Therefore

$$(6.1)\qquad \big(1+|G^{-1}(x)-E|\big)^{N/2} = \max\big(\beta_1^{-N/2}, (2-1/\beta_N)^{N/2}\big) \geqslant \beta_1^{-N/2} \geqslant \sqrt{\frac{\beta_N(x)}{\beta_1(x)}}.$$

On the other hand

$$\beta_1^{-N/2} \leqslant \sqrt{\left(\frac{\beta_N}{\beta_1}\right)^{N-1}}$$

and

$$(2-1/\beta_N)^{N/2} \leqslant \beta_N^{N/2} \leqslant \sqrt{\left(\frac{\beta_N}{\beta_1}\right)^{N-1}}.$$

The last two inequalities give

$$(6.2)\qquad \big(1+|G_1^{-1}(x)-E|\big)^{N/2} \leqslant \sqrt{\left(\frac{\beta_N(x)}{\beta_1(x)}\right)^{N-1}}.$$

Since inequalities (6.1) and (6.2) hold for almost all $x \in \Omega$, we conclude by the definitions of $K_{G^{-1}}$ and K_f that

$$K_{G^{-1}} \geqslant \sqrt{K_f} \quad \text{and} \quad K_{G^{-1}} \leqslant \sqrt{K_f^{N-1}},$$

which is nothing but (0.22).

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On convergence in the Mikusiński operational calculus

by

JÓZEF BURZYK (Katowice)

Abstract. A new description of the convergence of type I′ in the field of Mikusiński operators is given in terms of some family of functionals on the space L of locally integrable functions on $[0, \infty)$. As a consequence, sequential completeness of $\mathscr{F}$ and characterizations of boundedness and precompactness in L, $\mathscr{F}$ and in some subalgebra $\mathscr{F}_0$ of $\mathscr{F}$ are obtained. In particular, it is shown that a set A in $\mathscr{F}$ is precompact if and only if A is bounded (with respect to type I′ convergence).

1. Introduction. The field $\mathscr{F}$ of Mikusiński operators, considered in [7], has various applications, and is interesting also from theoretical point of view. A convergence used in the Mikusiński operational calculus, called type I convergence, is not topological (see [2], [9]). In spite of this, it is sensible to consider completeness with respect to type I convergence. In fact, we can define Cauchy sequences in every abelian group endowed with a convergence. We shall give two definitions (see [8], [5]).

Let X be an abelian group with a convergence G. A sequence $\{x_n\}$ $(x_n \in X)$ is called

(i) *P-Cauchy* if $x_{p_{n+1}} - x_{p_n} \to 0$ in G as $n\to\infty$ for every increasing sequence $\{p_n\}$ of positive integers;

(ii) *Q-Cauchy* if $x_{p_n} - x_{q_n} \to 0$ in G as $n\to\infty$ for every pair of increasing sequences $\{p_n\}$ and $\{q_n\}$ of positive integers.

An abelian group X with a convergence G is called *P-complete* (or *Q-complete*) if every P-Cauchy (Q-Cauchy) sequence is convergent in G.

Of course, each P-complete group is also Q-complete but not conversely. The converse implication holds if the convergence G satisfies the Urysohn condition and, additionally, some other natural conditions (see [8]).

Professor J. Mikusiński has posed the problem of P-completeness and Q-completeness of the field $\mathscr{F}$ equipped with type I convergence.

In this paper (Section 9) we shall show that $\mathscr{F}$ with type I convergence (which does not satisfy the Urysohn condition) is Q-complete. The problem of P-completeness of $\mathscr{F}$ (wiht type I convergence) remains open.

On the other hand, we shall obtain (Section 9) P-completeness of $\mathscr{F}$ with respect to another convergence, introduced by T. K. Boehme and called by him type I′ *convergence*.

The definitions of type I and type I′ convergences will be given in Section 2.

The above results concerning the completeness of $\mathscr{F}$ are consequences of a new description of type I′ convergence, given in Section 6. In Section 2 we introduce a family $\{B_{T,\varepsilon}\}$ $(T, \varepsilon > 0)$ of nonnegative functionals on the space L of locally integrable functions on $[0, \infty)$, used in a characterization of type I′ convergence.

Our description of that convergence allows one to characterize boundedness and precompactness in the spaces $L, \mathscr{F}$ (Sections 7,8) and some subalgebra $\mathscr{F}_0$ of $\mathscr{F}$ (Section 10) introduced by T. K. Boehme in [3].

In particular, the following result seems to be especially interesting (Theorem 5 in Section 8): a set $A \subset \mathscr{F}$ is precompact iff A is bounded. Here boundedness is meant with respect to type I′ convergence and precompactness with respect to any of types I and I′, because both types of precompactnes are equivalent.

More exactly, a set $A \subset \mathscr{F}$ is said to be ***bounded*** if for each sequence $\{x_n\}$ of elements of A and each sequence $\{\lambda_n\}$ of numbers such that $\lambda_n \to 0$, we have $\lambda_n x_n \to 0$ type I′; a set $A \subset \mathscr{F}$ is said to be ***precompact*** if each sequence $\{x_n\}$ of elements of A has a subsequence which is convergent type I or type I′, not necessary to an element of A.

Note that the above definition of precompactness is equivalent to the definition with using a topology introduced for type I′ convergence by T. K. Boehme (see [3]).

In Section 10, we shall extend the functionals $B_{T,\varepsilon}$ onto the subalgebra $\mathscr{F}_0$ and characterize, in terms of those functionals, type I′ convergence in $\mathscr{F}_0$. As a simple consequence, we shall obtain some results of T. K. Boehme concerning type I′ convergence in $\mathscr{F}_0$. Finally, let us mention about the methods used in the proofs.

In the proof of the main theorems (Theorems 1 and 2 in Section 6), which are characterizations of type I′ convergence in L and $\mathscr{F}$, we shall apply as a fundamental tool Lemma 6 proved in Section 4. In this lemma as well as in several other situations, we shall need various auxiliary facts; they are collected in Section 3 in the form of lemmas. As a matter of fact, Lemmas 2–5 are slight modifications of lemmas proved by T. K. Boehme in [1]. They concern an approximation property connected with Foiaş' theorem [6] (see Section 2) and a construction of a common multiple for a sequence of functions of L. We present these lemmas in a slightly stronger form than in [1] and therefore we give complete proofs of them.

Moreover, some facts about a common denominator of operators will be needed. We prove them in Section 5.

There and also in other places, the Titchmarsh theorem (see Section 2) will be applied.

2. Notation and definitions. It will be convenient to consider the field $\mathscr{F}$ of the Mikusiński operators as the quotient field formed from the ring L of locally integrable functions on $[0, \infty)$ with the usual addition and the followig multiplication:

$$(fg)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau \quad \text{where} \quad f, g \in L.$$

As in [7], the symbol l will be used for the function which is equal identically 1 for $t \geqslant 0$ and the symbol h^λ with $\lambda > 0$ for the translation operator:

$$h^\lambda\{f(t)\} = \begin{cases} 0 & \text{for} \quad 0 \leqslant t \leqslant \lambda, \\ f(t-\lambda) & \text{for} \quad t > \lambda. \end{cases}$$

We endow the space L with the seminorms:

$$\|f\|_T = \int_0^T f(t)\,dt \qquad (T > 0).$$

We have

$$\|fg\|_T \leqslant \|f\|_T \cdot \|g\|_T \tag{1}$$

for all $f, g \in L$ and $T > 0$.

The subspace of L consisting of all functions f such that $\|f\|_T > 0$ for any $T > 0$ will be denoted by L_0. Using Boehme's concept of the support number of function f, i.e. a maximal number $\Lambda(f)$ such that f vanishes a.e. (almost everywhere) in $[0, \Lambda(f)]$ (see [2]), we can write

$$L_0 = \{f \in L:\ \Lambda(f) = 0\}.$$

The convergence in all the seminorms $\|\cdot\|_T$, $T > 0$, will be called the *convergence in* L.

We can say that the sequence $x_n \in \mathscr{F}$ *converges to* $x \in \mathscr{F}$ *type* I if there exist representations $x_n = f_n/g$, $x = f/g$ $(f_n, f, g \in L,\ g \neq 0)$ such that $f_n \to f$ as $n \to \infty$ in L. Now, we say that $x_n \in \mathscr{F}$ *converges to* $x \in \mathscr{F}$ *type* I′ if each subsequence of $\{x_n\}$ possesses a subsequence which converges to x type I.

Since every function from L can be treated as an operator, we shall also consider operational convergences in L: type I and type I′.

The functionals $B_{T,\varepsilon}$ on L, mentioned in Introduction are defined in the following way:

$$B_{T,\varepsilon}(f) = \inf\{\|fg\|_T:\ g \in L_0,\ \|g\|_T < 1,\ \|l - lg\|_T < \varepsilon\} \tag{2}$$

for each $f \in L$ and $T, \varepsilon > 0$.

It is evident that

(i) $$B_{T,\varepsilon}(0) = 0$$

and

(ii) $$B_{T,\varepsilon}(\lambda f) = |\lambda| B_{T,\varepsilon}(f) \quad (\lambda \in R, f \in L)$$

for any $T, \varepsilon > 0$. Note that $B_{T,\varepsilon}(\cdot)$ is not a seminorm for fixed $T, \varepsilon > 0$, but we have the following inequality:

(iii) $$B_{T,\varepsilon_1+\varepsilon_2}(f_1+f_2) \leqslant B_{T,\varepsilon_1}(f_1) + B_{T,\varepsilon_2}(f_2).$$

In fact, for arbitrary $\eta > 0$ there exist functions $g_1, g_2 \in L_0$ such that

(3) $$\|g_1\|_T < 1, \quad \|g_2\|_T < 1,$$

(4) $$\|l - lg_1\|_T < \varepsilon_1, \quad \|l - lg_2\|_T < \varepsilon_2$$

and

(5) $$\|f_1 g_1\|_T < B_{T,\varepsilon_1}(f_1) + \eta/2, \quad \|f_2 g_2\|_T < B_{T,\varepsilon_2}(f_2) + \eta/2.$$

We have

(6) $$\|g_1 g_2\|_T \leqslant \|g_1\|_T \cdot \|g_2\|_T < 1,$$

by (1) and (3), and

(7) $$\|l - lg_1 g_2\|_T \leqslant \|l - lg_1\|_T + \|lg_1 - lg_1 g_2\|_T < \varepsilon_1 + \|g_1\|_T \|l - lg_2\|_T < \varepsilon_1 + \varepsilon_2$$

by (1), (3) and (4). Moreover,

(8) $$\begin{aligned}\|(f_1+f_2) g_1 g_2\|_T &\leqslant \|f_1 g_1 g_2\|_T + \|f_2 g_1 g_2\|_T \\ &< \|f_1 g_1\|_T \cdot \|g_2\|_T + \|f_2 g_2\|_T \|g_1\|_T \leqslant B_{T,\varepsilon_1}(f_1) + B_{T,\varepsilon_2}(f_2) + \eta\end{aligned}$$

by (1), (3) and (5). Relations (6), (7) and (8) imply

$$B_{T,\varepsilon_1+\varepsilon_2}(f_1+f_2) \leqslant B_{T,\varepsilon_1}(f_1) + B_{T,\varepsilon_2}(f_2) + \eta,$$

which proves (iii) since $\eta > 0$ was chosen arbitrarily.

It can be proved that

$$\lim_{\varepsilon \to 0} B_{T,\varepsilon}(f) = \|f\|_T$$

for any $T > 0$ and $f \in L$.

Note that the family of functionals $B_{T,\varepsilon}(T, \varepsilon > 0)$ induces a convergence different from that in L. For instance, if $f_n(t) = e^{nt}$, then it can be shown that $\|f_n\|_T \to \infty$ for every $T > 0$, but $B_{T,\varepsilon}(f_n) \to 0$ for every $T, \varepsilon > 0$.

We shall further apply the following theorem of C. Foiaş ([6]):

For any fixed $f \in L$, $g \in L_0$, $T > 0$ and $\varepsilon > 0$ there exists a function $k \in L$ such that $\|f - kg\|_T < \varepsilon$;

and the Titchmarsh theorem:

If $fg = 0$ a.e. in $(0, T)$, where $f, g \in L$, then there are $T_1, T_2 \geqslant 0$ such that $T_1 + T_2 \geqslant T$ and $f = 0$ a.e. in $(0, T_1)$ and $g = 0$ a.e. in $(0, T_2)$.

For a proof of the Titchmarsh theorem see e.g. [7].

Applying the concept of the support number, we can write the Titchmarsh theorem in the form:

$$\Lambda(fg) \leqslant \Lambda(f) + \Lambda(g) \quad (f, g \in L)$$

or, since the converse inequality is obvious, in the form

$$\Lambda(fg) = \Lambda(f) + \Lambda(g) \quad (f, g \in L).$$

As corollaries, we obtain the following two implications:

(9) If $f \in L_0$ and $g \in L_0$, then $fg \in L_0$.

(10) If $f \in L_0$, $g \in L$ and $fg = 0$ a.e. in $(0, T)$, then $g = 0$ a.e. in $(0, T)$.

The Titchmarsh theorem ensures the consistency of the following definition of the support number of an operator:

$$\Lambda(f/g) = \Lambda(f) - \Lambda(g).$$

This concept will be used in Section 4.

3. Some auxiliary facts. We start with the following lemma:

LEMMA 1. *Let $g \in L$. The transformation $f \to fg$ from L into L takes bounded sets into precompact sets.*

Proof. If a set A is bounded in L, then for every $T > 0$ there exists a constant number M_T such that $\|f\|_T < M_T$ for all $f \in A$ (of course, we can assume that $M_S \leqslant M_T$ for $S < T$). By (1), the set $\{fg : f \in A\}$ is bounded in L. Moreover, by simple calculations, we obtain

$$\int_0^T |(fg)(t+h) - (fg)(t)|\, dt \leqslant M_{T+|h|} \Big[\int_0^T |g(u+h) - g(u)|\, du + \int_0^{|h|} |g(u)|\, du \Big].$$

Hence, by the Lebesgue theorem, it follows that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_0^T |(fg)(t+h) - (fg)(t)|\, dt < \varepsilon$$

for all $f \in A$, $T > 0$ and $|h| < \delta$. This proves the lemma, in view of M. Riesz's theorem on precompactness in L.

Now we shall deduce, from Foiaş's theorem ([6]), a lemma of Boehme ([1]) in a little modified form.

LEMMA 2 (cf. [1]). *Given $g \in L_0$ and $T, \varepsilon > 0$, there is a function $k \in L_0$ such that $\|kg\|_T < 1$ and $\|l - lkg\|_T < \varepsilon$.*

Proof. Given ε and T, let a be a positive number such that

$$1 - \varepsilon/4T < a < 1.$$

There exists a function $f \in L_0$ such that $\|f\|_T = a$ and $\|l - lf\|_T < \varepsilon/2$. In fact, it suffices to choose $f \in L_0$ such that $f \geqslant 0$, $\int_0^{t_0} f = a$ and $f(t) = 0$ for $t > t_0$, where $0 < t_0 < \min(\varepsilon/4, T)$. Then

$$\|l - lf\|_T \leqslant \int_0^{t_0} \Big[1 - \int_0^t f(\tau)\,d\tau\Big]\,dt + (1-a)T < \varepsilon/2$$

and, of course, $\|f\|_T = a$.

By Foiaş's theorem, there is a function $k \in L$ such that $\|f - kg\|_T < 1 - a$. We have

$$\|kg\|_T \leqslant \|f\|_T + \|f - kg\|_T < 1$$

and

$$\|l - lkg\|_T \leqslant \|l - lf\|_T + \|l\|_T \cdot \|f - kg\|_T < (3/4)\varepsilon.$$

If $k \notin L_0$, i.e., $k = 0$ a.e. in some neighbourhood of 0, then we can put $\tilde{k} = k + \eta$, where $\eta \in L_0$ with

$$\|\eta\|_T < \|g\|_T^{-1} \min(\varepsilon/4T, 1 - \|kg\|_T)$$

and the statement of the lemma is satisfied for $k = \tilde{k} \in L_0$.

The next Lemma concerns type I convergence in L; we formulate and prove it in two alternative cases: for the convergence to an arbitrary function in L and to 0.

LEMMA 3. *If the sequence $\{f_n\}$, $f_n \in L$ $(n = 1, 2, \ldots)$ is type I convergent (to 0), then given $T, \varepsilon > 0$ there is a function g such that $g \in L_0$, $\|g\|_T < 1$, $\|l - lg\|_T < \varepsilon$ and the sequence $\{gf_n\}$ converges (to 0) in L.*

Proof. By the definition of type I convergence, there exists a function $\bar{g}$ such that $\bar{g} \in L$, $\bar{g} \neq 0$ and $\bar{g}f_n$ converges (to 0) in L. Let $\lambda = \sup\{T\colon \|\bar{g}\|_T = 0\}$ and $\tilde{g} = h^{-\lambda}\bar{g}$, where the symbol $h^{-\lambda}$ denotes the translation operator (see Introduction). Obviously $\tilde{g} \in L_0$ and the sequence $\tilde{g}f_n = h^{-\lambda}\bar{g}f_n$ converges (to 0) in L.

By Lemma 2, we can find k such that $k \in L_0$, $\|k\tilde{g}\|_T < 1$ and $\|l - lk\tilde{g}\|_T < \varepsilon$. Owing to Titchmarsh's theorem (9), we have $g = k\tilde{g} \in L_0$ and the sequence $gf_n = k\tilde{g}f_n$ converges (to 0) in L, as desired.

Now we prove, also in two alternative versions, a lemma which is a modification of Theorem 2 from [1].

LEMMA 4 (cf. [1]). *Let $T > 0$. If $g_n \in L_0$, $\|g_n\|_T < 1$, $\|l - lg_n\|_T < \varepsilon_n$ (or $\|g_n\|_n < 1$, $\|l - lg_n\|_n < \varepsilon_n$) for $n = 1, 2, \ldots$, where $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, then the sequence $\{lg_1g_2 \ldots g_n\}$ is convergent in the seminorm $\|\cdot\|_T$ (or in L) to a function $g \in L_0$ such that*

(11) $$g = \varphi_n g_n \quad \text{on} \quad [0, T] \quad (\text{or on } [0, \infty))$$

with $\varphi_n \in L_0$ for $n = 1, 2, \ldots$

Moreover, the inequality $\|fg\|_T \leqslant \|lfg_n\|_T$ holds for every $f \in L$ and $n = 1, 2, \ldots$

Proof. First assume that $g_n \in L_0$, $\|g_n\|_T < 1$, $\|l - lg_n\|_T < \varepsilon_n$ $(n = 1, 2, \ldots)$ with $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ for a fixed $T > 0$. For $m > n$ we have

$$\begin{aligned}\|lg_1g_2 \ldots g_m - lg_1g_2 \ldots g_n\|_T &\leqslant \|g_1g_2 \ldots g_n\|_T \cdot \|lg_{n+1} \ldots g_m - l\|_T \\ &\leqslant \sum_{i=n}^{m-1} \|[lg_{n+1} \ldots g_{i+1} - lg_{n+1} \ldots g_i]\|_T \\ &\leqslant \sum_{i=1}^{m-1} \|g_{n+1} \ldots g_i\|_T \cdot \|lg_{i+1} - l\|_T \\ &\leqslant \sum_{i=n+1}^{m} \varepsilon_i\end{aligned}$$

and this means, by the assumption, that $\{lg_1g_2 \ldots g_n\}$, $n = 1, 2, \ldots$, is a Cauchy sequence in the norm $\|\cdot\|_T$.

Let $lg_1g_2 \ldots g_n \to g$ as $n \to \infty$ (in $\|\cdot\|_T$). We shall show that $g \in L_0$. Given any positive $S < T$, we can find an index n_0 such that $\sum_{n=n_0}^{\infty} \varepsilon_n < S/2$. The sequence $g_{n_0}, g_{n_0+1}, g_{n_0+2}, \ldots$ satisfies the assumptions of the lemma. Therefore

$$lg_{n_0}g_{n_0+1} \ldots g_n \to h \quad \text{as} \quad n \to \infty$$

in the seminorm $\|\cdot\|_T$ and, consequently, in $\|\cdot\|_S$. Moreover,

$$\|h\|_S \geqslant \|l\|_S - \|l - h\|_S = S - \lim_{n\to\infty} \|l - lg_{n_0} \ldots g_n\|_S > S/2$$

in a similar way as previously.

In view of Titchmarsh's theorem ((9) and (10)) we have $g_1 \ldots g_{n_0} \in L_0$ and thus $\|g_1 \ldots g_{n_0}h\|_S > 0$. Since S is arbitrary, it follows that

$$g = \lim_{n\to\infty} lg_1g_2 \ldots g_n = g_1g_2 \ldots g_{n_0}h \in L_0,$$

as desired.

Thus we have proved the first part of the lemma under the assumption that $\|g_n\|_T < 1$, $\|l - lg_n\|_T < \varepsilon_n$ $(n = 1, 2, \ldots)$. If the alternative assumptions are satisfied, i.e., $\|g_n\|_n < 1$ and $\|l - lg_n\|_n < \varepsilon_n$ $(n = 1, 2, \ldots)$, the assertion immediately follows from the case just proved.

Given an index n, the sequence $g_1, g_2, \ldots, g_{n-1}, g_{n+1}, \ldots$ also satisfies the assumptions of the lemma (in the both versions). Denoting by φ_n the limit of this sequence, we get (11).

Finally we have, for $m > n$,

$$\|lfg_1 \ldots g_m\|_T \leqslant \|lfg_n\|_T$$

and, consequently,

$$\|fg\|_T = \lim_{m\to\infty} \|lfg_1 \ldots g_m\|_T \leqslant \|lfg_n\|_T$$

for every $n = 1, 2, \ldots$

At last we prove a lemma connected with Theorem 3 of [1].

LEMMA 5 (cf. [1]). *If $g_n \in L$ and $\|g_n\|_T > 0$ for some $T > 0$ $(n = 1, 2, \ldots)$, then there exists a function g in L such that $\|g\|_S > 0$ for every $S > T$ and $g = \varphi_n g_n$ $(n = 1, 2, \ldots)$ for some functions φ_n in L.*

Proof. At first let $g_n \in L_0$ $(n = 1, 2, \ldots)$. By Lemma 2, there are functions $k_n \in L_0$ such that

$$\|k_n g_n\|_n < 1, \quad \|l - lk_n g_n\|_n < 2^{-n} \quad \text{for} \quad n = 1, 2, \ldots$$

The functions $s_n = k_n g_n$ $(n = 1, 2, \ldots)$ satisfy the assumptions of Lemma 4. Thus there exist functions $g \in L_0$ and $\psi_n \in L_0$ such that

$$g = \psi_n s_n = \varphi_n g_n \quad (n = 1, 2, \ldots),$$

where $\varphi_n = \psi_n k_n$ and, by Titchmarsh's theorem (9), we have $\varphi_n \in L_0$.

Now let us suppose that $\|g_n\|_T > 0$ for $n = 1, 2, \ldots$ and some $T > 0$.

Let $\lambda_n = \sup\{S\colon \|g_n\|_S = 0\}$. Of course, we have $\lambda_n \leqslant T$. It is easy to see that $\tilde{g}_n = h^{-\lambda_n} g_n \in L_0$, where $h^{-\lambda_n}$ are translation operators.

It follows from the first part of the proof that there are $\tilde{g} \in L_0$ and $\tilde{\varphi}_n \in L_0$ such that $\tilde{g} = \tilde{\varphi}_n \hat{g}_n = \tilde{\varphi}_n h^{-\lambda_n} g_n$. Hence $g = \varphi_n g_n$, where $\varphi_n = h^T l \tilde{\varphi}_n h^{-\lambda_n}$, $g = h^T l \tilde{g}$.

Since $\lambda_n \leqslant T$, we have $\varphi_n \in L_0$. Moreover, $\|g\|_S > 0$ for $S > T$, by Titchmarsh's theorem. Thus the proof is complete.

4. The main lemma. The basic tool in what follows is the following lemma.

LEMMA 6. *Let $T > 1$. If for every $\varepsilon > 0$ the sequence $\{B_{2T,\varepsilon}(f_n)\}$, $n = 1, 2, \ldots$ is bounded, then there exist a subsequence $\{f_{p_n}\}$ of $\{f_n\}$ and a function $h \in L$ such that $\|h\|_1 > 0$ and the sequence $\{hf_{p_n}\}$, $n = 1, 2, \ldots$ is convergent in the seminorm $\|\cdot\|_T$.*

Proof. By the assumption, we can choose functions $g_{kn} \in L_0$ such that $\|g_{kn}\|_{2T} < 1$, $\|l - lg_{kn}\|_{2T} < 2^{-(k+1)}$ $(n, k = 1, 2, \ldots)$ and for every k the sequence $\{\|f_n g_{kn}\|_{2T}\}$, $n = 1, 2, \ldots$, is bounded. Using Lemma 1 and applying the diagonal process, we can choose a sequence $\{q_n\}$ such that for every k the sequences $\{lf_{q_n} g_{kq_n}\}$ and $\{lg_{kq_n}\}$, $n = 1, 2, \ldots$ are convergent in the seminorm $\|\cdot\|_{2T}$. Let $\bar{f}_n = f_{q_n}$, $\bar{g}_{kn} = g_{kq_n}$ and

$$l\bar{f}_n \bar{g}_{kn} \to F_k, \quad l\bar{g}_{kn} \to G_k \quad \text{as} \quad n\to\infty \tag{12}$$

in $\|\cdot\|_{2T}$. Multiplying the above convergent sequences by G_k and F_k, respectively and subtracting them, respectively, we obtain

$$l(\bar{f}_n G_k - F_k)\bar{g}_{kn} \to 0 \quad \text{as} \quad n\to\infty \tag{13}$$

in $\|\cdot\|_{2T}$ for every $k = 1, 2, \ldots$, by virtue of (1).

Since $\|l - lg_{kn}\|_{2T} < 2^{-(k+1)}$, it follows from (12) that $\|l - G_k\|_{2T} < 1/2$ for all k. Hence $\|G_k\|_{1/2} \geqslant \|l\|_{1/2} - \|l - G_k\|_{1/2} > 0$ for $k = 1, 2, \ldots$

According to Lemma 5, there exist functions $g, \varphi_k \in L$ $(k = 1, 2, \ldots)$ such that $\|g\|_1 > 0$ and $g = \varphi_k G_k$ for every k.

Let $\tilde{f}_k = \varphi_k F_k$. Multiplying the sequence in (13) by φ_k, we get for every k

$$l(\bar{f}_n g - \tilde{f}_k)\bar{g}_{kn} \to 0 \quad \text{as} \quad n\to\infty \tag{14}$$

in $\|\cdot\|_{2T}$.

Now, we shall show that all functions f_k are equal one to another on $[0, 2T-1]$. Let i and j be arbitrary indices. Of course,

$$\|\bar{g}_{in}\bar{g}_{jn}\|_{2T} \leqslant \|\bar{g}_{in}\|_{2T}\|\bar{g}_{jn}\|_{2T} < 1$$

and thus, by Lemma 1, the sequence $\{l\bar{g}_{in}\bar{g}_{jn}\}$, $n = 1, 2, \ldots$, contains a subsequence which converges in $\|\cdot\|_{2T}$ to some function ψ from L. Since for arbitrary n

$$\|l - l\bar{g}_{in}\bar{g}_{jn}\|_{2T} \leqslant \|l - l\bar{g}_{in}\|_{2T} + \|\bar{g}_{in}\|_{2T} \cdot \|l - l\bar{g}_{jn}\|_{2T} \leqslant 2^{-(i+1)} + 2^{-(j+1)} \leqslant 1/2,$$

we have $\|l - \psi\|_1 \leqslant \|l - \psi\|_{2T} \leqslant 1/2$ and thus $\|\psi\|_1 > 0$.

On the other hand, it follows from (14) that for arbitrary indices i, j we have

$$l(\bar{f}_n g - \tilde{f}_i)\bar{g}_{in}\bar{g}_{jn} \to 0, \quad l(\bar{f}_n g - \tilde{f}_j)\bar{g}_{jn}\bar{g}_{in} \to 0$$

and, consequently,

$$l(\tilde{f}_i - \tilde{f}_j)\bar{g}_{in}\bar{g}_{jn} \to 0$$

as $n\to\infty$ in $\|\cdot\|_{2T}$. Hence

$$(\tilde f_i-\tilde f_j)\psi = 0 \quad \text{on} \quad [0,2T].$$

Now since $\|\psi\|_1>0$, we obtain, owing to Titchmarsh's theorem,

$$\tilde f_i = \tilde f_j \quad \text{on} \quad [0,2T-1],$$

as desired.

Let f denote the common value of the functions $\tilde f_k$ $(k=1,2,\ldots)$ on $[0,2T-1]$. Evidently, by (14),

$$l(\bar f_n g-f)\bar g_{kn}\to 0 \quad \text{as} \quad n\to\infty$$

in $\|\cdot\|_{2T-1}$ and thus also in $\|\cdot\|_T$ for $k=1,2,\ldots$

We can choose a sequence $r_1<r_2<\ldots$ of positive integers such that

$$\|l(\bar f_{r_n}g-f)\bar g_{nr_n}\|_T<1/n. \tag{15}$$

By Lemma 4, we have

$$\lim_{n\to\infty}(l\bar g_{1r_1}\bar g_{2r_2}\ldots\bar g_{nr_n}) = \bar g\in L_0$$

and

$$\|(\bar f_{r_n}g-f)\bar g\|_T\leqslant \|l(\bar f_{r_n}g-f)\bar g_{nr_n}\|_T$$

for every $n=1,2,\ldots$

Therefore, putting $h=g\bar g$, we have

$$h\bar f_{r_n}\to f\bar g \quad \text{as} \quad n\to\infty$$

in $\|\cdot\|_T$, owing to (15).

Since $\bar g\in L_0$ and $\|g\|_1>0$, we obtain, by virtue of Titchmarsh's theorem (10), $\|h\|_1>0$. This completes the proof, because $\{\bar f_{r_n}\}$ is a subsequence of $\{f_n\}$.

5. Common denominator. Now we shall show situations, needed in the next sections, when all operators involved in a given sequence $\{x_n\}$ have a common denominator, i.e. $x_n=f_n/g$ with $f_n, g\in L$, $g\neq 0$ for $n=1,2,\ldots$

We start with some fact relative to support numbers of operators.

PROPOSITION 1. *If $\Lambda(x)<\Lambda(y)$, then $\Lambda(x+y)=\Lambda(x)$.*

Proof. If x and y are functions from L, then the assertion is obvious. Now let $x=f_1/g_1$, $y=f_2/g_2$ $(f_1,f_2,g_1,g_2\in L,\ g_1,g_2\neq 0)$ and

$$\Lambda(x) = \Lambda(f_1)-\Lambda(g_1)<\Lambda(y) = \Lambda(f_2)-\Lambda(g_2).$$

Hence, by Titchmarsh's theorem, we have

$$\Lambda(f_1g_2) = \Lambda(f_1)+\Lambda(g_2)<\Lambda(f_2)+\Lambda(g_1) = \Lambda(f_2g_1)$$

and, consequently,

$$\begin{aligned}\Lambda(x+y) &= \Lambda(f_1g_2+f_2g_1)-\Lambda(g_1g_2)\\ &= \Lambda(f_1g_2)-\Lambda(g_1g_2) = \Lambda(f_1)-\Lambda(g_1) = \Lambda(x).\end{aligned}$$

Thus the proposition is proved.

PROPOSITION 2. *The operators x_n $(n=1,2,\ldots)$ have a common denominator if and only if the sequence $\{\Lambda(x_n)\}$ is bounded from the left.*

Proof. If $x_n=f_n/g$, f_n, $g\in L$, $g\neq 0$, then

$$\Lambda(x_n) = \Lambda(f_n)-\Lambda(g)\geqslant -\Lambda(g),$$

i.e., the sequence $\{\Lambda(x_n)\}$ is bounded from below.

Note that if

$$\Lambda(x)>-c \quad (c>0) \tag{16}$$

then x can be represented in the form $x=f/g$ $(f,g\in L,\ g\neq 0)$, where $\Lambda(g)<c$, i.e. $\|g\|_c>0$. In fact, if $x=\tilde f/\tilde g$ with $\Lambda(\tilde g)\leqslant\Lambda(\tilde f)$, then, applying the shift operator, we can defined functions f, g such that $x=f/g$ and $\Lambda(g)=0<c$. On the contrary, if $\Lambda(\tilde f)<\Lambda(\tilde g)$, then we can find $f,g\in L$, $g\neq 0$ such that $x=f/g$ and $\Lambda(f)=0$, which implies, by (16), $\Lambda(g)<\Lambda(f)+c=c$.

Therefore, if $\Lambda(x_n)>-c$ $(c>0)$, then we can find the representation $x_n=f_n/g_n$, where $\|g_n\|_c>0$. By Lemma 5, there exists a function $g\in L$, $g\neq 0$ such that $g=g_nh_n$. Hence

$$x_n=f_nh_n/g_nh_n=f_nh_n/g,$$

i.e., the operators $\{x_n\}$ have a common denominator.

A sequence x_n of operators is called *precompact* if the set $\{x_n\colon n=1,2,\ldots\}$ is precompact, i.e., if each subsequence of $\{x_n\}$ has a convergent subsequence (not necessary to the same limit). Of course, both the convergences: type I and type I′ lead to the same precompact sequences.

PROPOSITION 3. *If a sequence $\{x_n\}$ of operators is precompact, then the operators x_n $(n=1,2,\ldots)$ have a common denominator. In particular, if the sequence $\{x_n\}$ is convergent (type I or I′), then the operators $\{x_n\}$ have a common denominator.*

Proof. Suppose that the operators x_n have no common denominator. By Proposition 2, there is an increasing sequence $\{p_n\}$ of positive integers such that $\Lambda(x_{p_n})\to-\infty$. On the other hand, one can select from $\{p_n\}$ a subsequence $\{q_n\}$ such that the sequence $\{x_{q_n}\}$ is convergent type I. Thus the operators x_{q_n} have a common denominator and, by Proposition 2, $\Lambda(x_{q_n})>c>-\infty$. This contradiction proves our assertion.

A sequence $\{x_n\}$ of operators is called *bounded type* I if the set $\{x_n\colon n=1,2,\ldots\}$ is bounded type I′, i.e., if for every subsequence $\{x_{k_n}\}$ of $\{x_n\}$

and sequence $\{\lambda_n\}$ of numbers such that $\lambda_n \to 0$ we have $\lambda_n x_{k_n} \to 0$ type I′.

PROPOSITION 4. *If $\{x_n\}$ is bounded type I′, then the operators x_n have a common denominator.*

Proof. Since $(1/n)x_n \to 0$ type I′, the operators $(1/n)x_n$ $(n = 1, 2, \ldots)$ have a common denominator, by virtue of Proposition 3. This implies that also the operators x_n $(n = 1, 2, \ldots)$ have a common denominator.

PROPOSITION 5. *If $\{x_n\}$ is a P-Cauchy sequence (with respect to type I or I′ convergence), then the operators x_n have a common denominator.*

Proof. Suppose, on the contrary, that the operators x_n have not a common denominator. By Proposition 2, it follows that there exists a sequence of positive integers $p_1 < p_2 < \ldots$ such that

$$\Lambda(x_{p_n}) \to -\infty \quad \text{as} \quad n \to \infty.$$

Since $x_{p_{n+1}} - x_{p_n} \to 0$ with respect to type I or I′ convergence, the sequence $\Lambda(x_{p_{n+1}} - x_{p_n})$ is, by Propositions 3 and 2, bounded from the left. On the other hand, by Proposition 1, we have

$$\Lambda(x_{p_{n+1}} - x_{p_n}) = \Lambda(x_{p_{n+1}}) \to \infty$$

which proves Proposition 5.

6. Type I′ convergence. In this section we give some characterizations of type I′ convergence in L and in $\mathcal{F}$.

THEOREM 1. *A sequence $\{f_n\}$ converges type I′ to f $(f_n, f \in L)$ if and only if $B_{T,\varepsilon}(f_n - f) \to 0$ as $n \to \infty$, for any $T, \varepsilon > 0$ (see Introduction).*

Proof. Of course, it suffices to prove the theorem for $f = 0$.

By Lemma 3, if $f_n \to 0$ type I′, then each subsequence of $\{B_{T,\varepsilon}(f_n)\}$ $(T, \varepsilon > 0)$ has a subsequence tending to 0, i.e., $B_{T,\varepsilon}(f_n) \to 0$ for all $T, \varepsilon > 0$.

Now let $B_{T,\varepsilon}(f_n) \to 0$ for any $T, \varepsilon > 0$. It suffices to prove that $\{f_n\}$ possesses a subsequence $\{f_{p_n}\}$ which converges to 0 type I.

There exists an increasing sequence of positive integers $\{p_n\}$ and functions $g_n \in L_0$ such that

$$\|g_n\|_n < 1, \qquad \|l - lg_n\|_n < 1/2^n, \qquad \|f_{p_n} \cdot g_n\|_n < 1/n.$$

By Lemma 4, we have $lg_1 \ldots g_n \to g$ in L as $n \to \infty$ and $g \in L_0$. We shall show that $\|f_{p_n} \cdot g\|_T \to 0$ as $n \to \infty$ for arbitrary $T > 0$.

Let $m > n_0 > T$. Since $\|g_i\|_i < 1$ $(i = 1, 2, \ldots)$, we may write

$$\|f_{p_n} \cdot lg_1 \ldots g_m\|_T < \|f_{p_n} \cdot g_n\|_T \cdot \|lg_1 \ldots g_{n_0}\|_T$$
$$< \|f_{p_n} \cdot g_n\|_n \cdot \|lg_1 \ldots g_{n_0}\|_{n_0} < (1/n)\|lg_1 \ldots g_{n_0}\|_{n_0}.$$

Thus $\|f_{p_n} \cdot g\|_T \to 0$ as $n \to \infty$ and the proof is complete.

As a simple consequence of Theorem 1 and of the second part of Proposition 3 we obtain the following

THEOREM 2. *A sequence $\{x_n\}$ of Mikusiński operators converges type I′ to x if and only if there exist representations $x_n = f_n/g$, $x = f/g$, where $f_n, f, g \in L$, $g \neq 0$ and $B_{T,\varepsilon}(f_n - f) \to 0$ for every $T, \varepsilon > 0$.*

7. Precompactness. Now we establish a characterization of precompact subsets of L considered in $\mathcal{F}$ and of precompact sequences in $\mathcal{F}$. Precompactness is meant in the sense of type I or I′ convergence (see Sections 1 and 4).

THEOREM 3. *A set $A \subset L$ is precompact in $\mathcal{F}$ if and only if the set $\{B_{T,\varepsilon}(f) \colon f \in A\}$ is bounded for arbitrary $T, \varepsilon > 0$.*

Proof. Because of Lemma 3, the precompactness of A implies boundedness of the set $\{B_{T,\varepsilon}(f) \colon f \in A\}$ for any $T, \varepsilon > 0$.

In order to prove the converse implication, it suffices to prove that if a sequence $\{f_n\}$, $f_n \in L$, has the property that for each $T, \varepsilon > 0$ the sequence $\{B_{T,\varepsilon}(f_n)\}$ is bounded, then there is a subsequence $(f_{p_n}\}$ and a function $h \in L$, $h \neq 0$ such that $\{hf_{p_n}\}$ converges in L to a function of the form hf, $f \in L$.

By Lemma 6, for every integer $k \geq 2$ there exists a sequence $\{p_{kn}\}$ $(n = 1, 2, \ldots)$ of positive integers such that $p_{kn} \to \infty$ as $n \to \infty$ and a function $h_k \in L$, $\|h_k\|_1 > 0$ such that $\{h_k f_{p_{kn}}\}$ is convergent in the seminorm $\|\cdot\|_k$. Of course, we can assume that $\{p_{k+1,n}\}$ $(n = 1, 2, \ldots)$ is a subsequence of $\{p_{kn}\}$ $(n = 1, 2, \ldots)$ for every $k = 2, 3, \ldots$

By Lemma 5, there exist functions $\varphi_k, h \in L$ such that $h = \varphi_k h_k$ $(k = 2, 3, \ldots)$ and $\|h\|_T > 0$ for $T > 1$, i.e., $h \neq 0$.

Multiplying $h_k f_{p_{kn}}$ by φ_k, we get thus the convergence of the sequence $\{hf_{p_{kn}}\}$ in $\|\cdot\|_k$ as $n \to \infty$ for $k = 2, 3, \ldots$ By diagonal method, putting $p_n = p_{nn}$, we obtain the convergence of sequence $\{hf_{p_n}\}$ in all the seminorms $\|\cdot\|_k$ $(k = 2, 3, \ldots)$, i.e., in L. This completes the proof.

Directly from Theorem 3 and Proposition 3, we obtain

THEOREM 4. *A sequence $\{x_n\}$ of operators is precompact if and only if there exist representations $x_n = f_n/g$ with $f_n, g \in L$, $g \neq 0$, and the sequence $\{B_{T,\varepsilon}(f_n)\}$ is bounded for all $T, \varepsilon > 0$.*

8. Boundedness and compactness. Now we shall prove that the boundedness of sets in L and in $\mathcal{F}$ with respect to type I′ convergence (see Introduction) is equivalent to their precompactness (type I or I′).

In other words, a set A in $\mathscr{F}$ (in L) is compact type I′ if and only if A is bounded and closed type I′.

THEOREM 5. *A set $A \subset \mathscr{F}$ is precompact if and only if A is bounded type* I′.

In particular, the same statement holds for sets in L.

Proof. At first we shall prove the theorem in the case of sets in L. Let A be a precompact set in L (type I or I'). By Theorem 3, the set $\{B_{T,\varepsilon}(f)\colon f \in A\}$ is bounded for any $T, \varepsilon > 0$. Hence, for any sequence of numbers $\{\lambda_n\}$, $\lambda_n \to 0$ and any sequence $\{x_n\}$, $x_n \in A$ we have

$$B_{T,\varepsilon}(\lambda_n f_n) = |\lambda_n| B_{T,\varepsilon}(f_n) \to 0 \quad \text{as} \quad n \to \infty.$$

This means, by Theorem 1, that $\lambda_n f_n \to 0$ type I′, which shows that A is type I′ bounded.

Conversely, suppose that A is not precompact (type I or I′). By Theorem 3, there are $T, \varepsilon > 0$ and a sequence $\{f_n\}$, $n = 1, 2, \ldots$ with $f_n \in A$ such that $B_{T,\varepsilon}(f_n) \to \infty$. Let $\lambda_n = [B_{T,\varepsilon}(f_n)]^{-1}$. Of course, $\lambda_n \to 0$ and $B_{T,\varepsilon}(\lambda_n f_n) = \lambda_n B_{T,\varepsilon}(f_n) = 1$, i.e., $B_{T,\varepsilon}(\lambda_n f_n) \not\to 0$. It follows, by Theorem 1, that $\lambda_n f_n \not\to 0$ type I′. Thus A is not bounded type I′, as desired.

Now we pass to the general case. Suppose that $A \subset \mathscr{F}$ is precompact and let $x_n \in A$, $\lambda_n \to 0$. Of course, the sequence $\{x_n\}$, $n = 1, 2, \ldots$ is also precompact. By Theorem 4, we have $x_n = f_n/g$ with $f_n, g \in L$, $g \neq 0$ and the sequence $\{B_{T,\varepsilon}(f_n)\}$ is bounded for all $T, \varepsilon > 0$. Now, by Theorem 3, the sequence $\{f_n\}$, $n = 1, 2, \ldots$ is precompact and, in view of the first part of the proof, we obtain $\lambda_n f_n \to 0$ type I′. Consequently we obtain $\lambda_n x_n = \lambda_n f_n / g \to 0$ type I′, which means that the set A is bounded type I′.

In turn, suppose that $A \subset \mathscr{F}$ is bounded type I′ and let $x_n \in A$, $n = 1, 2, \ldots$ Of course, the sequence $\{x_n\}$, $n = 1, 2, \ldots$ is also bounded. By Proposition 4, we have $x_n = f_n/g$ with $f_n, g \in L$, $g \neq 0$. Hence by the definition of boundedness the sequence $\{f_n\}$ is bounded type I′. In view of the first part of the proof, the sequence $\{f_n\}$ is precompact and thus it possesses a subsequence $\{f_{k_n}\}$, which is convergent (type I or I′). Hence the sequence $x_{k_n} = f_{k_n}/g$ is also convergent, which means that the set A is precompact. Thus Theorem 5 is proved.

From Theorems 3, 4 and 5 we obtain the following two corollaries.

COROLLARY 1. *A set $A \subset L$ is bounded type* I′ *if and only if the set $\{B_{T,\varepsilon}(f)\colon f \in A\}$ is bounded for any $T, \varepsilon > 0$.*

COROLLARY 2. *A sequence $\{x_n\}$ of operators is bounded type* I′ *if and only if there exists a representation $x_n = f_n/g$ with $f_n, g \in L$, $g \neq 0$ and the sequence $\{B_{T,\varepsilon}(f_n)\}$ is bounded for all $T, \varepsilon > 0$.*

9. Completeness. In this section, we shall prove that the field of Mikusiński operators is P-complete with respect to type I′ convergence and Q-complete with respect to type I convergence.

We start with the auxiliary

PROPOSITION 6. *Suppose $x_n \in \mathscr{F}$ ($f_n \in L$) for $n = 1, 2, \ldots$ If $\{x_n\}$ ($\{f_n\}$) is a P-Cauchy sequence, type* I *or* I′, *then the sequence $\{x_n\}$ ($\{f_n\}$) is precompact, type* I *or* I′.

Proof. First we shall prove the proposition for a sequence $\{f_n\}$ of functions from L.

Suppose that $\{f_n\}$ is not precompact, i.e., the sequence $\{B_{T,\varepsilon}(f_n)\}$, $n = 1, 2, \ldots$ is not bounded for some $T, \varepsilon > 0$, by virtue of Theorem 3.

We can choose a sequence of positive integers $p_1 < p_2 < \ldots$ such that

$$B_{T,\varepsilon}(f_{p_{n+1}}) - \|f_{p_n}\|_T > 1 \qquad (n = 1, 2, \ldots). \tag{17}$$

Let $g \in L_0$, $\|g\|_T < 1$, and $\|l - lg\|_T < \varepsilon$. We have

$$\|(f_{p_{n+1}} - f_{p_n}) g\|_T \geqslant \|f_{p_{n+1}} g\|_T - \|f_{p_n}\|_T \|g\|_T \geqslant B_{T,\varepsilon}(f_{p_{n+1}}) - \|f_{p_n}\|_T$$

and, by (17),

$$\|(f_{p_{n+1}} - f_{p_n})\, g\|_T > 1.$$

Since g from L_0 (satisfying the above inequalities) was chosen arbitrarily, we have

$$B_{T,\varepsilon}(f_{p_{n+1}} - f_{p_n}) \geqslant 1. \tag{18}$$

If $\|f_n\|$ is a P-Cauchy sequence with respect to type I′ convergence, then $f_{p_{n+1}} - f_{p_n} \to 0$ as $n \to \infty$ in type I′ convergence. Hence, by Theorem 1, we obtain $B_{T,\varepsilon}(f_{p_{n+1}} - f_{p_n}) \to 0$ as $n \to \infty$, which contradicts inequality (18).

If $\{f_n\}$ is a P-Cauchy sequence type I, then it is also P-Cauchy type I′ and the proof is reduced to the preceding case.

In the general case, if a sequence $\{x_n\}$ of operators is P-Cauchy (type I or I′), then we have $x_n = f_n/g$ with $f_n, g \in L$, $g \neq 0$, $n = 1, 2, \ldots$, by Proposition 5. Hence also $\{f_n\}$ is a P-Cauchy sequence. By the first part of the proof, the sequence $\{f_n\}$ is precompact. Consequently, $\{x_n\}$ is precompact, too, and the proof is finished.

THEOREM 6. *The field $\mathscr{F}$ endowed with type* I′ *convergence is P-complete.*

Proof. According to Proposition 6 and since type I′ convergence satisfies the Urysohn condition, it remains to prove that two different subsequences of a P-Cauchy sequence $\{x_n\}$ cannot possess different limits.

Let $x_{p_n} \to x$ and $x_{p_n} \to y$ for some increasing sequence $\{p_n\}$ and $\{q_n\}$ of positive integers. We can choose subsequences $\{\bar{p}_n\}$, $\{\bar{q}_n\}$ of p_n, q_n, respectively such that

$$\bar{p}_1 < \bar{q}_1 < \bar{p}_2 < \bar{q}_2 < \ldots$$

Since $\{x_n\}$ is a P-Cauchy sequence, the sequence

$$x_{\bar{q}_1} - x_{\bar{p}_1},\ x_{\bar{p}_2} - x_{\bar{q}_1},\ x_{\bar{q}_2} - x_{\bar{p}_2},\ \ldots$$

and its subsequence $\{x_{\bar{q}_n} - x_{\bar{p}_n}\}$ converge type I′ to 0. On the other hand, $x_{\bar{q}_n} - x_{\bar{p}_n} \to x - y$. This yields the identity $x = y$ and the proof is over (see also [8]).

As the corollary, we obtain finally

THEOREM 7. *The field $\mathscr{F}$ endowed with type* I *convergence is Q-complete.*

Proof. Let $\{x_n\}$ be a Q-Cauchy sequence of operators with respect to type I convergence. Of course, the sequence $\{x_n\}$ is Q-Cauchy and, consequently, P-Cauchy with respect to type I′ convergence (see [8]). By Theorem 6, there is an operator x such that $x_n \to x$ type I′. By the definition of type I′ convergence, there is a sequence $\{p_n\}$ of positive integers such that $x_{p_n} \to x$ type I. Since $\{x_n\}$ is Q-Cauchy type I, we have $x_n - x_{p_n} \to 0$ type I. Finally, we obtain $x_n = x_n - x_{p_n} + x_{p_n} \to x$ type I as $n \to \infty$, which completes the proof (cf. [5]).

10. The subalgebra $\mathscr{F}_0$. Following T. K. Boehme (see [2]), we denote by $\mathscr{F}_0$ the algebra of all operators of the form f/g, where $f \in L$ and $g \in L_0$. We can introduce the following functions $B_{T,\varepsilon}(x)$ for $x \in \mathscr{F}_0$:

$$B_{T,\varepsilon}(x) = \inf\{\|f\|_T \colon x = f/g,\ \|g\|_T < 1,\ \|l - lg\|_T < \varepsilon\}$$

which evidently coincide with those introduced earlier for $x \in L$.

Now we shall give, in terms of the functions $B_{T,\varepsilon}$, characterizations of type I′ convergence and of precompactness in $\mathscr{F}_0$.

THEOREM 8. *A sequence $\{x_n\}$ converges type* I′ *to x ($x_n, x \in \mathscr{F}_0$) if and only if $B_{T,\varepsilon}(x_n - x) \to 0$ as $n \to \infty$ for any $T, \varepsilon > 0$.*

Proof. Of course, we can restrict ourselves to the case $x = 0$.

Assume first that $B_{T,\varepsilon}(x_n) \to 0$ as $n \to \infty$ for all $T, \varepsilon > 0$. To prove that $x_n \to 0$ type I′ it suffices to select a subsequence $\{x_{p_n}\}$ of $\{x_n\}$ such that $x_{p_n} \to 0$ as $n \to \infty$, with respect type I convergence. We have, in particular,

$$B_{k,1/k}(x_n) \to 0 \quad \text{as} \quad n \to \infty$$

for $k = 1, 2, \ldots$

We therefore can choose an increasing sequence $\{p_n\}$ of positive integers and functions f_k, g_k $(k = 1, 2, \ldots)$ such that

$$\|f_k\|_k < 1/k, \quad \|g_k\|_k < 1, \quad \|l - lg_k\|_k < 2^{-k}, \quad f_k \in L, \quad g_k \in L_0$$

and

$$x_{p_k} = f_k/g_k \quad (k = 1, 2, \ldots). \tag{19}$$

By Lemma 4, if $n \to \infty$, then

$$lg_1 g_2 \ldots g_n \to g \quad \text{in} \quad L,$$

where $g \in L_0$. Moreover,

$$g = \varphi_k g_k \quad (k = 1, 2, \ldots), \tag{20}$$

where $\varphi_k = \lim\limits_{n\to\infty} lg_1 g_2 \ldots g_{k-1} g_{k+1} \ldots g_n$ in L.

By (19) and (20), we have

$$x_{p_k} = f_k \varphi_k / g \quad k = 1, 2, \ldots \tag{21}$$

Now, we shall prove that

$$\|f_k \varphi_k\|_T \to 0 \quad \text{as} \quad k \to \infty \tag{22}$$

for any $T > 0$. Let T be fixed and let $k_0 > T$. Since $\|g_i\|_T < \|g_i\|_i$ for $i > k_0$, we have

$$\|lg_1 \ldots g_{k-1} g_{k+1} \ldots g_n\|_T < \|lg_1 \ldots g_{k_0}\|_T$$

and, consequently,

$$\|\varphi_k\|_T < \|lg_1 \ldots g_{k_0}\|_T .$$

We therefore obtain, for $k > k_0$,

$$\|f_k \varphi_k\|_T \leqslant \|f_k\|_T\ \|\varphi_k\|_T \leqslant \|f_k\|_k \|lg_1 \ldots g_{k_0}\|_T \leqslant (1/k)\|lg_1 \ldots g_{k_0}\|_{k_0},$$

which implies (22). Hence, by (21) $x_{p_k} \to 0$ type I, as $k \to \infty$ as desired.

Now we assume, conversely, that $x_n \to 0$ type I′. We shall show that $B_{T,\varepsilon}(x_n) \to 0$ as $n \to \infty$ for any $T, \varepsilon > 0$. By Theorem 2, we have

$$x_n = f_n/g \quad (f_n, g \in L,\ g \neq 0) \tag{23}$$

and, for any $T, \varepsilon > 0$,

$$B_{T,\varepsilon}(f_n) \to 0 \quad \text{as} \quad n \to \infty,$$

i.e., there exist functions $g_n \in L_0$ $(n = 1, 2, \ldots)$ such that

$$\|g_n\|_T < 1, \quad \|l - lg_n\|_T < (1/2)\varepsilon \quad (n = 1, 2, \ldots) \tag{24}$$

and

$$\|f_n g_n\|_T \to 0 \quad \text{as} \quad n \to \infty \tag{25}$$

for any $T, \varepsilon > 0$.

On the other hand, since $x_n \in \mathscr{F}_0$, we have $x_n = \tilde{f}_n/\tilde{g}_n$ with $\tilde{f}_n \in L$, $\tilde{g}_n \in L_0$ for $n = 1, 2, \ldots$ Hence, by (23),

$$f_n/g = \tilde{f}_n/\tilde{g}_n \quad (n = 1, 2, \ldots),$$

which implies the equalities

$$\Lambda(g) + \Lambda(\tilde{f}_n) = \Lambda(g\tilde{f}_n) = \Lambda(f_n \tilde{g}_n) = \Lambda(f_n).$$

Thus

$$A(g) \leqslant A(f_n) \quad (n = 1, 2, \ldots)$$

and consequently,

$$x_n = hf_n/hg \quad (n = 1, 2, \ldots)$$

where $h = h^{-A(g)}$ (the shift operator), $hg \in L_0$ and $hf_n \in L$ $(n = 1, 2, \ldots)$.

By Lemma 2, there is a function $k \in L_0$ such that

$$\text{(26)} \qquad \|khg\|_T < 1, \quad \|l - lkhg\|_T < (1/2)\varepsilon$$

for any $T, \varepsilon > 0$.

Finally, we have

$$x_n = khf_n g_n / khgg_n \quad (n = 1, 2, \ldots),$$

where $khgg_n \in L_0$ and, in view of inequalities (24), (26), we get

$$\|khgg_n\|_T \leqslant \|khg\|_T \cdot \|g_n\|_T < 1$$

and

$$\|l - lkhgg_n\|_T \leqslant \|l - lkhg\|_T + \|lkhg - lkhgg_n\|_T < \varepsilon.$$

Moreover,

$$B_{T,\varepsilon}(x_n) \leqslant \|khf_n g_n\|_T \leqslant \|kh\|_T \cdot \|f_n g_n\|_T$$

and, by (25), the desired assertion is proved.

THEOREM 9. *A set $A \subset \mathscr{F}_0$ is precompact if and only if the set $\{B_{T,\varepsilon}(x): x \in A\}$ is bounded for every $T, \varepsilon > 0$ (precompactness of type I or I′).*

Proof. Note that for any $T, \varepsilon > 0$ the boundedness of the sets $\{B_{T,\varepsilon}(x): x \in A\}$ is equivalent to the boundedness of all sequences $\{B_{T,\varepsilon}(x_n)\}$, where $\{x_n\}$ is a sequence from A. In turn, this is equivalent to the fact that $x_n \in A$ $(n = 1, 2, \ldots)$ and $\lambda_n \to 0$ implies

$$|\lambda_n| B_{T,\varepsilon}(x_n) = B_{T,\varepsilon}(x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

However in view of Theorem 8, the last statement is equivalent to the assertion that $\lambda_n x_n \to 0$ type I′, provided $\lambda_n \to 0$ and $x_n \in A$ $(n = 1, 2, \ldots)$. This means that the set A is bounded, and, by Theorem 5, that A is precompact. Thus the chain of equivalent assertions is closed and the proof is finished.

In [4], the following theorem is proved:

Let X be a group with a convergence. If there is a function $A: X \to R^+$ such that

(i) $A(x_n) \to 0$ and $A(y_n) \to 0$ implies $A(x_n - y_n) \to 0$,

(ii) $A(x) = 0$ iff $x = 0$,

then there exists a norm $\|\cdot\|$ in X (i.e. the conditions: (i) $\|x\| = 0$ iff $x = 0$, (ii) $\|-x\| = \|x\|$, (iii) $\|x + y\| \leqslant \|x\| + \|y\|$ are satisfied), such that $A(x_n) \to 0$ iff $\|x_n\| \to 0$.

Applying this theorem and Theorem 8, we obtain in a simple way the fact of metrizability of $\mathscr{F}_0$ (see [2]).

THEOREM 10. *$\mathscr{F}_0$ is a Montel complete metric space.*

Proof. It suffices to see that the function

$$A(x) = \sum_{n=1}^{\infty} 2^{-n} B_{n,1/n}(x) / (1 + B_{n,1/n}(x)), \quad x \in \mathscr{F}_0$$

satisfies the assumptions of the theorem given above and that type I′ convergence in $\mathscr{F}_0$ is equivalent to the convergence defined by A (see Theorem 8).

Now, we shall apply Theorem 8 to the proof of the following theorem of T. K. Boehme (see [3]):

THEOREM 11 (Boehme [3]). *Let $x_n, x \in \mathscr{F}$ $(n = 1, 2, \ldots)$; $x_n \to x$ type I′ if and only if there exist representations $x_n = f_n/g_n$, $x = f/g$ such that $f_n \xrightarrow{L} f$, $g_n \xrightarrow{L} g$ and $A(g_n) \to A(g)$.*

The proof will be preceeded by a lemma.

LEMMA 7 (Boehme [2]). *If $f_n \xrightarrow{L} f$, $g_n \xrightarrow{L} g \in L_0$, then $f_n/g_n \to f/g$ type I′.*

Proof. Note that it suffices to prove our assertion for $f = 0$. Let $h_n = h^{-A(g_n)} g_n$ and $y_n = f_n/h_n \in L_0$. We have

$$\text{(27)} \qquad h_n \xrightarrow{L} g.$$

Since $g \in L_0$, for any $T, \varepsilon > 0$ there exists $k \in L_0$ such that

$$\|kg\|_T < 1, \quad \|l - lkg\|_T < \varepsilon,$$

in view of Lemma 2. By (27), we have also

$$\|kh_n\|_T < 1, \quad \|l - lkh_n\|_T$$

for sufficiently large n.

Now, we can write $y_n = kf_n/kh_n$ and thus

$$\text{(28)} \qquad B_{T,\varepsilon}(y_n) \leqslant \|kf_n\|_T,$$

by the definition of $B_{T,\varepsilon}(x)$ for $x \in \mathscr{F}_0$, given at the beginning of this section. By (28), we have

$$B_{T,\varepsilon}(y_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Owing to Theorem 8, we get $y_n \to 0$ type I′ and, consequently, $x_n = h^{-\Lambda(g_n)} y_n \to 0$ type I′ as desired.

Proof of Theorem 11. Suppose that $x_n = f_n/g_n$, $x = f/g$, $f_n \xrightarrow{L} f$, $g_n \xrightarrow{L} g$ and $\Lambda(g_n) \to \Lambda(g)$.

Let $y_n = f_n/\varphi_n$, where $\varphi_n = h^{-\Lambda(g_n)} g_n$. By Lemma 7, we have $y_n \to f/\varphi$ type I′, where $\varphi = h^{-\Lambda(g)} g$, i.e., $x_n = h^{-\Lambda(g_n)} y_n \to x$ type I′.

Now let $x_n \to x$ type I′. We can assume that $x = 0$. By Proposition 3, we have $x_n = f_n/g$, where $f_n, g \in L$ $(n = 1, 2, \ldots)$. Of course, we have $y_n \to 0$ type I', where

$$y_n = f_n/h^{-\Lambda(g)} g.$$

By Theorem 8, there exists an increasing sequence of positive integers r_k such that

$$B_{k,1/k}(y_n) < 1/k$$

for $n > r_k$. That means, there exist functions $\varphi_n \in L$, $\psi_n \in L_0$ such that

$$y_n = \varphi_n/\psi_n, \quad \|\varphi_n\|_k < 1/k, \quad \|\psi_n\|_k < 1, \quad \|l - l\psi_n\|_k < 1/k. \tag{29}$$

or $n > r_k$. Hence we can write

$$y_n = l\varphi_n/l\psi_n$$

and we have

$$l\varphi_n \xrightarrow{L} 0, \quad l\psi_n \xrightarrow{L} l, \quad \Lambda(l\psi_n) = 0 = \Lambda(f),$$

in view of (29).

Finally, we have $x_n = \bar{f}_n/g_n$, where

$$f_n = h^{-\Lambda(g)} l f_n \xrightarrow{L} 0, \quad g_m = l\psi_n \xrightarrow{L} l, \quad \text{and} \quad \Lambda(\bar{g}_n) \to \Lambda(l),$$

i.e., the proof is finished.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
40-013 Katowice, Wieczorka 8, Poland

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