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On tensor products of spaces of continuous functions

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Abstract. We obtain a necessary and sufficient condition for a subalgebra of $C(X\times Y)$ to be identifiable as the weak tensor product of a subalgebra of $C(X)$ with a subalgebra of $C(Y)$ and show that the Bohr compactification of the product of a family of topological groups is (isomorphic to) the product of their Bohr compactifications.

For a topological space X and a Banach space B, let $C(X, B)$ denote the sup-normed Banach space of all B-valued bounded continuous functions on X. If $B = \boldsymbol{R}$, the reals, $C(X, \boldsymbol{R}) = C(X)$. The following facts are well known: For a closed subspace E of $C(X)$ and a Banach space B, the weak tensor product $E\check{\otimes}B$ may be defined as a closed subspace of $C(X, B)$ [7], p. 356. Thus for any X and Y, $C(X)\check{\otimes}C(Y)$ is $C(X, C(Y))$, which can be identified as a subspace of $C(X\times Y)$, [7], p. 89. Moreover, if X and Y are compact, then by the Stone–Weierstrass theorem $C(X)\check{\otimes}C(Y) \equiv C(X\times Y)$, [7], p. 357, where the symbol $\equiv$ denotes linear isometric bijection. In this note we exploit these facts to obtain a characterization of subspaces of $C(X\times Y)$ which can be identified as weak tensor products of subspaces of $C(X)$ and $C(Y)$, and note some consequences of this characterization.

Let X and Y be Hausdorff and completely regular (hence Tychonoff and uniformizable). A *subbocof* is a closed subalgebra containing constant functions, [7], p. 236. A *full subbocof* is one which separates points from closed sets. Let E and F be full subbocofs of $C(X)$ and $C(Y)$, respectively and let S, V denote the E-compactification of X and F-compactification of Y, respectively, so that $E \equiv C(S)$ and $F \equiv C(V)$ where, as usual, S and V may be taken as subsets of E^* and F^* (the normed conjugates of E and F) with the weak* topologies. Now

$$E\check{\otimes}F \equiv C(S)\check{\otimes}C(V) \equiv C(S, C(V)) \equiv C(S\times V)$$

and $C(S\times V)$ is of course identified with a full subbocof G of $C(X\times Y)$.

* Some of the results reported in this paper are included in a thesis for the Ph.D. at the University of Nebraska–Lincoln.

Moreover,

$$E = \{g(\cdot, y) | \ g \in G, y \in Y\} \quad \text{and} \quad F = \{g(x, \cdot) | \ g \in G, \ x \in X\}.$$

Suppose (conversely) that (i) G is a full subbocof of $C(X \times X)$; and (ii) E and F (defined as above) are full subbocofs of $C(X)$ and $C(Y)$, respectively; and (iii) $E \hat{\otimes} F$ may be canonically embedded into G. The following lemma gives a necessary and sufficient condition for $G = E \hat{\otimes} F$ under this embedding.

CHARACTERIZATION LEMMA. *With G, E, F as above, $G = E\hat{\otimes} F$ if and only if for each $g \in G$, the set $\hat{g} = \{g(x, \cdot) | \ x \in X\}$ is norm precompact in F (or equivalently, in $C(Y)$).*

Proof. The necessity of the condtion is easy to check. Thus, assume that each set $\hat{g}$ is precompact. We need only to prove that every $g \in G$ can be identified as an element of $E \hat{\otimes} F$. For given $g \in G$, define $T_g[s](y) = s(g(\cdot, y))$ for each $s \in S$ and $y \in Y$ where S is the E-compactification of X. Clearly $T_g[s]$ is a real-valued function on Y. We show that $T_g[s] \in F$ for each $s \in S$, as follows. Let $\{x_a\}$ be a net in X such that $s = w^* - \lim \tilde{x}_a$, where $\tilde{x}_a$ is the image of x_a in S. Then $s(g(\cdot, y)) = \lim \tilde{x}_a(g(\cdot, y)) = \lim g(x_a, y)$. Now since the set $\hat{g}$ is precompact, there exists a subnet $\{x_\beta\}$ such that $\lim g(x_\beta, \cdot) = \varphi$ exists in the sup norm and $\varphi \in F$. This means that $\lim g(x_\beta, y) = \varphi(y)$ uniformly in y and so

$$s(g(\cdot, y)) = \lim g(x_a, y) = \lim g(x_\beta, y) = \varphi(y)$$

for each $y \in Y$ and $T_g[s] \in F$ as claimed. Thus, T_g maps S into F Moreover, $T_g[s]$ is an element in the norm closure $\mathrm{cl}\,\hat{g}$ of the set g for each $s \in S$. A similar argument shows that every element in $\mathrm{cl}\,\hat{g}$ is a $T_g[s]$ for some $s \in S$. Thus $\mathrm{cl}\,\hat{g}$ is $T_g[S]$, a compact subset of F. Hence, given $\varepsilon > 0$ there exists a finite set $\{s_1, \ldots, s_N\} \subseteq S$ such that for each fixed $s \in S$,

$$\|T_g[s] - T_g[s_i]\| < \varepsilon \tag{1}$$

for some i, $1 \leqslant i \leqslant N$. Also (from the definition of sup norm) there exist $y_i \in Y$ $(1 \leqslant i \leqslant N)$ such that

$$\|T_g[s] - T_g[s_i]\| < |T_g[s](y_i) - T_g[s_i](y_i)| + \varepsilon. \tag{2}$$

We shall show that $T_g\colon S \to F$ is continuous. Suppose that $\lim s_a = s$ in S (w^*-topology) and

$$\|T_g[s_a] - T_g[s_{i_a}]\| < \varepsilon, \quad 1 \leqslant i_a \leqslant N. \tag{3}$$

Then $\lim s_a(g(\cdot, y_i)) = s(g(\cdot, y_i))$ for $1 \leqslant i \leqslant N$ and so there exists a_0 such that for $a > a_0$,

$$|s_a(g(\cdot, y_i)) - s(g(\cdot, y_i))| < \varepsilon.$$

That is,

$$|T_g[s_a](y_i) - T_g[s](y_i)| < \varepsilon, \quad 1 \leqslant i \leqslant N. \tag{4}$$

Thus if $a > a_0$, we have from (1)–(4),

$$\begin{aligned}
&\|T_g[s_a] - T_g[s]\| \\
&\qquad \leqslant \|T_g[s_a] - T_g[s_{i_a}]\| + \|T_g[s_{i_a}] - T_g[s]\| \\
&\qquad < \varepsilon + |T_g[s_{i_a}](y_{i_a}) - T_g[s](y_{i_a})| + \varepsilon \\
&\qquad \leqslant 2\varepsilon + |T_g[s_{i_a}](y_g) - (Tg)[s_a](y_{i_a})| + |T_g[s_a](y_{i_a}) - T_g[s](y_{i_a})| \\
&\qquad \leqslant 2\varepsilon + \|T_g[s_{i_a}] - T_g[s_a]\| + \varepsilon < 4\varepsilon.
\end{aligned}$$

This proves that $T_g \in C(S, F)$ if $\hat{g}$ is precompact. Hence g can be identified with an element of $C(S, C(V)) = E \hat{\otimes} F$ and the lemma is completely proved.

Finally, we note that the roles of X and Y can be interchanged and so an equivalent characterization is that the set $\check{g} = \{g(\cdot, y) | \ y \in Y\}$ is norm precompact in E (or equivalently, in $C(X)$) for each $g \in G$.

The following consequences are noteworthy:

(1) $C(X)\hat{\otimes} C(Y)$ is identified by the lemma as the subspace of $C(X \times Y)$ consisting of the set of all functions f with the property that the set f is norm precompact in $C(Y)$ (equivalently, $\check{f}$ is norm precompact in $C(X)$). Thus, in general, $C(X)\hat{\otimes} C(Y)$ is a proper subspace of $C(X \times Y)$.

In the next three examples X and Y are Hausdorff topological groups.

(2) Let $A(X)$ denote the space of almost periodic functions on the group X, [4], p. 247. From the lemma it follows that

$$A(X) \hat{\otimes} A(Y) = A(X \times Y),$$

a fact proved also by T.-W. Ma [6], Thm 2.1, by using the property that each almost periodic function can be uniformly approximated by trigonometric polynomials which are finite linear combinations of entry functions of continuous unitary representations of the group. Our proof, via the lemma, is more direct and elementary; and the Banach–Stone theorem [1], p. 115, implies that the Bohr compactification of the product group $X \times Y$ is (isomorphic to) the product of the Bohr compactifications of X and Y.

(3) Let $W(X)$ denote the space of weakly almost periodic functions on X, [3], p. 38. From the lemma it follows that, in general, $W(X) \hat{\otimes} W(Y)$ is a proper subspace of $W(X \times Y)$. This can be seen, for example, as follows: Let $\varphi \in W(X) \setminus A(X)$ and write $f(x, y) = \varphi(xy)$. Then $f \in W(X \times X)$ but $\hat{f}$ is not norm precompact as required in the lemma.

(4) Let $U(X)$ denote the space of uniformly continuous functions on the group X; similarly $U(Y)$ and $U(X\times Y)$ (see [5], p. 210, for uniform continuity on topological groups). From the lemma it follows that, in general, $U(X)\hat{\otimes} U(Y)$ is a proper subspace of $U(X\times Y)$. This is immediate from (3) since every w.a.p function is uniformly continuous [3], p. 38.

(5) The examples (1), (3) and (4) can be rephrased in terms of compactifications as follows: From (1), in general, the Stone–Čech compactification $\beta(X\times Y)$ is not equal to $\beta(X)\times\beta(Y)$; similarly for the w.a.p and uniform compactifications. By Glicksberg's theorem [2], p. 371, if $X\times Y$ is pseudocompact, then $\beta(X\times Y)=\beta(X)\times\beta(Y)$; and so for any full subbocof G of $C(X\times Y)$ and the corresponding E and F satisfying the conditions (i)–(iii) of the lemma, it follows that the G-compactification of $X\times Y$ is (homeomorphic to) the product of the E-compactification of X with the F-compactification of Y. Thus, in particular, if the group $X\times Y$ is pseudocompact, then the w.a.p or the uniform compactification of $X\times Y$ is (isomorphic to) the product of the respective compactifications of X and Y.

(6) The equation $A(X)\hat{\otimes}A(Y)\cong A(X\times Y)$ can be extended to an arbitrary family of topological groups. First of all, it is clear that $\hat{\otimes}_{\gamma\in F}A(X_\gamma)\cong A\left(\mathsf{P}_{\gamma\in F}X_\gamma\right)$ for any finite family $\{X_\gamma\}_{\gamma\in F}$ of topological groups. If $\{X_\gamma\}_{\gamma\in\Gamma}$ is any family of Hausdorff topological groups, then the family $\{\hat{\otimes}_{\gamma\in F}A(X_\gamma)\mid F \text{ a finite subset of } \Gamma\}$ is a direct family [7], p. 212, p. 359, of Banach spaces and $\hat{\otimes}_{\gamma\in\Gamma}A(X_\gamma)$ is the direct limit. With product topology, $X=\mathsf{P}_{\gamma\in\Gamma}X_\gamma$ is a Hausdorff group and

$$A(X)=A\left(\mathsf{P}_{\gamma\in\Gamma}X_\gamma\right)\cong\hat{\otimes}_{\gamma\in\Gamma}A(X_\gamma).$$

This can be proved by using the uniform continuity of almost periodic functions and the following approximation property (which follows from definitions in [5], p. 182): Let $\{X_\gamma\}_{\gamma\in\Gamma}$ be a family of uniform spaces and $X=\mathsf{P}_{\gamma\in\Gamma}X_\gamma$ be the product uniform space, [5], p. 182. If f is uniformly continuous on X and $\varepsilon>0$, there exist: a finite subset F of Γ and a function f_F uniformly continuous on the finite product $\mathsf{P}_{\gamma\in F}X_\gamma$ such that $\|f-f_F\|<\varepsilon$. In other words, every uniformly continuous function on a product space can be uniformly approximated by a u.c function on a finite subproduct. Again, the Banach–Stone theorem implies that the Bohr compactification of an arbitrary family of topological groups is (isomorphic to) the product of their Bohr compactifications.

(7) For locally compact spaces X and Y, Grothendieck proved that

$$C_0(X)\hat{\otimes}C_0(Y)\cong C_0(X\times Y),$$

[7], p. 357. This does not follow from the characterization lemma by one-point compactifications. Indeed, if we write $C_1(X)=C_0(X)\oplus\boldsymbol{R}$ and similarly for $C_1(Y)$ and $C_1(X\times Y)$, then $C_1(X\times Y)$ is a proper subspace of $C_1(X)\hat{\otimes}C_1(Y)$ and condition (iii) of the lemma cannot be fulfilled. However, it is clear that one can prove a similar characterization for subalgebras of $C_0(X\times Y)$ and show, for example, that

$$A_0(X\times Y)\cong A_0(X)\hat{\otimes}A_0(Y)$$

for any groups X and Y, where $A_0(X)$ denotes the space of almost periodic functions on X with mean zero.

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