characterized by Vogt and Wagner (see [24], [26] and also Dubinsky [11], where further references are given). They have shown that a Fréchet space E is isomorphic to a subspace (resp. a quotient space of $\Lambda_\infty(\alpha)$) iff E is $\Lambda_N(\alpha)$-nuclear and has property (DN) (resp. (Ω)). The same characterization holds true for $\Lambda_1(\alpha)$ provided that "$\Lambda_N(\alpha)$-nuclearity" is replaced by $\Lambda_1(\alpha)$-nuclearity and that (DN) (resp. (Ω)) is replaced by $(\underline{\mathrm{DN}})$ (resp. $(\overline{\Omega})$). Since we do not want to give the definition of these properties, we just indicate that they are linear topological invariants and that (DN) and $(\underline{\mathrm{DN}})$ are inherited by topological linear subspaces, whereas (Ω) and $(\overline{\Omega})$ are inherited by separated quotient spaces. For Köthe sequence spaces $\Lambda(P)$ these properties can be expressed in terms of the Köthe set P. In this form some of them have already been introduced by Dragilev [10].

1.3. *Analytic functions.* Let E be an l.c. space and let $\Omega \neq \emptyset$ be an open subset of E. A function $f\colon \Omega \to C$ is called

(a) *G-analytic* if for any $a, b \in E$ the function $z \mapsto f(az+b)$ is a holomorphic function in one variable on its natural domain of definition,

(b) *hypoanalytic* if f is G-analytic and continuous on any compact subset of Ω,

(c) *holomorphic* if f is G-analytic and continuous on Ω.

$H_{hy}(\Omega)$ (resp. $H(\Omega)$) denotes the vector space of all hypoanalytic (resp. holomorphic) functions on Ω. The compact-open topology on $H_{hy}(\Omega)$ and $H(\Omega)$ is denoted by τ_0.

For further details concerning analytic functions on l.c. spaces we refer to the book of Dineen [9].

2. Some fundamental lemmas and results. In this section we introduce some more notation and give a sequence space representation of the space of hypoanalytic functions on open polydiscs in the strong dual of a reflexive nuclear Köthe space $\Lambda(P)$. From this we draw a corollary and then we provide several lemmas which will be applied in the subsequent sections.

2.1. *Notation and remarks.*

(a) Let $\Lambda(P)$ be a nuclear Köthe space. For any $a \in \Lambda(P)$ with $a \geqslant 0$ the set

$$\boldsymbol{D}_a := \{x \in \Lambda(P)' \mid \sup_{n\in N} |x_n| a_n < 1\}$$

is an open subset of $\Lambda(P)'_b$, called an *open polydisc*. For a given open polydisc $\boldsymbol{D}_a$ we call r_j the *j-th radius of* $\boldsymbol{D}_a$, where r_j is defined as $1/a_j$ for $a_j > 0$ and as ∞ for $a_j = 0$.

(b) We put

$$\boldsymbol{M} := \{m \in \boldsymbol{N}_0^{\boldsymbol{N}} \mid m_j = 0 \text{ for almost all } j \in \boldsymbol{N}\}$$

and define for any $x \in \omega$ and any $m \in \boldsymbol{M}$ the m-th power of x as

$$x^m := \prod_{j\in N} x_j^{m_j}.$$

(c) If α is an exponent sequence with $\sup_{n\in N}\big(\ln(n+1)/\alpha_n\big) < \infty$, we put $(\alpha|m) = \sum_{j\in N} \alpha_j m_j$. Then we define the exponent sequence $\beta = \beta(\alpha)$ as the increasing arrangement of the family $\big((\alpha|m)\big)_{m\in M}$ and we fix a bijection $b\colon \boldsymbol{N}\to\boldsymbol{M}$ with the property $\beta_n = \big(\alpha|b(n)\big)$ for any $n \in \boldsymbol{N}$. We remark that $\beta(\alpha)$ has been determined — up to equivalence — for a large number of sequences α which are of importance in analysis in Börgens, Meise and Vogt [3]. Explicit formulas and examples are given in Section 5 of [3]. Moreover, it was shown in [3], 3.4 (b), that $\beta(\alpha)$ is always a stable sequence.

(d) If $\boldsymbol{D}_a$ is an open polydisc in $\Lambda(P)'_b$ and $f \in H(\varphi \cap \boldsymbol{D}_a)$, then for any $m = (m_1, \ldots, m_n, 0, \ldots) \in \boldsymbol{M}$ the m-th Taylor coefficient of f (with respect to the origin) is given by

$$a_m(f) = \left(\frac{1}{2\pi i}\right)^n \int_{|z_1|=r_1} \cdots \int_{|z_n|=r_n} \frac{f(z_1, \ldots, z_n, 0, \ldots)}{z_1^{m_1+1} \cdots z_n^{m_n+1}}\, dz_1 \ldots dz_n,$$

where $0 < r_j < 1/a_j$ for $1 \leqslant j \leqslant n$ are arbitrarily chosen real numbers.

Since $\Lambda(P)$ is assumed to be nuclear, $\varphi \cap \boldsymbol{D}_a$ is sequentially dense in $\boldsymbol{D}_a$ with respect to the topology $\beta\big(\Lambda(P)', \Lambda(P)\big)$. Hence f is uniquely determined by the family of its Taylor coefficients.

The importance of nuclear sequence spaces in connection with the Taylor expansion by monomials in the coordinate functions was demonstrated by Boland and Dineen [8]. The following theorem is essentially due to them [8], Thm. 11. We repeat it here because it is fundamental for our results; its proof is given by an easy modification of the proof of Börgens, Meise and Vogt [3], Thm. 2.1.

2.2. THEOREM. *Let $\Lambda(P)$ be nuclear and reflexive and let $\boldsymbol{D}_a$ be an open polydisc in $\Lambda(P)'_b$. Let Q be a subset of $\Lambda(P)'$ consisting of non-negative sequences, such that $\{N_q \mid q \in Q\}$ is a fundamental system for the compact subsets of $\boldsymbol{D}_a$.*

(a) *For $f \in H(\varphi \cap \boldsymbol{D}_a)$ let $(a_m)_{m\in \boldsymbol{M}}$ denote the Taylor coefficients of f, defined as in 2.1 (d). Equivalent are:*

(1) $f = g|(\varphi \cap \boldsymbol{D}_a)$ *for some* $g \in H_{hy}(\boldsymbol{D}_a)$;

(2) $f|(N_q \cap \varphi)$ *is bounded for any* $q \in Q$;

(3) $\sup_{m\in \boldsymbol{M}} |a_m| q^m < \infty$ *for any* $q \in Q$;

(4) $\sum_{m\in \boldsymbol{M}} |a_m| q^m < \infty$ *for any* $q \in Q$.

(b) *The mapping $T: (H_{hy}(\boldsymbol{D}_a), \tau_0) \to \Lambda(\boldsymbol{M}, Q^{\boldsymbol{M}})$, $T(f) = (a_m(f))_{m\in \boldsymbol{M}}$, is a topological isomorphism, where $Q^{\boldsymbol{M}} = \{(q^m)_{m\in \boldsymbol{M}} \mid q \in Q\}$. The space $\Lambda(\boldsymbol{M}, Q^{\boldsymbol{M}})$ is nuclear.*

Remark. (a) Since $\Lambda(P)$ is reflexive and nuclear, it follows from 1.1 that there exists a set Q having the properties required in 2.2.

(b) In 2.2 the reflexivity of $\Lambda(P)$ is only used to get a convenient formulation. If $\Lambda(P)$ is assumed to be only nuclear, then 2.2 holds if $H_{hy}(\boldsymbol{D}_a)$ is replaced by the space of all G-analytic functions on $\boldsymbol{D}_a$ which are bounded on the equicontinuous subsets of $\boldsymbol{D}_a$, endowed with the topology of uniform convergence on these sets.

The following corollary of 2.2 has already been stated in Börgens, Meise and Vogt [3], 2.5.

2.3. COROLLARY. *Let $\Lambda_1(\alpha)$ be nuclear and let $\mathbf{1} \in \Lambda_1(\alpha)$ denote the sequence identically 1. Then $(H(\boldsymbol{D}_1), \tau_0)$ is nuclear and isomorphic to $\Lambda_1(\beta(\alpha))$ by the mapping*

$$T: f \mapsto (a_{b(n)}(f))_{n\in \boldsymbol{N}}.$$

Proof. It is easily seen that $Q = \{(r^{\alpha_n})_{n\in \boldsymbol{N}} \mid 0 < r < 1\}$ is a fundamental system for the compact subsets of $\boldsymbol{D}_1$. Since any hypoanalytic function on a (DFN)-space is already continuous, we get from 2.2 that $(H(\boldsymbol{D}_1), \tau_0)$ is isomorphic to $\Lambda(\boldsymbol{M}, Q^{\boldsymbol{M}})$. However, this space is isomorphic to $\Lambda_1(\beta(\alpha))$ by the mapping

$$(a_m)_{m\in \boldsymbol{M}} \mapsto (a_{b(n)})_{n\in \boldsymbol{N}}.$$

In the subsequent sections we shall use the sequence space representation given in 2.2 for a number of computations. For this purpose it will be useful to have a convenient description for the systems $Q = Q(P, a)$ appearing in 2.2. Therefore we now indicate how such systems can be obtained for nuclear Fréchet spaces $\Lambda(P)$.

2.4. LEMMA. *Let $\Lambda(P)$ be a nuclear Fréchet space and let $a \in \Lambda(P)$, $a \geqslant 0$, be given. We assume that the Köthe set*

$$P = \{(p_{j,k})_{j\in \boldsymbol{N}} \mid k \in \boldsymbol{N}\}$$

satisfies

(α) $p_{j,k} \leqslant p_{j,k+1}$ *for all* $j \in \boldsymbol{N}$ *and all* $k \in \boldsymbol{N}$;

(β) $\lim_{j\to\infty} (p_{j,k}/p_{j,k+1}) = 0$ *for all* $k \in \boldsymbol{N}$.

(i) *Then there exist strictly increasing sequences $(n_k)_{k\in \boldsymbol{N}}$ in $\boldsymbol{N}$ and $(\varrho_k)_{k\in \boldsymbol{N}}$ in $(0, 1)$ with the following properties:*

(a) $\sup_{j\geqslant n_k} a_j p_{j,k} < \varrho_k$ *for any* $k \in \boldsymbol{N}$;

(b) $1 > \varrho_k > \sqrt{\varrho_{k-1}} > 0$ *for any* $k > 1$.

(ii) *The set $Q = \{q_k \mid k \in \boldsymbol{N}\}$, where $q_k = (q_{j,k})_{j\in \boldsymbol{N}}$ is defined by*

$$q_{j,k} = \begin{cases} \varrho_k/a_j & \text{if} \quad a_j \neq 0 \text{ and } 1 \leqslant j < n_k, \\ k p_{j,k} & \text{if} \quad a_j = 0 \text{ and } 1 \leqslant j < n_k, \\ p_{j,k} & \text{if} \quad j \geqslant n_k, \end{cases}$$

is contained in $\boldsymbol{D}_a$ and $(N_{q_k})_{k\in \boldsymbol{N}}$ is an increasing fundamental system for the compact subsets of $\boldsymbol{D}_a$.

Proof. (i): The sequences $(n_k)_{k\in \boldsymbol{N}}$ and $(\varrho_k)_{k\in \boldsymbol{N}}$ are defined inductively as follows. First we choose ϱ_1, with $0 < \varrho_1 < 1$. Since a belongs to $\Lambda(P)$, we have $\lim_{j\to\infty} a_j p_{j,1} = 0$. Hence there exists $n_1 \in \boldsymbol{N}$ such that $\sup_{j\geqslant n_1} a_j p_{j,1} < \varrho_1$. Now assume that n_k and ϱ_k are defined for $1 \leqslant k \leqslant m$ in such a way that (a) and (b) hold for $1 \leqslant k \leqslant m$ (putting $\varrho_0 = 0$). Then we choose ϱ_{m+1} with $\sqrt{\varrho_m} < \varrho_{m+1} < 1$. Since $\lim_{j\to\infty} a_j p_{j,m+1} = 0$, there exists $n_{m+1} > n_m$ such that $\sup_{j\geqslant n_{m+1}} a_j p_{j,m+1} < \varrho_{m+1}$. Hence (a) and (b) are satisfied for $1 \leqslant k \leqslant m+1$ and the existence of the sequences follows by induction.

(ii): From the definition of Q, (a) and (b) it follows that Q is a subset of $\boldsymbol{D}_a$. Since $\Lambda(P)$ is nuclear and reflexive, this implies that N_q is compact in $\boldsymbol{D}_a$ for any $q \in Q$.

If L is an arbitrary compact subset of $\boldsymbol{D}_a$, by (α) there exists $l \in \boldsymbol{N}$ with $L \subset N_{l\cdot p_l} \cap \boldsymbol{D}_a$. Since (b) implies that $\varrho_{k+1} > \varrho_1^{2^{-k}}$, we have $\lim_{k\to\infty} \varrho_k = 1$. Hence ($\alpha$) and ($\beta$) imply that we can find $k > l$ with

$$\sup_{x\in L} \sup_{j\in \boldsymbol{N}} |x_j| a_j \leqslant \varrho_k \quad \text{and} \quad \sup_{j\geqslant n_k} (p_{j,l}/p_{j,k}) \leqslant 1/l.$$

From this it follows easily that $L \subset N_{q_k}$, hence $(N_{q_k})_{k\in \boldsymbol{N}}$ is a fundamental system for the compact subsets of $\boldsymbol{D}_a$. This system is increasing since we have $q_{j,k} \leqslant q_{j,k+1}$ for all $j \in \boldsymbol{N}$ and all $k \in \boldsymbol{N}$. Since ϱ is increasing, this follows from (α) and (a).

The following two lemmas will be needed in the next sections.

2.5. Lemma. *Let E be an* l.c. *space and let F be a nuclear Fréchet space.*

(a) *If E is a subspace of F, then $(H(E_b'), \tau_0)$ is a subspace of $(H(F_b'), \tau_0)$.*

(b) *If E is a quotient space of F, then $(H(E_b'), \tau_0)$ is a quotient space of $(H(F_b'), \tau_0)$.*

Proof. (a): Let $\hat{E}$ denote the completion of E. Then it follows from Köthe [13], § 29,6 (1) and § 27,2 (5), that $E_b' = \hat{E}_b'$. Hence it follows from the Hahn–Banach theorem that the restriction map $\pi\colon F_b' \to E_b'$ is surjective and open. If one defines

$$^T\pi\colon (H(E_b'), \tau_0) \to (H(F_b'), \tau_0)$$

by

$$^T\pi(f) = f \circ \pi,$$

then $^T\pi$ is an injective topological homomorphism since any compact set in E_b' is the π-image of a compact set in F_b'.

(b): Let $\pi\colon F \to E$ denote the quotient map. Then E_b' can be regarded as a subspace of F_b' by means of the adjoint $^t\pi$ of π and hence the extension theorem of Boland [5], 3.1, tells that

$$q\colon (H(F_b'), \tau_0) \to (H(E_b'), \tau_0),$$
$$q(f) := f \circ {}^t\pi,$$

is surjective and open.

The proof of the following lemma is an easy consequence of the Cauchy inequalities (see e.g. Aron and Schottenloher [1], Thm. 2.2, (c) $\Rightarrow$ (d)).

2.6. Lemma. *Let E be a Fréchet–Montel space and let $U \neq \emptyset$ be an open subset of E_b'. Then E is a complemented subspace of $(H(U), \tau_0)$.*

Remark. From 2.6 and the inheritance properties of (DN), ($\underline{\mathrm{DN}}$), (Ω) and ($\bar{\Omega}$) it easily follows that a necessary condition for $(H(U), \tau_0)$ having one of these properties is that E has the corresponding one.

3. Subspaces and quotients of finite type power series spaces. This section in which we deal with the properties ($\bar{\Omega}$) and ($\underline{\mathrm{DN}}$) contains the main result of this article, namely a characterization of those open polydiscs $\boldsymbol{D}_a$ in a nuclear space $\Lambda_1(\alpha)_b'$ for which $(H(\boldsymbol{D}_a), \tau_0)$ is a power series space. The proof of the main theorem is prepared by the following two lemmas.

3.1. Lemma. *Let $\Lambda_1(\alpha)$ be nuclear. A diagonal map $D\colon \Lambda_1(\alpha) \to \Lambda_1(\alpha)$, $Dx = (d_j x_j)_{j\in\boldsymbol{N}}$, is an automorphism iff $d \in \Lambda_1(\alpha)$ and $1/d \in \Lambda_1(\alpha)$.*

Proof. Since $\Lambda_1(\alpha)$ is nuclear, we have $\boldsymbol{1} \in \Lambda_1(\alpha)$. Hence $d = D(\boldsymbol{1}) \in \Lambda_1(\alpha)$ and $1/d = D^{-1}(\boldsymbol{1}) \in \Lambda_1(\alpha)$ if D is an automorphism. In order to prove the converse implication it suffices to show that D is continuous for any $d \in \Lambda_1(\alpha)$. However, this is a consequence of the following estimate which holds for all $x \in \Lambda_1(\alpha)$ and all r with $0 < r < 1$:

$$\pi_r(Dx) = \sum_{j=1}^{\infty} |d_j x_j| r^{\alpha_j} \leqslant (\sup_{j\in\boldsymbol{N}} |d_j| \sqrt{r}^{\alpha_j}) \sum_{j=1}^{\infty} |x_j| \sqrt{r}^{\alpha_j} \leqslant \pi_{\sqrt{r}}(d)\,\pi_{\sqrt{r}}(x).$$

3.2. Lemma. *Let $\Lambda(P)$ be a nuclear Fréchet space, where $P = \{(p_{j,k})_{j\in\boldsymbol{N}} \mid k \in \boldsymbol{N}\}$. For $a \in \Lambda(P)$, $a \geqslant 0$, the following are equivalent:*

(1) *$(H(\boldsymbol{D}_a), \tau_0)$ has ($\bar{\Omega}$);*

(2) *For any $s \in \boldsymbol{N}$ there exists $t > s$ and $j_s \in \boldsymbol{N}$ such that*

$$p_{j,s} \leqslant a_j p_{j,t}^2 \quad \text{for all } j \geqslant j_s.$$

Proof. We may assume that P satisfies 2.4 (α) and (β). Then we have by 2.4 and 2.2 that $(H(\boldsymbol{D}_a), \tau_0)$ is isomorphic to $\Lambda(\boldsymbol{M}, Q^{\boldsymbol{M}})$, where Q is defined as in 2.4.

(1) $\Rightarrow$ (2): First we show that (1) implies $a_j \neq 0$ for all $j \in \boldsymbol{N}$. In order to prove this we put $D_j = \{z \in \boldsymbol{C} \mid |z| < r_j\}$, where $r_j = 1/a_j$. Then it is easy to see that $H(D_j) \simeq \Lambda_{r_j}(n)$ is a quotient space of $(H(\boldsymbol{D}_a), \tau_0)$. Hence (1) implies that $\Lambda_{r_j}(n)$ has ($\bar{\Omega}$). By Wagner [28], 1.11, this shows $r_j < \infty$, i.e. $a_j > 0$.

Then we remark that property ($\bar{\Omega}$) of $\Lambda(\boldsymbol{M}, Q^{\boldsymbol{M}})$ implies by Wagner [28], 1.11, that the following holds true:

(3) For any $s \in \boldsymbol{N}$ there exists $t > s$ such that for any $k \in \boldsymbol{N}$ there exists $C > 0$ such that

$$q_k^m q_s^m \leqslant C q_t^{2m} \quad \text{for all } m \in \boldsymbol{M}.$$

By choosing $m = ne_j$ $(e_j = (\delta_{ij})_{i\in\boldsymbol{N}})$ for $n \in \boldsymbol{N}$ and taking n-th roots we get from (3) by going to the limit $n \to \infty$:

(4) For any $s \in \boldsymbol{N}$ there exists $t > s$ such that for any $k \in \boldsymbol{N}$

$$q_{j,k} q_{j,s} \leqslant q_{j,t}^2 \quad \text{for all } j \in \boldsymbol{N}.$$

In order to see that (4) implies (2) we choose $j \geqslant n_t$ arbitrarily. Then the definition of Q in 2.4 and (4) imply that for any k with $j < n_k$ we have

$$q_{j,k} p_{j,s} = \frac{\varrho_k}{a_j} p_{j,s} \leqslant p_{j,t}^2.$$

Because of $\lim_{k\to\infty} \varrho_k = 1$, $\lim_{k\to\infty} n_k = \infty$ and $a_j \neq 0$ this implies

$$p_{j,s} \leqslant a_j p_{j,t}^2 \quad \text{for all } j \geqslant n_t,$$

hence (2) holds.

(2) ⇒ (1): As the proof of 2.4 shows, we may assume that the sequence $(n_k)_{k\in \boldsymbol{N}}$ in 2.4 (i) is constructed in such a way that $n_s \geqslant j_s$ for all $s \in \boldsymbol{N}$. Then we get from (2):

(5) For any $s \in \boldsymbol{N}$ there exists $t > s$ such that

$$p_{j,s} \leqslant a_j p_{j,t}^2 \quad \text{for all } j \geqslant n_s.$$

Now we show that (5) can be used to prove that Q satisfies (4). This implies that $Q^{\boldsymbol{M}}$ satisfies (3), which gives by Wagner [28], 1.10, that $\Lambda(\boldsymbol{M}, Q^{\boldsymbol{M}})$ has $(\bar{\Omega})$, i.e. that (1) holds.

In order to prove (4) we first remark that (5) implies $a_j \neq 0$ for all $j \in \boldsymbol{N}$. Then, for a given $s \in \boldsymbol{N}$, we choose $t > s$ such that (5) holds. Now we remark that for any l with $j \geqslant n_l$ we get from 2.4 (a) that $p_{j,l} \leqslant \varrho_l/a_j$. Hence $q_{j,l} \leqslant \varrho_l/a_j$ for all $j \in \boldsymbol{N}$ and all $l \in \boldsymbol{N}$ by the definition of Q. Using this and 2.4 (b) we get for $j < n_t$ and $k \in \boldsymbol{N}$

$$q_{j,k}\, q_{j,s} \leqslant \frac{\varrho_k}{a_j}\,\frac{\varrho_s}{a_j} \leqslant \frac{\varrho_{t-1}}{a_j^2} \leqslant \left(\frac{\varrho_t}{a_j}\right)^2 = q_{j,t}^2.$$

If $j \geqslant n_t$, then we get by the same arguments and (5) that for any $k \in \boldsymbol{N}$ we have

$$q_{j,k}\, q_{j,s} \leqslant \frac{1}{a_j}\, p_{j,s} \leqslant p_{j,t}^2 = q_{j,t}^2.$$

This shows that Q satisfies (4), which completes the proof.

Remark. (a) If in 3.2 $\Lambda(P)$ has a continuous norm, we may assume that $p_{j,k} > 0$ for all j and k in $\boldsymbol{N}$. Then 3.2 (2) is equivalent to

(2)′ For $a > 0$ and for any $s \in \boldsymbol{N}$ there exists $t > s$ such that

$$\liminf_{j\to\infty} (a_j p_{j,t}^2/p_{j,s}) \geqslant 1.$$

(b) In the proof of 3.2 we have shown that (1) implies $a_j > 0$ for all $j \in \boldsymbol{N}$. Hence it follows from the definition of Q that $\Lambda(P) = \Lambda(Q)$. But then (4) in connection with Wagner [28], 1.11, proves that (1) implies in particular that $\Lambda(P)$ has $(\bar{\Omega})$, which has already been remarked to be a consequence of 2.6.

In 2.3 we have seen that for any nuclear space $\Lambda_1(\alpha)$ we have $(H(\boldsymbol{D}_1), \tau_0) = \Lambda_1(\beta(\alpha))$ for the open polydisc $\boldsymbol{D}_1$ in $\Lambda_1(\alpha)_b'$. The following theorem gives a characterization of all polydiscs $\boldsymbol{D}_a$ in $\Lambda_1(\alpha)_b'$ for which $(H(\boldsymbol{D}_a), \tau_0)$ is isomorphic to $\Lambda_1(\beta(\alpha))$ and shows that this property also characterizes the polydiscs $\boldsymbol{D}_a$ for which $(H(\boldsymbol{D}_a), \tau_0)$ has $(\bar{\Omega})$.

3.3. Theorem. *Let $\Lambda_1(\alpha)$ be nuclear and let $a \in \Lambda_1(\alpha)$ satisfy $a \geqslant 0$. Then the following are equivalent:*

(1) *$a > 0$ and $\lim_{j\to\infty} a_j^{-1} \ln a_j = 0$;*

(2) *$a > 0$ and $1/a \in \Lambda_1(\alpha)$;*

(3) *$(H(\boldsymbol{D}_a), \tau_0) \simeq \Lambda_1(\beta(\alpha))$;*

(4) *$(H(\boldsymbol{D}_a), \tau_0)$ is a power series space;*

(5) *$(H(\boldsymbol{D}_a), \tau_0)$ is a power series spaces of finite type;*

(6) *$(H(\boldsymbol{D}_a), \tau_0)$ is a quotient of a power series space of finite type;*

(7) *$(H(\boldsymbol{D}_a), \tau_0)$ has $(\bar{\Omega})$.*

Proof. (1) ⇒ (2): For an arbitrary r with $0 < r < 1$ we put $\varepsilon := \ln(1/r) > 0$. Then there exists $J = J(\varepsilon)$ such that $|a_j^{-1} \ln a_j| \leqslant \varepsilon$ for all $j \geqslant J$. This implies $\ln(1/a_j) \leqslant \varepsilon a_j$ and hence $1/a_j \leqslant e^{\varepsilon \alpha_j} = (1/r)^{\alpha_j}$ for all $j \geqslant J$. Since $\Lambda_1(\alpha)$ is nuclear, we get $1/a \in \Lambda_1(\alpha)$.

(2) ⇒ (3): Since a and $1/a$ belong to $\Lambda_1(\alpha)$, we get from Lemma 3.1 that the diagonal map $A\colon \Lambda_1(\alpha) \to \Lambda_1(\alpha)$, $Ax := (a_j x_j)_{j\in \boldsymbol{N}}$, is an automorphism of $\Lambda_1(\alpha)$. Obviously we have $A(\boldsymbol{D}_1) = \boldsymbol{D}_a$. Hence A induces an isomorphism between $(H(\boldsymbol{D}_a), \tau_0)$ and $(H(\boldsymbol{D}_1), \tau_0)$. Because of 2.3 this implies that (3) holds.

(3) ⇒ (4) and (5) ⇒ (6) hold trivially.

(4) ⇒ (5): By 2.6 $(H(\boldsymbol{D}_a), \tau_0)$ has a subspace which is isomorphic to $\Lambda_1(\alpha)$. Hence (4) implies that (5) holds.

(6) ⇒ (7): This follows from the remark that any power series space of finite type has $(\bar{\Omega})$ and that $(\bar{\Omega})$ is inherited by quotient spaces (see Wagner [28], 1.11, 1.2).

(7) ⇒ (1): We choose a strictly increasing sequence σ in $(0, 1)$ with $\lim_{k\to\infty} \sigma_k = 1$. Then $P = \{(\sigma_k^{\alpha_j})_{j\in \boldsymbol{N}} \mid k \in \boldsymbol{N}\}$ satisfies 2.4 (α) and (β) and we have $\Lambda(P) = \Lambda_1(\alpha)$. Hence we can apply 3.2, which shows that $a > 0$. Since $a \in \Lambda_1(\alpha)$ implies $\limsup_{j\to\infty} a_j^{-1} \ln a_j \leqslant 0$, it suffices to show $\liminf_{j\to\infty} a_j^{-1} \ln a_j \geqslant 0$. This follows from 3.2 (2), which implies that for any $s \in \boldsymbol{N}$ there exist $t = t(s) > s$ and $j_s \in \boldsymbol{N}$ such that

$$a_j \geqslant \frac{p_{j,s}}{p_{j,t}^2} = \left(\frac{\sigma_s}{\sigma_t^2}\right)^{\alpha_j} \quad \text{for all } j \geqslant j_s,$$

and consequently

$$\liminf_{j\to\infty} a_j^{-1} \ln a_j \geqslant \liminf_{s\to\infty} \ln \frac{\sigma_s}{\sigma_{t(s)}^2} = 0.$$

3.4. Corollary. *Let $\Lambda_1(\alpha)$ be nuclear and let $a, b \in \Lambda_1(\alpha)$ satisfy $a \geqslant 0$ and $b \geqslant 0$. If $1/a \in \Lambda_1(\alpha)$ and $1/b \notin \Lambda_1(\alpha)$, then $(H(\boldsymbol{D}_a), \tau_0)$ and $(H(\boldsymbol{D}_b), \tau_0)$ are not isomorphic.*

Remark. (a) Theorem 3.3 holds if $\Lambda_1(\alpha)$ is replaced by $\Lambda_R(\alpha)$, $0 < R < \infty$, provided that (1) (resp. (2)) is replaced by (1)′ (resp. (2)′):

(1)′ $\alpha > 0$ and $\lim_{j\to\infty} \alpha_j^{-1} \ln a_j = \ln R$;

(2)′ $a > 0$ and $1/a \in \Lambda_{1/R}(\alpha)$.

(b) Theorem 3.3 is optimal in the following sense: If E is a nuclear Fréchet space for which there exists an open subset U in E_b' such that $(H(U), \tau_0)$ is isomorphic to a power series space and contains a bounded function which is not constant, then E is isomorphic to a power series space of finite type. This is obtained in the following way: As the proof of Vogt [21], 2.6, shows, $(H(U), \tau_0)$ does not have (DN), hence $(H(U), \tau_0)$ is isomorphic to some $\Lambda_1(\gamma)$, which is nuclear by the theorem of Boland [7] and Waelbroeck [27]. From 2.6 we get that E is isomorphic to a subspace and a quotient space of $\Lambda_1(\gamma)$. Hence E is isomorphic to some nuclear $\Lambda_1(\delta)$ by a result of Mityagin [15] (see also Vogt [24], 1.6).

(c) If $\Lambda_1(\alpha)$ is nuclear and if for the open polydiscs $\boldsymbol{D}_a$ and $\boldsymbol{D}_b$ in $\Lambda_1(\alpha)_b'$ the spaces $(H(\boldsymbol{D}_a), \tau_0)$ and $(H(\boldsymbol{D}_b), \tau_0)$ have $(\overline{\Omega})$, then they are not only isomorphic as l.c. spaces but even as topological algebras. This follows from the proof of 3.3 since it shows the existence of an automorphism A of $\Lambda_1(\alpha)$ with $A(\boldsymbol{D}_a) = \boldsymbol{D}_b$. This can also be derived from Meise and Vogt [14], where a classification of the algebra isomorphisms between $(H(\boldsymbol{D}_a), \tau_0)$ and $(H(\boldsymbol{D}_b), \tau_0)$ is given.

(d) It is an obvious consequence of 3.3 that for any open subset U of $\Lambda_1(\alpha)$ which is biholomorphically equivalent to $\boldsymbol{D}_1$, the space $(H(U), \tau_0)$ is isomorphic to $\Lambda_1(\beta(\alpha))$. Furthermore, it follows from 3.3 that for any analytic subvariety V of $\boldsymbol{D}_1$ (whatever the right definition will be) for which $\varrho\colon (H(\boldsymbol{D}_1), \tau_0) \to (H(V), \tau_0)$, ϱ the restriction, is surjective and open, $(H(V), \tau_0)$ has $(\overline{\Omega})$. Up to now almost nothing is known in this situation, except if V is the intersection of $\boldsymbol{D}_1$ with a closed hyperplane (see Raboin [17], Cor. 3 of Thm. 3).

3.5. Proposition. *Let $\Lambda(P)$ be a quotient of a nuclear space $\Lambda_1(\alpha)$, where $P = \{(p_{j,k})_{j\in N} \mid k \in N\}$. For $a \in \Lambda(P)$, $a \geqslant 0$, the following are equivalent:*

(1) *$(H(\boldsymbol{D}_a), \tau_0)$ is a quotient of $\Lambda_1(\beta(\alpha))$;*

(2) *a satisfies 3.2 (2).*

There exists $b \in \Lambda(P)$ satisfying condition (2).

Proof. Since $\Lambda(P)$ is a quotient of $\Lambda_1(\alpha)$, it is $\Lambda_1(\alpha)$-nuclear. Hence it follows from Börgens, Meise and Vogt [4], Thm. 4.1, that $(H(\boldsymbol{D}_a), \tau_0)$ is $\Lambda_1(\beta(\alpha))$-nuclear. Since $\beta(\alpha)$ is stable by Börgens, Meise and Vogt [3], 3.4 (b), we get from Wagner [28], Thm. 2.5, that $(H(\boldsymbol{D}_a), \tau_0)$ satisfies (1) iff it has $(\overline{\Omega})$. By 3.2 this is equivalent to (2).

In order to show the existence of some $b \in \Lambda(P)$ satisfying (2) we remark that $\Lambda(P)$ has $(\overline{\Omega})$. Hence we may assume by Wagner [28], 1.11,

(3) $p_{j,k} p_{j,s} \leqslant p_{j,s+1}^2$ for all $j \in N$, all $s \in N$ and all $k \in N$.

For any $j \in N$ there exists $q \in N$ such that $p_{j,q} > 0$, hence (3) implies that

$$\pi_j := \sup_{k\in N} p_{j,k} \leqslant \frac{p_{j,q+1}^2}{p_{j,q}} < \infty$$

and that $\pi_j > 0$. Moreover, (3) implies that $b := 1/\pi$ satisfies

$$p_{j,s} \leqslant b_j p_{j,s+1}^2 \quad \text{for all } j \in N \text{ and all } s \in N.$$

Hence b satisfies 3.2 (2) and its definition shows $b \in \Lambda(P)$.

In Theorem 3.3 it is surprising that $(H(\boldsymbol{D}_a), \tau_0)$ is isomorphic to $\Lambda_1(\beta(\alpha))$ iff it is a quotient of some $\Lambda_1(\gamma)$. The reason for this is explained by the following proposition which shows in particular that for any open polydisc $\boldsymbol{D}_b$ in $\Lambda_1(\alpha)_b'$ the space $(H(\boldsymbol{D}_b), \tau_0)$ is a subspace of $\Lambda_1(\beta(\alpha))$. Hence the equivalence of 3.3 (3) and 3.3 (6) also follows from the result of Mityagin [15] mentioned in Remark (b) behind 3.4.

3.6. Proposition. *Let $\Lambda(P)$ be a subspace of a nuclear space $\Lambda_1(\alpha)$. Then $(H(\boldsymbol{D}_a), \tau_0)$ is a subspace of $\Lambda_1(\beta(\alpha))$ for any open polydisc $\boldsymbol{D}_a$ in $\Lambda(P)_b'$.*

Proof. Since $\Lambda(P)$ is a subspace of $\Lambda_1(\alpha)$, it follows from Vogt [23], 2.3, that it has $(\underline{\mathrm{DN}})$. From Vogt [23], 4.1 and the nuclearity of $\Lambda(P)$ it follows that we may assume that $P = \{(p_{j,k})_{j\in N} \mid k \in N\}$ has the properties stated in 2.4 (α) and (β) and satisfies, moreover, (γ) and (δ):

(γ) For any $k \in N$ there is $q > k$ such that

$$k p_{j,k} \leqslant p_{j,q} \quad \text{for all } j \in N.$$

(δ) For any $k \in N$ there exists $\varepsilon > 0$ such that

$$p_{j,k}^{1+\varepsilon} \leqslant p_{j,k+1} p_{j,1}^{\varepsilon} \quad \text{for all } j \in N.$$

Then we get from 2.4 and 2.2 that for any $a \in \Lambda(P)$, $a \geqslant 0$, we have $(H(\boldsymbol{D}_a), \tau_0) \simeq \Lambda(\boldsymbol{M}, Q^{\boldsymbol{M}})$, where $Q = \{(q_{j,k})_{j\in N} \mid k \in N\}$ is given according to 2.4 and (γ) by

$$q_{j,k} = \begin{cases} \varrho_k / a_j & \text{if } \quad a_j \neq 0 \text{ and } 1 \leqslant j < n_k, \\ p_{j,k} & \text{if } \quad j \geqslant n_k \text{ or if } a_j = 0. \end{cases}$$

We claim that (δ) holds for the Köthe set Q. This will be a consequence of the following statements (1)–(4) in which $k \in N$ is arbitrary but fixed.

(1) Since $\lim_{\varepsilon\to 0}(\varrho_1/\varrho_k)^\varepsilon = 1$ and since $\varrho_k < \varrho_{k+1}$, there exists $\varepsilon_1 > 0$ such that for all ε with $0 < \varepsilon \leqslant \varepsilon_1$ we have $\varrho_k/\varrho_{k+1} \leqslant (\varrho_1/\varrho_k)^\varepsilon$. Hence we have for $0 < \varepsilon \leqslant \varepsilon_1$ and all j with $a_j \neq 0$

$$\left(\frac{\varrho_k}{a_j}\right)^{1+\varepsilon} \leqslant \frac{\varrho_{k+1}}{a_j}\left(\frac{\varrho_1}{a_j}\right)^\varepsilon.$$

(2) Since $\Lambda(P)$ is Hausdorff, (γ) implies that $p_{j,1} \neq 0$ all $j \in \boldsymbol{N}$. Hence we have $\lim_{\varepsilon\to 0}(a_j p_{j,1})^\varepsilon = 1$ for all $j \in \boldsymbol{N}$ with $a_j \neq 0$. Since $\varrho_k < \varrho_{k+1}$, this implies that there exists $\varepsilon_2 > 0$ such that for all ε with $0 < \varepsilon \leqslant \varepsilon_2$ and all j with $n_1 \leqslant j < n_k$ and $a_j \neq 0$ we have

$$\left(\frac{\varrho_k}{a_j}\right)^{1+\varepsilon} \leqslant \frac{\varrho_{k+1}}{a_j} p_{j,1}^\varepsilon.$$

(3) Since $\lim_{\varepsilon\to 0}(p_{j,1}/p_{j,k})^\varepsilon = 1$ for all $j \in \boldsymbol{N}$ and since $\sup_{j \geqslant n_k} a_j p_{j,k} \leqslant \varrho_k < \varrho_{k+1}$ by 2.4 (i) (a) there exists $\varepsilon_3 > 0$ such that for all ε with $0 < \varepsilon \leqslant \varepsilon_3$ and all j with $a_j \neq 0$ and $n_k \leqslant j < n_{k+1}$ we have

$$p_{j,k}^{1+\varepsilon} \leqslant \frac{\varrho_{k+1}}{a_j} p_{j,1}^\varepsilon.$$

(4) From (δ) we get that the existence of ε_4 satisfying $p_{j,k}/p_{j,k+1} \leqslant (p_{j,1}/p_{j,k})^{\varepsilon_4}$ for all $j \in \boldsymbol{N}$. Since $p_{j,1}/p_{j,k} \geqslant 1$ by (α), we even have for all $0 < \varepsilon \leqslant \varepsilon_4$ and all $j \in \boldsymbol{N}$

$$p_{j,k}^{1+\varepsilon} \leqslant p_{j,k+1} p_{j,1}^\varepsilon.$$

If we choose $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, then it follows from the definition of Q and (1)–(4) that

$$q_{j,k}^{1+\varepsilon} \leqslant q_{j,k+1} q_{j,1}^\varepsilon \quad \text{for all } j \in \boldsymbol{N}.$$

Since k was arbitrary, we have shown that for any $k \in \boldsymbol{N}$ there exist $\varepsilon > 0$ such that

$$(5) \qquad q_k^{(1+\varepsilon)m} \leqslant q_{k+1}^m q_1^{\varepsilon m} \quad \text{for all } m \in \boldsymbol{M}.$$

Because of Vogt [23], 4.1, (5) implies that $\Lambda(\boldsymbol{M}, Q^{\boldsymbol{M}})$ and consequently $(H(\boldsymbol{D}_a), \tau_0)$ has ($\underline{\mathrm{DN}}$). Since $\Lambda(P)$ is a quotient of $\Lambda_1(\alpha)$, it is $\Lambda_1(\alpha)$-nuclear. Hence $(H(\boldsymbol{D}_a), \tau_0)$ is $\Lambda_1(\beta(\alpha))$-nuclear by Börgens, Meise and Vogt [4], Thm. 4.1. Since $\beta(\alpha)$ is stable, an application of Vogt [23], Satz 3.2, gives that $(H(\boldsymbol{D}_a), \tau_0)$ is a subspace of $\Lambda_1(\beta(\alpha))$.

From 3.6 we get the following corollary which also shows that 3.6 is in a sense optimal.

3.7. Corollary. *An* l.c. *space E is a subspace of a nuclear power series space of finite type iff $(H(E'_b), \tau_0)$ has this property.*

Proof. If E is a subspace of some nuclear space $\Lambda_1(\alpha)$, then by 2.5 (a) $(H(E'_b), \tau_0)$ is a subspace of $(H(\Lambda_1(\alpha)'_b), \tau_0)$, which is a subspace of $\Lambda_1(\beta(\alpha))$ by 3.6. If $(H(E'_b), \tau_0)$ is a subspace of some nuclear space $\Lambda_1(\gamma)$, then E is a subspace of $\Lambda_1(\gamma)$ by 2.6.

4. Subspaces and quotients of infinite type power series spaces. This section contains some results on subspaces and quotient spaces of power series spaces of infinite type. The proofs are more easy than in Section 3 since $(H(\Lambda_\infty(\alpha)'_b), \tau_0)$ is isomorphic to $\Lambda_\infty(\beta(\alpha))$ for any nuclear space $\Lambda_\infty(\alpha)$.

4.1. Proposition. *Let $\Lambda_\infty(\alpha)$ be nuclear and let E be an* l.c. *space.*

(a) *If E is a subspace of $\Lambda_\infty(\alpha)$, then $(H(E'_b), \tau_0)$ is a subspace of $\Lambda_\infty(\beta(\alpha))$.*

(b) *If E is a quotient space of $\Lambda_\infty(\alpha)$, then $(H(E'_b), \tau_0)$ is a quotient space of $\Lambda_\infty(\beta(\alpha))$.*

Proof. (a): If E is a subspace of $\Lambda_\infty(\alpha)$, then $(H(E'_b), \tau_0)$ is a subspace of $(H(\Lambda_\infty(\alpha)'_b), \tau_0)$ by 2.5 (a). Hence the result follows from Börgens, Meise and Vogt [3], Thm. 2.1, which tells that $(H(\Lambda_\infty(\alpha)'_b), \tau_0)$ is isomorphic to $\Lambda_\infty(\beta(\alpha))$.

(b): The same arguments as in part (a) apply if 2.5 (a) is replaced by 2.5 (b).

4.2. Corollary. *Let E be a Fréchet–Montel space.*

(a) *E is a subspace of s iff $(H(E'_b), \tau_0)$ is a subspace of s.*

(b) *E is a quotient space of s iff $(H(E'_b), \tau_0)$ is a quotient space of s.*

Proof. The "if" part follows from 4.1 since for $\alpha = (\ln(n+1))_{n\in\boldsymbol{N}}$ the sequence $\beta(\alpha)$ is equivalent to α by Börgens, Meise and Vogt [3], Thm. 2.4. The "only if" part follows easily from 2.6.

Remark. Because of the characterization of the subspaces (resp. quotient spaces) of s given by Vogt [21] (resp. Vogt and Wagner [25]) Corollary 4.2 tells that a nuclear Fréchet space E has (DN) (resp. (Ω)) iff $(H(E'_b), \tau_0)$ has (DN) (resp. (Ω)). Because of Vogt [21], Satz 2.6, it cannot be expected that (DN) holds for many open subsets of E'_b. Indeed, it follows from this result (resp. from Vogt [21], 2.4) and the preceding remark that for a nuclear Fréchet space $\Lambda(P)$ the space $(H(\boldsymbol{D}_a), \tau_0)$ has (DN) iff $\boldsymbol{D}_a = \Lambda(P)'_b$. We shall show now that more can be obtained for the property (Ω), since the dual form of (Ω) can be localized in a sense.

In order to do this we recall the definition of the space $H(K)$ of germs of holomorphic functions on a compact subset E of a metrizable l.c. space E. We choose a decreasing open neighbourhood basis $(U_n)_{n\in\boldsymbol{N}}$ of K and let $H^\infty(U_n)$ denote the space of all bounded holomorphic functions on U_n endowed with the sup-norm. Then $\{H^\infty(U_n), r_{nm}\}_{n\in \mathbf{N}}$ is an inductive system if r_{nm}: $H^\infty(U_n)\to H^\infty(U_m)$ denotes the restriction map for $m \geqslant n$. The inductive limit of this system is denoted by $H(K)$. We remark that $H(K)$ does not depend on the particular choice of the neighbourhood basis $(U_n)_{n\in\mathbf{N}}$.

4.3. PROPOSITION. *Let E be an* l.c. *space.*

(a) *If E is a quotient space of s, then $H(K)_b'$ is a quotient space of s for any compact set $K \neq \emptyset$ in E.*

(b) *If E is a Fréchet–Schwartz space and if there exists a compact set $K \neq \emptyset$ in E for which $H(K)_b'$ is a quotient space of s, then E is a quotient space of s.*

Proof. (a): If E is a quotient of s, then $(H(E_b'), \tau_0)$ is a quotient of s by 4.2 (b). Since E is nuclear, $H_E(\{0\})$ is a (DFN)-space by Bierstedt and Meise [2], Thm. 7. From this and the duality result of Boland [6], Thm. 1 and Remark 2 (a), we get that $H_E(\{0\})_b'$ is isomorphic to $(H(E_b'), \tau_0)$. Then Satz 1.8 and Lemma 2.1 of Vogt and Wagner [25] imply that $H_E(\{0\})$ has the following property:

(*) For any $p\in \mathbf{N}$ there exists $q \in \mathbf{N}$ such that for any $k \in \mathbf{N}$ there exist $n \in \mathbf{N}$ and $C > 0$ such that for all $r > 0$ and any $f\in H_E(\{0\})$

$$\sup_{x\in V_q}|f(x)| \leqslant Cr^n \sup_{x\in V_k}|f(x)| + \frac{1}{r}\sup_{x\in V_p}|f(x)|,$$

where $(V_j)_{j\in\mathbf{N}}$ is a decreasing absolutely convex neighbourhood basis of zero in E and where the supremum is allowed to be infinite.

Now let $K \neq \emptyset$ be a compact subset of E and define $U_j = K+V_j$ for all $j \in \mathbf{N}$. Then we have $H(K) = \operatorname{ind}_{n\to} H^\infty(U_n)$ and it follows from (*) that $H(K)$ satisfies (*) if we replace in (*) V by U. Since E is nuclear, it follows from Bierstedt and Meise [2], Thm. 7, that $H(K)$ is a (DFN)-space. Hence (*) implies by Vogt and Wagner [25], 2.1, that $H(K)_b'$ has (Ω). Since $H(K)_b'$ is nuclear, this proves by Vogt and Wagner [25], 1.8, that $H(K)_b'$ is a quotient of s.

(b): Because of Bierstedt and Meise [2], Prop. 10, E_b' is a complemented subspace of $H(K)$. Hence $E = E_{bb}''$ is a complemented subspace of $H(K)_b'$ and consequently a quotient space of s.

4.4. COROLLARY. *Let $\Lambda_\infty(\alpha)$ be nuclear and let E be a quotient space of $\Lambda_\infty(\alpha)$.*

(a) *$H(K)_b'$ is a quotient space of $\Lambda_\infty(\beta(\alpha))$ for any compact set K in E.*

(b) *If $E = \Lambda(P)$, then $(H(\mathbf{D}_a), \tau_0)$ is a quotient space of $\Lambda_\infty(\beta(\alpha))$ for any open polydisc $\mathbf{D}_a$ in $\Lambda(P)_b'$.*

Proof. (a): Since $\Lambda_\infty(\alpha)$ is a quotient of s, we get from 4.3 (a) and Vogt and Wagner [25], 1.8, that $H(K)_b'$ has (Ω) for any compact set K in E. Since $\Lambda_\infty(\alpha)$ is $\Lambda_{\mathbf{N}}(\alpha)$-nuclear (see Ramanujan and Terzioglu [18], 2.12), we get from Börgens, Meise and Vogt [4], Thm. 4.4, that $H(K)_b'$ is $\Lambda_{\mathbf{N}}(\beta(\alpha))$-nuclear. Hence Vogt and Wagner [25], 1.8, implies that $H(K)_b'$ is a quotient space of $\Lambda_\infty(\beta(\alpha))$.

(b): If $\mathbf{D}_a$ is any open polydisc in $\Lambda(P)_b'$, then it follows from the duality theorem of Boland and Dineen [8], Thm. 20, that $(H(\mathbf{D}_a), \tau_0) = H(\mathbf{D}_a^M)_b'$, where $\mathbf{D}_a^M$ is the multiplicative polar of $\mathbf{D}_a$, which is a compact subset of E. Hence we get from (a) that $(H(\mathbf{D}_a), \tau_0)$ is a quotient of $\Lambda_\infty(\beta(\alpha))$.

4.5. COROLLARY. *If $\Lambda(P)$ is a quotient space of s, then for any $a \in \Lambda(P)$, $a \geqslant 0$, there is a closed ideal I in the* l.c. *algebra $(H(s_b'), \tau_0)$ such that $(H(s_b'), \tau_0)/I$ is isomorphic to $(H(\mathbf{D}_a), \tau_0)$ as an* l.c. *algebra.*

Proof. By 4.4 (b) there exists a continuous linear surjection π: $s \to (H(\mathbf{D}_a), \tau_0)$. Its transpose ${}^t\pi$: $(H(\mathbf{D}_a), \tau_0)_b' \to s_b'$ is continuous and linear. Hence the mapping φ: $\mathbf{D}_a \to s_b'$ defined by $\varphi(z) = {}^t\pi\circ\delta_z$, where δ_z denotes the evaluation at the point z, is a holomorphic mapping, i.e. continuous and weakly Gâteaux-analytic. Since the composition of holomorphic mappings is holomorphic again, φ induces Φ: $(H(s_b'), \tau_0) \to (H(\mathbf{D}_a), \tau_0)$ by the definition $\Phi(f) = f\circ\varphi$. It is easy to see that Φ is a continuous algebra homomorphism. In order to show that Φ is surjective, we choose $g \in (H(\mathbf{D}_a), \tau_0)$ arbitrarily. Since π is surjective, there exists $f\in s = (s_b')' \subset H(s_b')$ with $g = \pi(f)$, which is equivalent to $g = {}^{tt}\pi(f) = f\circ{}^t\pi$. Hence Φ is surjective and the result follows from the open mapping theorem for Fréchet spaces.

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Norm inequalities relating singular integrals and the maximal function

by

ERIC T. SAWYER* (Hamilton, Ont.)

Abstract. We prove that if the weighted L^p norms $(1 < p < \infty)$ of the Riesz transforms are bounded by the weighted L^p norm of the maximal function, then the weight function satisfies the C_p condition of B. Muckenhoupt. Conversely we show that if the weight function satisfies the C_q condition for some $q > p$, then the weighted L^p norm of any standard singular integral is bounded by the weighted L^p norm of the maximal function.

§1. Introduction. We consider the problem of characterizing the non-negative weights w for which $(1 < p < \infty)$

(1) $$\int |Tf|^p w \leqslant C \int |Mf|^p w \quad \text{for all appropriate } f$$

where $Tf = K*f$ is a singular integral in R^n with kernel K satisfying the standard conditions

(i) $$\|\hat{K}\|_\infty \leqslant C,$$

(ii) $$|K(x)| \leqslant C|x|^{-n},$$

(iii) $$|K(x)-K(x-y)| \leqslant C|y||x|^{-n-1} \quad \text{for} \quad |y| < |x|/2.$$

R. Coifman and C. Fefferman have shown ([1]; Theorem III) that (1) holds for $1 < p < \infty$ provided the weight w satisfies the A_∞ condition. B. Muckenhoupt has shown ([7]; Theorem 2.1) that in the case when T is the Hilbert transform, inequality (1) does not imply that w satisfies the A_∞ condition. He has derived ([7]; Theorem 1.2) the following necessary condition for (1) (with T the Hilbert transform) which he has conjectured to be sufficient.

(C_p) There are positive constants C, ε such that
$$\int_E w \leqslant C(|E|/|Q|)^\varepsilon \int |M_{\chi_Q}|^p w$$
whenever E is a subset of a cube $Q \subset R^n$.

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