STUDIA MATHEMATICA, T. LXXV. (1983)

On permanent radicals in commutative locally convex algebras

by

W. ŻELAZKO (Warszawa)

Abstract. In this paper we introduce a concept of permanent radical in topological algebras and characterize it in the classes of locally convex algebras and B_0-algebras.

§ 1. Introduction. All algebras considered in this paper are commutative complex algebras possessing unit elements. The unit element of an algebra A will be denoted by e_A, or by e in case when there is no need of explicit mention of the algebra in question. The *radical* $\operatorname{rad} A$ of an algebra A is the intersection of all its maximal ideals. Denoting by $G(A)$ the group of all invertible elements in A we have

(1) $$\operatorname{rad} A = \{x \in A:\ e - ax \in G(A) \text{ for all } a \text{ in } A\}.$$

A topological algebra is a topological linear space together with an associative jointly continuous multiplication making of it an algebra over $\boldsymbol{C}$. In terms of neighbourhoods of the origin the joint continuity of multiplication means that for each such a neighbourhood U there exists another neighbourhood V of zero satisfying

(2) $$V^2 \subset U.$$

If A is, moreover, a locally convex space, then its topology can be given by means of a family $(\|x\|_\alpha)$ of seminorms, and (2) reads in the following way: for each α there is an index β such that

(3) $$\|xy\|_\alpha \leqslant \|x\|_\beta \|y\|_\beta$$

for all $x, y \in A$. It can be also assumed that for α and β satisfying relation (3) we have also

$$\|x\|_\alpha \leqslant \|x\|_\beta$$

for all $x \in A$. In case when the seminorms $(\|x\|_\alpha)$ can be chosen so that

(4) $$\|xy\|_\alpha \leqslant \|x\|_\alpha \|y\|_\alpha$$

we say that the algebra in question is *multiplicatively convex* (shortly *m-convex*).

In this paper we shall consider only complete algebras, i.e. topological algebras being complete topological linear spaces. Relation (2) implies that the completion of a topological algebra is again such an algebra under the naturally defined multiplication, thus every topological algebra is a dense subalgebra of a complete algebra. We shall consider also B_0-algebras, i.e. completely metrizable locally convex algebras. Their topology can be given by means of an increasing sequence $\|x\|_1 \leqslant \|x\|_2 \leqslant \ldots$ of seminorms satisfying

$$\|xy\|_i \leqslant \|x\|_{i+1}\|y\|_{i+1}, \tag{5}$$

for $i = 1, 2, \ldots$ and all $x, y \in A$. In case when a B_0-algebra is m-convex relation (5) is replaced by

$$\|xy\|_i \leqslant \|x\|_i\|y\|_i. \tag{6}$$

Denoting by $\mathscr{T}$, $\mathscr{LC}$, $\mathscr{M}$, $\mathscr{B}_0$, $\mathscr{M}_0$, $\mathscr{B}$ the classes of all topological, locally convex, m-convex, of type B_0, m-convex of type B_0, and Banach algebras, respectively, we have obvious inclusions $\mathscr{T} \supset \mathscr{LC} \supset \mathscr{M} \supset \mathscr{M}_0 \supset \mathscr{B}$ and $\mathscr{LC} \supset \mathscr{B}_0 \supset \mathscr{M}_0$.

Let $\mathscr{K}$ be any class of topological algebras and $A \in \mathscr{K}$. We say that an algebra $B \in \mathscr{K}$ is a *$\mathscr{K}$-extension* (*superalgebra*) of A if there is a topological isomorphism φ of A into B with $\varphi(e_A) = e_B$. In this case we can treat A as a subalgebra of B and write simply $A \subset B$.

1.1. Definition. Let $\mathscr{K}$ be a class of topological algebras and let $A \in \mathscr{K}$. The *$\mathscr{K}$-permanent radical of A* is the set

$$\operatorname{rad}_{\mathscr{K}} A = \{x \in A\colon x \in \operatorname{rad} B \text{ for each } \mathscr{K}\text{-extension } B \text{ of } A\}. \tag{7}$$

Obviously $\operatorname{rad}_{\mathscr{K}} A$ is an ideal in A contained in its radical. Also for $\mathscr{K}_1 \supset \mathscr{K}_2$ we have

$$\operatorname{rad}_{\mathscr{K}_1} A \subset \operatorname{rad}_{\mathscr{K}_2} A$$

for each $A \in \mathscr{K}_2$.

1.2. Definition. We say that for a class $\mathscr{K}$ of topological algebras the concept of $\mathscr{K}$-permanent radical has an *absolute character* if

$$\operatorname{rad}_{\mathscr{K}} A = \operatorname{rad}_{\mathscr{T}} A \tag{8}$$

for each $A \in \mathscr{K}$. If this fails we say that the concept of permanent radical has a *relative character* in the considered class $\mathscr{K}$.

In this paper we characterize $\mathscr{LC}$-permanent radicals as ideals consisting of elements possessing small powers (cf. definition below). In §2 we describe these ideals in an arbitrary topological algebra, while in Section 3 we show that for classes $\mathscr{LC}$ and $\mathscr{B}_0$ these ideals coincide with permanent radicals relative to these classes. We observe also that for the classes $\mathscr{LC}$ and $\mathscr{B}_0$ the concept of permanent radical has an absolute character, while in the classes $\mathscr{M}$, $\mathscr{M}_0$, $\mathscr{B}$ it has a relative character.

For an additional information on topological linear spaces and topological algebras the reader is refered to [2] and [4].

§2. Elements possessing small powers.

2.1. Definition. Let A be a topological algebra. We say that an element $x \in A$ has *small powers* if for each neighbourhood U of the origin in A there is a positive integer n such that

$$\lambda x^n \in U$$

for all complex scalars λ.

The set of all elements in A possessing small powers will be designated by $I_s(A)$. The concept of elements possessing small powers is related to the concept of short lines ([2], p. 114), i.e. families of elements (x_a) of a topological linear space X such that for each neighbourhood U of the origin in X there is an index a such that the (complex) line Cx_a is contained in U.

2.2. Lemma. *Let A be as above and let $x \in A$. Then the element x has small powers if and only if for each neighbourhood U of the origin in A there exists an integer n such that*

$$Ax^n \subset U, \tag{9}$$

i.e. the powers x^n generate "small ideals".

Proof. Let U be a neighbourhood of the origin in A and choose a neighbourhood V satisfying relation (2). Choose a positive integer n so that $Cx^n \subset V$. Let $a \in A$ and choose a positive scalar λ so that $\lambda a \in V$. We have

$$ax^n = \lambda a \cdot x^n/\lambda \in V \cdot V \subset U$$

and relation (9) follows. The converse statement is obvious.

2.3. Corollary. *If $x \in I_s(A)$, then relation (9) holds true for any extension B of A taken instead of A. Of course, U should be then an arbitrary neighbourhood of the origin in B.*

2.4. Corollary. *If $x \in I_s(A)$, then for each neighbourhood U of the origin in A there exists a positive integer $n(U)$ such that*

$$Cx^n \subset U \tag{10}$$

for all $n \geqslant n(U)$. The converse statement is also true.

2.5. Proposition. *Let A be as above. We have $x \in I_s(A)$ if and only if for each sequence $(a_n)_0^\infty$ of complex scalars the power series $\sum_0^\infty a_j x^j$ is convergent in A.*

Proof. Let $x \in I_s(A)$ and let (a_n) be any sequence of complex numbers. Let U be an arbitrary neighbourhood of the origin in A. By relation (9) there is a natural number n such that for any positive integer k we have

$$\sum_{j=n}^{n+k} a_j x^j = ax^n \in U,$$

what means that the series in question is convergent in A.

Suppose now that $x \notin I_s(A)$. By Corollary 2.4 there is a neighbourhood U of zero in A such that relation (10) fails for infinitely many natural numbers n, say for $n \in (n_j)_1^\infty$ (one can easily show that for some U relation (10) fails for all natural n). Thus there exist complex numbers $\lambda_j, j = 1, 2, \ldots$, such that $\lambda_j x^{n_j} \notin U$ for all j. This means that the series $\sum_{j=1}^{\infty} \lambda_j x^{n_j}$ is divergent in A and the conclusion follows.

Remark. The above proposition holds true (with the same proof) if we replace the sequence (a_n) by an arbitrary sequence (a_n) of elements of A.

2.6. Proposition. *For each topological algebra A the set $I_s(A)$ is an ideal in A and*

$$I_s(A) \subset \mathrm{rad}_{\mathscr{T}} A. \tag{11}$$

Proof. In order to show that $I_s(A)$ is an ideal in A it is sufficient, in view of formula (9), to show that $x, y \in I_s(A)$ implies $x+y \in I_s(A)$. Let U be an arbitrary neighbourhood of zero in A and find another such a neighbourhood V satisfying

$$V+V \subset U.$$

Using Lemma 2.2 we find a positive integer k such that $Ax^k \subset V$ and $Ay^k \subset V$. Setting $n = 2k$ we obtain

$$\lambda(x+y)^n = ax^k + by^k \in V+V \subset U,$$

where λ is an arbitrary complex number and a, b are suitable elements in A. Thus $x+y \in I_s(A)$.

Let $x \in I_s(A)$. By Proposition 2.5 the series $\sum_0^\infty x^j$ converges in A, its sum being $(e-x)^{-1}$. Thus relation (1) together with the fact that $I_s(A)$ is an ideal implies that $I_s(A) \subset \mathrm{rad}\, A$. If B is any extension of A, then obviously $I_s(A) \subset I_s(B)$. Consequently $I_s(A) \subset \mathrm{rad}\, B$ for any extension B of A and so relation (11) holds true. Conclusion follows.

Remark. It can happen that $I_s(A)$ is a non-closed ideal in A. For instance, if A is a Banach algebra, then $I_s(A)$ is the collection of all its nilpotent elements, which is a non-closed subset of the radical $\mathrm{rad}\, A$ in the case when A possesses nilpotents of arbitrarily high orders.

§ 3. A characterization of permanent radicals in locally convex and B_0-algebras. In this section we prove that for $\mathscr{K} = \mathscr{LC}$ or $\mathscr{K} = \mathscr{B}_0$ we have $\mathrm{rad}_{\mathscr{K}} A = I_s(A)$ for each $A \in \mathscr{K}$. We observe also that this is not true for $\mathscr{K} = \mathscr{M}, \mathscr{M}_0$, or $\mathscr{B}$.

We need the following simple lemma ([4], Lemma 3).

3.1. Lemma. *Let $(a_n)_0^\infty$ be a sequence of positive real numbers with $a_0 = 1$. Then there exists a sequence $(b_n)_0^\infty$ of positive real numbers, $b_0 = 1$, such that*

$$a_{i+j} \leqslant b_i b_j, \tag{12}$$

for all non-negative integers i, j. In particular

$$a_i \leqslant b_i, \quad i = 0, 1, 2, \ldots \tag{13}$$

Proof. Put $b_0 = 1$ and suppose that the numbers $b_1, b_2, \ldots, b_{n-1}$ are already constructed. Then put

$$b_n = \max\{a_n, a_{n+1}/b_1, a_{n+2}/b_2, \ldots, a_{2n-1}/b_{n-1}, a_{2n}^{1/2}\}.$$

3.2. Lemma. *Let A be a locally convex algebra and let $(\|x\|_a)$ be a system of seminorms defining its topology, so that for each a there is a β such that relation (3) holds true. Let $x_0 \in A$ and $x_0 \notin I_s(A)$. Then there exists an index a such that*

$$\|x_0^n\|_a \neq 0 \tag{14}$$

for $n = 0, 1, 2, \ldots$, where $x_0^0 = e_A$.

Proof. If for each index a there is a natural number $n(a)$ with $\|x_0^{n(a)}\|_a = 0$ then

$$Cx_0^{n(a)} \subset U_{\varepsilon,a} = \{x \in A : \|x\|_a < \varepsilon\}.$$

Since the neighbourhoods of the form $U_{\varepsilon,a}$ form a basis of neighbourhoods of the origin in A, we obtain $x_0 \in I_s(A)$ and this contradiction proves formula (14) for some a. Conclusion follows.

Our main result is based upon the following lemma. In its proof we use part of the construction given in the proof of the main result in [4].

3.3. Lemma. *Let $A \in \mathscr{K}$, where $\mathscr{K} = \mathscr{LC}$ or $\mathscr{B}_0$. Then*

$$\mathrm{rad}_{\mathscr{K}} A \subset I_s(A). \tag{15}$$

Proof. We have to show that if $x_0 \notin I_s(A)$ then $x_0 \notin \mathrm{rad}_{\mathscr{K}} A$. Assume then that $x_0 \notin I_s(A)$ and denote by $\|x\|_1$ a continuous seminorm on A satisfying relation (14) of Lemma 3.2. In the case where $\mathscr{K} = \mathscr{B}_0$ we can assume that it is the first seminorm in a sequence $(\|x\|_i)$ defining the topology of A and satisfying relation (5). In the case where $\mathscr{K} = \mathscr{LC}$ we simply assume that $\|x\|_1$ is a seminorm belonging to a system $(\|x\|_\alpha)$ defining the topology of A and satisfying relations (3).

In order to show $x_0 \notin \mathrm{rad}_{\mathscr{K}} A$ we shall construct an extension $B \supset A$, $B \in \mathscr{K}$, such that $x_0 \notin \mathrm{rad}\, B$. First we construct a certain matrix (a_j^n) of positive real numbers, $n = 0, 1, 2, \ldots, j = 1, 2, \ldots$ We put $a_1^0 = 1$ and $a_1^n = \|x_0^n\|_1^{-1}$ for $n = 1, 2, \ldots$ Setting now $a_i = a_1^i$, we obtain, by Lemma 3.1, a sequence (b_i) and define $a_2^n = b_n$. Setting again in Lemma 3.1 $a_i = a_2^i$ we obtain $a_3^i = b_i$ and so on. Obtained in this way matrix (a_j^n) satisfies, by formulas (12) and (13), the following relations

$$a_n^0 = 1, \quad n = 1, 2, \ldots, \tag{16}$$

$$a_n^i \leqslant a_{n+1}^i, \quad n = 1, 2, \ldots, \quad i = 0, 1, 2, \ldots, \tag{17}$$

and

$$a_n^{i+j} \leqslant a_{n+1}^i a_{n+1}^j, \quad n = 1, 2, \ldots, \quad i, j = 0, 1, 2, \ldots \tag{18}$$

Let B be the algebra of power series $\sum_i x_i t^i$ with coefficients in A and with the variable t, such that the defined below seminorms are finite. Assume first $A \in \mathscr{B}_0$ and put

$$\Big\|\sum_j x_j t^j\Big\|_k = \sum_j a_k^j \|x_j\|_k, \quad k = 1, 2, \ldots \tag{19}$$

Relation (17) shows that this is an increasing sequence of seminorms since the sequence $\|x\|_k$ increases for any $x \in A$. The space B of all power series for which the seminorms (19) are finite is a B_0-space. This is also a B_0-algebra under Cauchy multiplication of power series. The joint continuity of this multiplication follows from the estimation below. We take here into account relations (5) and (18).

$$\begin{aligned}\Big\|\sum_n x_n t^n \sum_l y_l t^l\Big\|_k &= \Big\|\sum_n \Big(\sum_l x_{n-l} y_l\Big) t^n\Big\|_k \\ &= \sum_n a_k^n \Big\|\sum_l x_{n-l} y_l\Big\|_k \leqslant \sum_{n,l} a_{k+1}^{n-l} a_{k+1}^l \|x_{n-l}\|_{k+1} \|y_l\|_{k+1} \\ &= \Big\|\sum_n x_n t^n\Big\|_{k+1} \Big\|\sum_n y_n t^n\Big\|_{k+1}.\end{aligned}$$

The algebra B is an extension of A under identification of elements of A with constant power series. This follows from relation (16) which implies that the seminorms (19) restricted to A became original seminorms defining the topology of A.

The above construction works as well when A is an m-convex B_0-algebra, or a Banach algebra. In the first case by the inequality

$$\|xy\|_i \leqslant \|x\|_i \|y\|_i \leqslant \|x\|_{i+1} \|y\|_{i+1}$$

we obtain the previous situation. In the second case all seminorms $\|x\|_i$ coincide with the Banach algebra norm $\|x\|$ of A, but then the seminorms (19) define, generally speaking, a B_0-algebra which in certain cases becomes m-convex, or even a Banach algebra, still being an extension of A.

Consider now the case of an arbitrary locally convex algebra. Starting from the seminorm $\|x\|_1$ and taking in (3) 1 as α we find a suitable β, which we call 2. Then for $\alpha = 2$ we find a β called 3 etc. In this way we obtain an increasing sequence of seminorms (which in certain instances can reduce to a single seminorm) satisfying relations (5). To this system there corresponds an at most countable system of seminorms of the form (19). The same process we can perform starting from any other continuous seminorm $\|x\|_\alpha$, satisfying relation (14), obtaining in this way corresponding seminorms of the form (19). The space B of power series for which all these seminorms are finite is a complete locally convex space and a locally convex algebra under convolution multiplication, what can be seen in exactly the same way as in the case of a B_0-algebra. Moreover, B is a superalgebra for A since the seminorms (19) restricted to A give there the original topology. It remains to be shown that $x_0 \notin \mathrm{rad}\, B$. Suppose then that $x_0 \in \mathrm{rad}\, B$ and try to get a contradiction. By relation (1) the element $e - x_0 t$ is invertible in B, and so there is a sequence $(z_i) \subset A$ such that $\sum_i z_i t^i \in B$ and

$$(e - x_0 t) \sum_{i=0}^{\infty} z_i t^i = e.$$

The above implies

$$(z_0 - e) + \sum_{j=1}^{\infty} (z_{j-1} x_0 - z_j) t^j = 0,$$

from which by an easy induction we obtain $z_k = x_0^k$, $k = 0, 1, \ldots$ But the series $\sum_k x_0^k t^k$ does not belong to B, since by (19) we have

$$\Big\|\sum_{k=0}^{\infty} x_0^k t^k\Big\|_1 = \sum_{k=0}^{\infty} a_1^k \|x_0^k\|_1 = \sum_{k=0}^{\infty} \|x_0^k\|_1^{-1} \|x_0^k\|_1$$

and the right hand series diverges. The contradiction proves formula (15).

The relation $\mathrm{rad}_{\mathscr{F}} A \subset \mathrm{rad}_{\mathscr{K}} A$ for any $A \in \mathscr{K}$ together with formulas (11), (15) and Proposition 2.5 imply our main result.

3.4. THEOREM. *Let A be a locally convex algebra (resp. a B_0-algebra). Then the permanent radical $\mathrm{rad}_{\mathscr{LC}} A$ (resp. $\mathrm{rad}_{\mathscr{B}_0} A$) is the ideal $I_s(A)$ of all* *elements in A possessing small powers. Or equivalently, for the classes of locally convex or B_0-algebras, the permanent radical of an algebra is the set of all its elements on which operate all formal power series.*

Remark. The above characterization shows that the concept of permanent radical in the classes $\mathscr{LC}$ or $\mathscr{B}_0$ has an absolute character (cf. Definition 1.2). This is not true for such classes as $\mathscr{M}$, $\mathscr{M}_0$, or $\mathscr{B}$. In fact, in these classes we have $\operatorname{rad}_{\mathscr{M}} A = \operatorname{rad} A$, what follows from the fact that for $A \in \mathscr{M}$ its radical is given by

$$\operatorname{rad} A = \bigcap_{\alpha} \{x \in A: \lim_n \|x^n\|_\alpha^{1/n} = 0\},$$

where the intersection is taken with respect to all continuous seminorms on A satisfying relation (4).

3.5. Corollary. *If A is a Banach algebra then its $\mathscr{LC}$-permanent radical, or $\mathscr{B}_0$-permanent radical coincides with the set of all its nilpotent elements, and equals to $\operatorname{rad}_{\mathscr{T}} A$.*

Let us remark (cf. remarks at the end of Section 2), that if $A \in \mathscr{B}$ and $\operatorname{rad} A$ contains elements of arbitrarily high orders, then the set $\operatorname{rad}_{\mathscr{B}_0} A$ is a non-closed ideal in A. This was, in fact, known to Rolewicz, who used it in [1] to the construction of a B_0-algebra possessing a non-closed radical.

We do not know whether Theorem 3.4 is true for the class of all topological algebras.

Problem. Let A be a topological algebra. Does the ideal $I_s(A)$ of all elements of A possessing small powers coincide with the $\mathscr{T}$-permanent radical $\operatorname{rad}_{\mathscr{T}} A$ of A?

References

[1] S. Rolewicz, *Some remarks on radicals in commutative B_0-algebras*, Bull. Acad. Polon. Sci. 15 (1967), 153–155.
[2] — *Topological Linear Spaces*, Monografie Matematyczne, vol 56, Warszawa 1972.
[3] W. Żelazko, *Selected topics in topological algebras*, Aarhus University Lecture Notes No 31 (1971).
[4] — *A characterization of $\mathscr{LC}$-non-removable ideals in commutative Banach algebras*, Pacific J. Math. 87 (1980), 241–247.

INSTYTUT MATEMATYCZNY PAN
MATHEMATICAL INSTITUTE
POLISH ACADEMY OF SCIENCES

Received October 12, 1981 (1714)

STUDIA MATHEMATICA, T. LXXV. (1983)

Holomorphic functional calculus and quotient Banach algebras

by

L. WAELBROECK (Bruxelles)

Abstract. Let $\mathscr{A}$ be a commutative associative Banach algebra with unit, and α an ideal of $\mathscr{A}$ with a Banach norm stronger than the norm induced by that of $\mathscr{A}$. Let $\bar{a}_1, \ldots, \bar{a}_n$ be elements of $\mathscr{A}/\alpha$. We define $\operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$. We construct a homomorphism $\mathcal{O}(\operatorname{sp}\bar{a}, \mathscr{A}/\alpha) \to \mathscr{A}/\alpha$, mapping z_i onto $\bar{a}_i$, and unit on unit. This relative holomorphic functional calculus ($\mathscr{A} \operatorname{mod} \alpha$) generalizes classical holomorphic functional calculus (where $\alpha = 0$).

Let $\mathscr{A}$ be a Banach algebra, which is commutative, associative, and with unit. Let α be a Banach ideal. Let $\bar{a}_1, \ldots, \bar{a}_n$ be elements of $\mathscr{A}/\alpha$, or, if you prefer $a_1, \ldots, a_n$ elements of $\mathscr{A}$, where of course $a_i \in \bar{a}_i$. The spectrum $\operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$, i.e. $\operatorname{sp}_\alpha(a_1, \ldots, a_n)$ is the set of $(s_1, \ldots, s_n) \in \mathbf{C}^n$ such that

$$\sum_1^n (\bar{a}_i - s_i)\mathscr{A}/\alpha \neq \mathscr{A}/\alpha,$$

i.e.

$$\sum_1^n (a_i - s_i)\mathscr{A} + \alpha \neq \mathscr{A}.$$

Let now $U \subseteq \mathbf{C}^n$ be open in $\mathbf{C}^n$, $U \supseteq \operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$, let

$$\mathcal{O}(U, \mathscr{A}/\alpha) = \mathcal{O}(U, \mathscr{A})/\mathcal{O}(U, \alpha).$$

Call 1 the constant function on U, equal to 1, and z_i the holomorphic mapping $z_i: (s_1, \ldots, s_n) \to s_i$. We shall construct a homomorphism

$$\mathcal{O}(U, \mathscr{A}/\alpha) \to \mathscr{A}/\alpha$$

which maps z_i on $\bar{a}_i$, 1 onto 1. This homomorphism is induced by a continuous linear mapping $\mathcal{O}(U, \mathscr{A}) \to \mathscr{A}$ which maps $\mathcal{O}(U, \alpha)$ into α. If U is a schlicht domain of holomorphy, the homomorphism above is unique.

This is the first of two papers. In the second paper, we shall prove that every ideal of a quasi-Banach algebra has at least one quasi-Banach