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Spline bases in classical function spaces on compact C^∞ manifolds
Part I

by

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Abstract. Sobolev and Besov spaces on smooth compact manifolds are treated. The main results are formulated in Theorems A and B. Theorem A gives existence of a biorthogonal system of functions with "all possible" nice properties as a Schauder basis in Sobolev spaces on smooth compact manifolds. Theorem B gives existence of Schauder bases in Besov spaces which in the same time give linear isomorphisms of those spaces with some sequence spaces. In both theorems the duality questions are considered.

This part of the paper contains suitable decomposition of the manifold which induces a decomposition of function spaces over the manifold into direct sum of spaces of the same type on cubes with boundary conditions.

1. Introduction. In this paper we study Sobolev and Besov spaces on a compact d-dimensional C^∞ manifold M (the relevant definitions are recalled in Sections 2 and 4). Our objective is to construct Schauder bases in these spaces with certain special properties. These properties elucidate the structure of the spaces and their embeddings. Some applications (e.g. improved Sobolev type embedding theorems, estimates for the eigenvalues of integral operators, and asymptotic estimates for the Kolmogorov diameters in the class of Besov spaces) are given in Section 11.

There are two principal results in this paper. In both theorems μ is a fixed finite measure on M (cf. Section 4) and m is a fixed positive integer.

THEOREM A. *There is a sequence $(f_n)_{n=1}^{\infty}$ of elements of $C^m(M)$ with the following properties:*

(A1) *(f_n) is a Schauder basis in $C^k(M)$ and also in the Sobolev spaces $W_1^k(M)$ for $0 \leqslant k \leqslant m$,*

(A2) *(f_n) is an unconditional Schauder basis in each of the spaces $W_p^k(M)$ for $0 \leqslant k \leqslant m$, $1 < p < \infty$,*

(A3) *there is a (unique) sequence $(g_n)_{n=1}^{\infty}$ in $\mathring{C}^m(M)$, i.e. the C^m closure of the set of smooth functions on M which vanish in a neighborhood of the boundary of M, such that for $i, j = 1, 2, \ldots,$*

$$\int_M f_i g_j \, d\mu = \delta_{i,j},$$

(A1′) (g_n) *is a Schauder basis in each of the spaces* $\overset{\circ}{C}{}^k(M)$ *and* $\overset{\circ}{W}{}^k_1(M)$ *(defined analogously to* $\overset{\circ}{C}{}^m(M)$*) for* $0 \leqslant k \leqslant m$,

(A2′) (g_n) *is an unconditional Schauder basis in each of the spaces* $\overset{\circ}{W}{}^k_p(M)$ *for* $0 \leqslant k \leqslant m$, $1 < p < \infty$.

Let us remark that the spaces listed in (A1) and (A1′) are known not to admit any unconditional Schauder basis.

It is important to note that, in view of (A3), the statement (A1′) (resp. (A2′), (A1), (A2)) is equivalent by duality to the assertion that (A1) (resp. (A2), (A1′), (A2′)) remains true if $0 \geqslant k \geqslant -m$ (cf. Section 4). (Throughout this paper we avoid working with the spaces W^k_p (resp. $B^s_{p,q}$) with $k < 0$ (resp. $s \leqslant 0$).) This can be achieved by proving the corresponding dual statements concerning the conjugate space (cf. Section 4). For necessary facts concerning duality and interpolation properties of these spaces we refer to [17].

The sequences (f_n), (g_n) constructed in Theorem A are also bases in the Besov spaces $B^s_{p,q}(M)$ and $\overset{\circ}{B}{}^s_{p,q}(M)$ (and they are unconditional if $1 < p < \infty$) for $-m < s < m$, $1 < p, q < \infty$.

This follows from the well-known interpolation formulae for the Besov spaces on M (cf. (4.1) below). A stronger result in the Besov case is contained in Theorem B below.

Given a numerical sequence $a = (a_n)_{n=1}^{\infty}$ and parameters $-\infty < \varrho < \infty$, $1 \leqslant p, q \leqslant \infty$ we let

$$\|a\|_{b^{\varrho}_{p,q}} = \Big(\sum_{m=0}^{\infty}\Big(2^{m\varrho}\Big(\sum_{n=2^m}^{2^{m+1}-1}|a_n|^p\Big)^{1/p}\Big)^q\Big)^{1/q}$$

with the usual modifications if p or q equals ∞. By $b^{\varrho}_{p,q}$ we denote the space of those sequences a such that $\|a\|_{b^{\varrho}_{p,q}} < \infty$.

THEOREM B. *There is a sequence* $(\varphi_n)_{n=1}^{\infty}$ *which is a Schauder basis in the spaces* $W^0_p(M)$ *for* $1 \leqslant p \leqslant \infty$. *Moreover, for* $-m < s < m$, $1 \leqslant p$, $q \leqslant \infty$, *if* $a = (a_n)_{n=1}^{\infty}$ *is a numerical sequence (with* $a_n = 0$ *for large* n*), then*

$$\text{(B1)} \qquad C^{-1}\|a\|_{b^{\varrho}_{p,q}} \leqslant \Big\|\sum_{n=1}^{\infty} a_n\varphi_n\Big\|_{B^s_{p,q}(M)} \leqslant C\|a\|_{b^{\varrho}_{p,q}},$$

where $\varrho = s/d - 1/p + 1/2$ *and* $C < \infty$ *depends only on* m, s *and the manifold.*

Consequently, the biorthogonal functions $(\psi_n)_{n=1}^{\infty}$ (defined analogously as in (A3)) form a Schauder basis in the spaces $\overset{\circ}{W}{}^0_p(M)$ for $1 \leqslant p \leqslant \infty$. Also the estimates

$$\text{(B2)} \qquad C^{-1}\|b\|_{b^{\varrho}_{p,q}} \leqslant \Big\|\sum_{n=1}^{\infty} b_n\psi_n\Big\|_{\overset{\circ}{B}{}^s_{p,q}(M)} \leqslant C\|b\|_{b^{\varrho}_{p,q}}$$

are satisfied (with the same values of C and ϱ) for $-m < s < m$, $1 \leqslant p, q \leqslant \infty$ and any (finitely non-zero) sequence $b = (b_n)_{n=1}^{\infty}$.

In particular, (φ_n) and (ψ_n) form unconditional bases in the respective Besov spaces (here by $B^s_{p,\infty}(M)$ or $\overset{\circ}{B}{}^s_{p,\infty}(M)$ we mean the closure of the smooth functions in the respective norm). Moreover, the map

$$f \mapsto \Big(\int_M f\psi_n\,d\mu\Big)_{n=1}^{\infty}$$

establishes an isomorphism between the scale of Besov spaces $B^s_{p,q}(M)$, $|s| < m$, $1 \leqslant p, q \leqslant \infty$, and the corresponding subset of the (much simpler) scale of the spaces $b^{\varrho}_{p,q}$.

We have not settled the question whether the sequence (φ_n) is a basis in the Sobolev spaces $W^k_p(M)$ for $1 < p < 2$ or $2 < p < \infty$.

Our construction of bases in a function space $\mathscr{F}(M)$ (where $\mathscr{F}$ is either W^k_p or $B^s_{p,q}$) on the manifold M is done in two stages. First we find a decomposition of $\mathscr{F}(M)$ into a finite direct sum whose summands are linearly isomorphic to some standard spaces. When this is accomplished, it suffices to construct appropriate Schauder bases in those standard spaces.

It turns out that a good choice of the standard spaces are the spaces which we denote by $\mathscr{F}(Q)_Z$. (Here $Q = \langle 0, 1\rangle^d$ is the d-dimensional cube and Z is the union of some $(d-1)$-dimensional closed faces of Q.) They consist of those functions in $\mathscr{F}(Q)$ which "vanish" on Z.

The main point is that the study of $\mathscr{F}(Q)_Z$ can be reduced to the case where $d = 1$ if one considers suitable spaces of vector-valued functions. This case is treated using spline function theory.

For technical reasons the paper has been divided into two parts. In Part II we construct bases in the standard spaces $\mathscr{F}(Q)_Z$. In Part I we give preliminary material and construct the decomposition of $\mathscr{F}(M)$.

In Section 3 we decompose the manifold M, using basic facts from Morse theory, into finitely many non-overlapping cubes. (The use of Morse theory was suggested to us by Professor K. Gęba.) This decomposition has some special properties which enable us to construct (in Section 6) a sequence of extension operators which is used in order to decompose $\mathscr{F}(M)$.

The technical tools for Section 6 are developed in Section 5 where we give explicit constructions of various extension operators and show how to prove their continuity.

Finally, Section 4 contains a general description of the whole construction of the bases.

We should mention that in [16] we sketched the earlier less elaborate version of the construction. At that time we could prove parts (A1), (A2) (or (A1′), (A2′)) of Theorem A. Also Theorem B was obtained only for (φ_n) and $0 < s < m$ (or else for (ψ_n) and $0 < s < m$).

Constructions of simultaneous Schauder bases in spaces of smooth functions on a manifold M were previously known in case where M is a product of 1-dimensional compact manifolds (cf. [14], [33]). It seems that our results are new even in the case where M is the 2-dimensional sphere or disk.

A novelty in our approach is also the fact that at no place in the proofs we use the methods of the complex functions theory.

2. Preliminaries on function spaces on subsets of $\boldsymbol{R}^d$. In this section we give some definitions and describe certain basic properties of spaces of Sobolev and Besov type defined on an open subset Ω of the space $\boldsymbol{R}^d$, $d \geqslant 1$. We are interested in properties of certain extension operators and in characterizations in terms of moduli of smoothness and interpolation spaces.

For the purposes of this paper the case where Ω is a bounded parallelepiped is the most important. We include, however, some results for more general sets. This is necessary if one wants to extend the results to the case of subsets of manifolds.

Actually, for some technical reasons, we often prefer to use closed sets in this context. Let us agree that if $\Omega = \operatorname{Int} F$, where F is a closed set and we have defined a function space $\mathscr{F}(\Omega)$, then $\mathscr{F}(F)$ will be just another notation for $\mathscr{F}(\Omega)$. (This convention will be used only for "good" sets F, e.g. always one has $F = \bar{\Omega}$ and $F \setminus \Omega$ has measure zero.) In particular, we do this when we quote below some results of H. Johnen and K. Scherer (their conditions on Ω are obviously satisfied when $\Omega = \operatorname{Int} F$, where $F \subset \boldsymbol{R}^d$ is compact and is a proper set in the sense of Definition 3.1 below).

Our definition of Sobolev and Besov spaces differs somewhat from the classical one. This assures that the spaces we consider are separable (which is necessary for the existence of Schauder bases). More precisely, the symbols W^k_∞ and $B^s_{p,\infty}$, where

$$k = 0, 1, \ldots, \quad s > 0, \quad 1 \leqslant p \leqslant \infty,$$

have a modified meaning (this amounts to taking the closure of smooth functions in the classical space).

Let Z_+ denote the set of all non-negative integers. For $\boldsymbol{a} = (a_1, \ldots, a_d) \in Z^d_+$ we let, as usual, $|\boldsymbol{a}| = a_1 + \ldots + a_d$ and

$$D^{\boldsymbol{a}} = \frac{\partial^{|\boldsymbol{a}|}}{\partial t_1^{a_1} \ldots \partial t_d^{a_d}}$$

(this may be the generalized or the classical partial derivative).

We write $\boldsymbol{a} \leqslant \boldsymbol{m}$ for $\boldsymbol{a}, \boldsymbol{m} \in Z^d_+$ iff $a_i \leqslant m_i$ for $i = 1, \ldots, d$. In $L_p(\Omega)$, $1 \leqslant p < \infty$, we have the norm

$$\|f\|_p(\Omega) = \Big(\int_\Omega |f|^p\, dx\Big)^{1/p}$$

and in $L_\infty(\Omega)$ one has

$$\|f\|_\infty(\Omega) = \operatorname*{ess\,sup}_\Omega |f|.$$

Let $C(\Omega)$ denote the space of all uniformly continuous functions on Ω which vanish at ∞. The *Sobolev spaces* $W^m_p(\Omega)$ and $W^{\boldsymbol{m}}_p(\Omega)$, where $m \in Z_+$, $\boldsymbol{m} \in Z^d_+$, $1 \leqslant p < \infty$, are defined in terms of the generalized derivatives

$$W^m_p(\Omega) = \{f \in L_p(\Omega)\colon D^{\boldsymbol{a}} f \in L_p(\Omega) \text{ for } \boldsymbol{a} \in Z_+, |\boldsymbol{a}| \leqslant m\},$$
$$W^{\boldsymbol{m}}_p(\Omega) = \{f \in L_p(\Omega)\colon D^{\boldsymbol{a}} f \in L_p(\Omega) \text{ if } \boldsymbol{0} \leqslant \boldsymbol{a} \leqslant \boldsymbol{m}\}.$$

Replacing in these formulae $L_p(\Omega)$ by $C(\Omega)$, we obtain the definition of $W^m_\infty(\Omega)$ and $W^{\boldsymbol{m}}_\infty(\Omega)$. The norm in these spaces is defined, for $m \in Z_+$, $\boldsymbol{m} \in Z^d_+$ and $1 \leqslant p \leqslant \infty$, by the formulae

$$\|f\|^{(m)}_p(\Omega) = \sum_{|\boldsymbol{a}| \leqslant m} \|D^{\boldsymbol{a}} f\|_p(\Omega),$$
$$\|f\|^{(\boldsymbol{m})}_p(\Omega) = \sum_{\boldsymbol{a} \leqslant \boldsymbol{m}} \|D^{\boldsymbol{a}} f\|_p(\Omega),$$

respectively.

We shall sometimes write C^m and $C^{\boldsymbol{m}}$ instead of W^m_∞ and $W^{\boldsymbol{m}}_\infty$.

The jth unit vector in $\boldsymbol{R}^d$ is denoted by $\boldsymbol{e}_j$ and the symbols $W^m_{p,i}$, $\| \; \|^{(m)}_{p,i}$ are used for $W^{m\boldsymbol{e}_i}_p$, $\| \; \|^{(m\boldsymbol{e}_i)}_p$.

The spaces $W^m_p(\Omega)$, $W^m_{p,i}(\Omega)$, where $m \in Z_+$, $1 \leqslant p \leqslant \infty$, $i = 1, \ldots, d$, are separable Banach spaces.

The modulus of smoothness of order $k \in Z_+$ in the direction $\boldsymbol{u} \in \boldsymbol{R}^d$ in $W^0_p(\Omega)$ is defined for $0 < \delta < D/k$, $D = \operatorname{diam} \Omega$, $1 \leqslant p \leqslant \infty$, as follows:

$$\omega^{(\boldsymbol{u})}_{k,p}(f; \delta)_\Omega = \sup_{0 < h \leqslant \delta} \|\Delta^k_{\boldsymbol{u}} f\|_p\big(\Omega(hk\boldsymbol{u})\big),$$

where we put

$$\Omega(\boldsymbol{u}) = \{x \in \Omega;\ x + \lambda \boldsymbol{u} \in \Omega \text{ for } 0 \leqslant \lambda \leqslant 1\},$$

$$\Delta^k_{\boldsymbol{u}} f(x) = \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} f(x + j\boldsymbol{u}).$$

The isotropic modulus of smoothness is defined as

$$\omega_{k,p}(f; \delta)_\Omega = \sup_{\|\boldsymbol{u}\| = 1} \omega^{(\boldsymbol{u})}_{k,p}(f; \delta)_\Omega,$$

where $\|u\|$ is the Euclidean norm in $\boldsymbol{R}^d$. Moreover, we put
$$\omega_{k,p}^{(i)} = \omega_{k,p}^{(e_i)}.$$
Now, let for $0 < s < k \in Z_+$, $1 \leqslant p, q \leqslant \infty$,
$$\|f\|_{p,q}^{(s)}(\Omega) = \|f\|_p(\Omega) + \left(\int_0^{D/k} \left(\frac{\omega_{k,p}(f;t)_\Omega}{t^s}\right)^q \frac{dt}{t}\right)^{1/q},$$
$$\|f\|_{p,q,i}^{(s)}(\Omega) = \|f\|_p(\Omega) + \left(\int_0^{D/k} \left(\frac{\omega_{k,p}^{(i)}(f;t)_\Omega}{t^s}\right)^q \frac{dt}{t}\right)^{1/q}$$
(cf. Corollary 2.6 below); if $q = \infty$, then the integral
$$\left(\int_0^{D/k} |\varphi(t)|^q \frac{dt}{t}\right)^{1/q}$$
is interpreted as $\operatorname*{ess\,sup}_{0<t<D/k} |\varphi(t)|$.

Define, for $0 < s < k$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$,
$$B_{p,q}^s(\Omega) = \{f \in W_p^0(\Omega):\ \|f\|_{p,q}^{(s)}(\Omega) < \infty\},$$
$$B_{p,q,i}^s(\Omega) = \{f \in W_p^0(\Omega):\ \|f\|_{p,q,i}^{(s)}(\Omega) < \infty\},$$
$$B_{p,\infty}^s(\Omega) = \{f \in W_p^0(\Omega):\ \omega_{k,p}(f;t)_\Omega = o(t^s)\},$$
$$B_{p,\infty,i}^s(\Omega) = \{f \in W_p^0(\Omega):\ \omega_{k,p}^{(i)}(f;t)_\Omega = o(t^s)\}.$$
Moreover, for $\boldsymbol{s} = (s_1, \ldots, s_d)$, $0 < s_i < k$, let
$$B_{p,q}^{\boldsymbol{s}}(\Omega) = \bigcap_{i=1}^d B_{p,q,i}^{s_i}(\Omega),$$
and let the norm be
$$\|f\|_{p,q}^{(\boldsymbol{s})}(\Omega) = \sum_{i=1}^d \|f\|_{p,q,j}^{(s_i)}(\Omega).$$
The spaces $B_{p,q}^s$, $B_{p,q,i}^s$, $B_{p,q}^{\boldsymbol{s}}$ are separable Banach spaces for $1 \leqslant p, q \leqslant \infty$, $s > 0$.

Let $I \subset \boldsymbol{R}$ be a compact interval and let X be a Banach space. The Sobolev spaces $W_p^m(I;X)$ of X-valued functions on I, where $m \in Z_+$, $1 \leqslant p \leqslant \infty$, are defined similarly as $W_p^m(I)$ (cf. [30]). (Instead of $L_p(I)$ one uses $L_p(I;X)$, the space of strongly measurable p-integrable functions, and $C(I)$ is replaced by $C(I;X)$, the space of strongly continuous X-valued functions on I.) The norm in $W_p^m(I;X)$ is defined as in the scalar case and is denoted by $\|\ \|_p^m(I;X)$. We define the modulus of smoothness of order k for a function $f \in W_p^0(I,X)$ as in the real-valued case (here $d = 1$) and denote it by $\omega_{k,p}(f;X;\delta)_I$.

Let $Q = I_1 \times \ldots \times I_d$ be a closed parallelepiped in $\boldsymbol{R}^d$. For any j, $1 \leqslant j \leqslant d$, we can identify Q with the product $I \times Q_0$, where $I = I_j$ and $Q_0 = \prod_{i \neq j} I_i$. Then the space $W_{p,j}^m(Q)$ can be identified with $W_p^m(I;X)$, where $X = \boldsymbol{R}$ if $d = j = 1$ and $X = W_p^0(Q_0)$ otherwise. We write
$$W_{p,j}^m(Q) = W_p^m\big(I;\ W_p^0(Q_0)\big).$$
Also one has
$$\omega_{k,p}^{(j)}(f;\delta)_Q = \omega_{k,p}\big(f;\ W_p^0(Q_0);\ \delta\big)_I.$$
Hence the formula, for $0 < s < k$, $1 \leqslant p, q \leqslant \infty$,
$$B_{p,q,j}^s(Q) = B_{p,q}^s\big(I;\ W_p^0(Q_0)\big)$$
makes sense with the obvious meaning of the right-hand side.

Now we are ready to state the Marchaud type inequalities in both the isotropic and the anisotropic cases.

Proposition 2.1. *Let $Q \subset \boldsymbol{R}^d$ be a compact parallelepiped of the form $Q = I \times Q_0$, where $I \subset \boldsymbol{R}$. Moreover, let $X = W_p^0(Q_0)$ (in case $d = 1$: $Q = I$ and $X = \boldsymbol{R}$). Then there is a constant $C = C(I, k, m)$ such that for $0 < \delta < c = |I|/(k+m)$, $k, m \geqslant 1$,*

(2.2) $$\omega_{k,p}(f;X;\delta)_I := \omega_{k,p}(f;X;\delta)_{\mathrm{Int}\,I} \leqslant C\delta^k \left(\|f\|_p(I;X) + \int_\delta^c \frac{\omega_{k+m,p}(f;X;t)_I}{t_{k+1}}\,dt\right)$$

holds for $f \in W_p^0(I;X)$ with $\|f\|_p(I;X) = \|f\|_p(Q)$, $1 \leqslant p \leqslant \infty$.

The proof of Proposition 2.1 as presented for $d = 1$ in Johnen [27], Proposition 3.1, can be easily adapted to our vector-valued situation.

Proposition 2.3. *Let F be a compact proper set in $\boldsymbol{R}^d$ and let $c = \operatorname{diam} F/(k+m)$, $1 \leqslant p \leqslant \infty$. Then there is a constant $C = C(F, k, m)$ such that for $0 < \delta < c$ and $f \in W_p^0(F)$ we have*

(2.4) $$\omega_{k,p}(f;\delta)_F := \omega_{k,p}(f;\delta)_{\mathrm{Int}\,F} \leqslant C\delta^k \left(\|f\|_p(F) + \int_\delta^c \frac{\omega_{k+m,p}(f;t)_F}{t^{k+1}}\,dt\right).$$

This result is proved in Johnen–Scherer [28], Theorems 1 and 2. The following Hardy's inequality appears to be useful. For measurable $\varphi \geqslant 0$ and for $0 < s < k$, $1 \leqslant q \leqslant \infty$, we have

(2.5) $$\left(\int_0^\infty \left(\frac{1}{t^s}\, t^k \int_t^\infty \frac{\varphi(u)}{u^{k+1}}\,du\right)^q \frac{dt}{t}\right)^{1/q} \leqslant \frac{1}{k-s} \left(\int_0^\infty \left(\frac{\varphi(u)}{u^s}\right)^q \frac{du}{u}\right)^{1/q}.$$

This follows from (9.9.9) in [24]. Applying this inequality we obtain

COROLLARY 2.6. *Let $F \subset \mathbf{R}^d$ be a compact proper set in $\mathbf{R}^d$ and let $Q \subset \mathbf{R}^d$ be a parallelepiped. Let $s > 0$ and let $k_1, k_2 > s$ be integers. Denote by $B^s_{p,q}(F)_j$ and $B^s_{p,q,i}(Q)_j$, $i = 1, \ldots, d$, the Besov spaces corresponding to the integer k_j for $j = 1, 2$. Then one has, for $1 \leqslant p, q \leqslant \infty$, $i = 1, \ldots, d$,*

$$B^s_{p,q}(F)_1 = B^s_{p,q}(F)_2,$$

$$B^s_{p,q,i}(Q)_1 = B^s_{p,q,i}(Q)_2,$$

i.e. the respective sets are equal, the corresponding norms being equivalent.

In the sequel we often make use of extension operators. In case of function spaces on proper sets one may apply Stein's extension theorem (Theorem 6.5 in [35]). In some situations extension operators with desired additional properties can be custom-made using the method of Hestenes [25]. We shall call them *H-operators*. Here is the simplest case.

For a given $m \in Z_+$ and $0 < a < \infty$, let $I = \langle 0, a \rangle$ and $J = \langle -a, a \rangle$. Moreover, let $Q = I \times Q_0$ be a parallelepiped in $\mathbf{R}^d$ and let $X = W^0_p(Q_0)$ if $d > 1$ and $X = \mathbf{R}$ if $d = 1$. Now, for $f \in W^0_p(I; X)$, the Hestenes extension operator is defined by the formula

$$Tf(t) = \begin{cases} f(t) & \text{for } t \in I, \\ \sum_{j=0}^{m} c_j f(-t2^{-j}) & \text{for } t \in J \setminus I, \end{cases} \tag{2.7}$$

where the c_j's are the solution of

$$\sum_{j=0}^{m} c_j(-1/2)^{ij} = 1, \quad i = 0, \ldots, m.$$

Computing the generalized derivatives (or the classical ones if $p = \infty$) of Tf we find that $Tf \in W^k_p(J; X)$ if $f \in W^k_p(I; X)$, $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$ (cf. [2]). In fact one has the following easy estimate.

LEMMA 2.8. *There is a constant $C = C(a, m)$ such that the operator T defined in (2.7) satisfies*

$$\|Tf\|^{(k)}_p(J; X) \leqslant C\|f\|^{(k)}_p(I; X)$$

for $f \in W^k_p(I; X)$, $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$.

Repeating step by step the argument presented in the proof of Proposition 5.1 of [27], but adapted to X-valued functions we obtain

PROPOSITION 2.9. *The extension operator defined as in (2.7) preserves the vector-valued ($X = W^0_p(Q_0)$) modulus of smoothness of order m. More precisely, there is a constant $C = C(a, m)$ such that*

$$\omega_{m,p}(Tf; X; \delta)_J \leqslant C\omega_{m,p}(f; X; \delta)_I \tag{2.10}$$

holds for $m \geqslant 1$, $0 < \delta \leqslant |I|/m$, $1 \leqslant p \leqslant \infty$.

Similarly, adapting the proof of Proposition 5.2 of [27] to X-valued functions, we get

PROPOSITION 2.11. *Let $Q \subset \mathbf{R}^d$ be a compact parallelepiped, $Q = I \times Q_0$, where $I \subset \mathbf{R}$. Let $J \supset I$ be a compact interval. Put $X = W^0_p(Q_0)$ if $d > 1$, and $X = \mathbf{R}$ if $d = 1$. Let $m \in Z_+$.*

Then there are extension operators

$$T\colon W^0_p(I; X) \to W^0_p(J; X), \qquad T_0\colon W^0_p(I; X) \to W^0_p(\mathbf{R}; X)$$

and a constant $C = C(m, I, J)$ such that for $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$, $f \in W^k_p(I, X)$ one has

$$\|Tf\|^{(k)}_p(J; X) \leqslant C\|f\|^{(k)}_p(I; X),$$

$$\|T_0 f\|^{(k)}_p(\mathbf{R}; X) \leqslant C\|f\|^{(k)}_p(I; X)$$

and, if $f \in W^0_p(I; X)$, $0 < \delta < |I|/m$, then

$$\omega_{m,p}(Tf; X; \delta)_J \leqslant C\omega_{m,p}(f; X; \delta)_I, \tag{2.12}$$

$$\omega_{m,p}(T_0 f; X; \delta)_{\mathbf{R}} \leqslant C\big(\delta^m\|f\|_p(I; X) + \omega_{m,p}(f; X; \delta)_I\big). \tag{2.13}$$

LEMMA 2.14. *Let Q and $S \supset Q$ be compact parallelepipeds in $\mathbf{R}^d$, and let $1 \leqslant p \leqslant \infty$, $m \in Z_+$. Then there are extension operators*

$$T\colon W^0_p(Q) \to W^0_p(S), \qquad T_0\colon W^0_p(Q) \to W^0_p(\mathbf{R}^d)$$

and constants $C = C(m, d, Q, S)$, $C_0 = C(m, d, Q)$ such that

$$\begin{aligned} &\|Tf\|^{(k)}_{p,j}(S) \leqslant C\|f\|^{(k)}_{p,j}(Q), \\ &\|T_0 f\|^{(k)}_{p,j}(\mathbf{R}^d) \leqslant C_0\|f\|^{(k)}_{p,j}(Q) \end{aligned} \tag{2.15}$$

hold for $f \in W^k_p(Q)$, $k = 0, \ldots, m$; $j = 1, \ldots, d$, and

$$\begin{aligned} &\omega^{(j)}_{m,p}(Tf; \delta)_S \leqslant C\omega^{(j)}_{m,p}(f; \delta)_Q, \\ &\omega^{(j)}_{m,p}(T_0 f; \delta)_{\mathbf{R}^d} \leqslant C_0\big(\delta^m\|f\|_p(Q) + \omega^{(j)}_{m,p}(f; \delta)_Q\big) \end{aligned} \tag{2.16}$$

hold for $f \in W^0_p(Q)$, $0 < \delta m < \operatorname{diam} Q$, $j = 1, \ldots, d$.

Proof. We shall consider the case of Q, S only. The proof in the case of Q, $\mathbf{R}^d$ is similar and it is omitted. Inequalities (2.15) follow from the construction of T and T_0. Let now

$$Q = I_1 \times \ldots \times I_d, \qquad S = J_1 \times \ldots \times J_d,$$

$$Q^{(i)} = J_1 \times \ldots \times J_{i-1} \times I_i \times \ldots \times I_d = Q^{(i)}_i \times I_i,$$

i.e.

$$Q^{(i)}_i = J_1 \times \ldots \times J_{i-1} \times I_{i+1} \times \ldots \times I_d.$$

Moreover, let $X_i = W_p^0(Q_i^{(i)})$ and let

$$T_i\colon\ W_p^0(Q^{(i)}) = W_p^0(I_i; X_i) \to W_p^0(J_i; X_i) = W_p^0(Q^{(i+1)})$$

be an extension operator given as in Proposition 2.11. We are going to check the following inequality:

$$\omega_{m,p}^{(j)}(T_i f;\ \delta)_{Q^{(i+1)}} \leqslant C\omega_{m,p}^{(j)}(f;\ \delta)_{Q^{(i)}}, \qquad i, j = 1, \ldots, d. \tag{2.17}$$

In case $i = j$ it is a consequence of (2.12). Now let $i \neq j$ and let, for $h > 0$, g_h denote the characteristic function of the set $Q^{(i+1)}(mhe_j)$. Then

$$Q^{(i+1)}(he_j) \cap Q^{(i)} = Q^{(i)}(he_j)$$

and

$$(\Delta_{he_j}^m T_i f) g_h = T_i\big((\Delta_{he_j}^m f) g_h\big).$$

Therefore, by Proposition 2.11, we get

$$\begin{aligned}\omega_{m,p}^{(j)}(T_i f;\ \delta)_{Q^{(i+1)}} &= \sup_{0<h<\delta} \big\|T_i\big((\Delta_{he_j}^m f) g_h\big)\big\|_p (Q^{(i+1)}) \\ &\leqslant C \sup_{0<h<\delta} \|\Delta_{he_j}^m f\|_p \big(Q^{(i+1)}(mhe_j) \cap Q^{(i)}\big) \\ &= C\omega_{m,p}^{(j)}(f;\ \delta)_{Q^{(i)}}.\end{aligned}$$

Now, defining $T = T_d \circ \ldots \circ T_1$ we obtain by repeated application of (2.17) the first inequality in (2.16), and this completes the proof.

In the isotropic case there is an extension theorem for proper sets in $\boldsymbol{R}^d$. Namely, in our notation, Johnen and Scherer proved in [28]:

PROPOSITION 2.18. *Let F_1 and $F_2 \supset F_1$ be compact proper sets in $\boldsymbol{R}^d$, and let $1 \leqslant p \leqslant \infty$, $m \in Z_+$. Then there are an extension operator $E\colon W_p^0(F_1) \to W_p^0(F_2)$ and a constant $C = C(m, d, F_1, F_2)$ such that*

$$\|Ef\|_p^{(k)}(F_2) \leqslant C\|f\|_p^{(k)}(F_1) \tag{2.19}$$

holds for $f \in W_p^k(F_1)$, $k = 0, \ldots, m$, and

$$\omega_{m,p}(Ef;\ \delta)_{F_2} \leqslant C\omega_{m,p}(f;\ \delta)_{F_1} \tag{2.20}$$

holds for $f \in W_p^0(F_1)$, $0 < \delta < \operatorname{diam} F_1$.

This result can be extended to the case $F_2 = \boldsymbol{R}^d$. For $d = 1$ it follows from Lemma 2.14, and for $d > 1$ we have

LEMMA 2.21. *Let F be a compact proper set in $\boldsymbol{R}^d$ and let $m \in Z_+$. Then there is an extension operator $E_0\colon W_p^0(F) \to W_p^0(\boldsymbol{R}^d)$ such that (2.19) holds with $F_1 = F$ and $F_2 = \boldsymbol{R}^d$.*

Moreover, there are constants $C_p = C(p, m, d, F)$ and $C = C(m, d, F)$ such that for $f \in W_p^0(F)$ one has

$$\omega_{m,p}(E_0 f;\ \delta)_{\boldsymbol{R}^d} \leqslant C_p[\delta^m \|f\|_p(F) + \omega_{m,p}(f;\ \delta)_F] \tag{2.22}$$

if $1 < p < \infty$, and

$$\omega_{m,p}(E_0 f;\ \delta)_{\boldsymbol{R}^d} \leqslant C\delta^m \Big[\|f\|_p(F) + \int_\delta^c \frac{\omega_{m,p}(f;\ t)_F}{t^{m+1}}\, dt\Big] \tag{2.23}$$

if $1 \leqslant p \leqslant \infty$; $0 < \delta < c = \operatorname{diam} F/m$.

Proof. Let us take an auxiliary parallelepiped $Q \supset F$. Let E be the extension operator from Proposition 2.18 corresponding to $F_1 = F$ and $F_2 = Q$. Moreover, let T_0 be the extension operator from Lemma 2.14, and let $E_0 = T_0 E$. On the one hand we have the trivial inequalities

$$\omega_{m,p}^{(j)} \leqslant \omega_{m,p}, \qquad j = 1, \ldots, d. \tag{2.24}$$

On the other hand we have the following two inequalities proved by Boman (cf. [5], Theorem 6.3 and Corollary 2.4, respectively):

There are constants $C_p' = C(m, d, p)$ and $C'' = C(m, d)$ such that for $f \in W_p^0(\boldsymbol{R}^d)$, $0 < \delta < \infty$

$$\omega_{m,p}(f;\ \delta)_{\boldsymbol{R}^d} \leqslant C_p' \sum_{j=1}^d \omega_{m,p}^{(j)}(f;\ \delta)_{\boldsymbol{R}^d} \tag{2.25}$$

holds for $1 < p < \infty$, and

$$\omega_{m,p}(f;\ \delta)_{\boldsymbol{R}^d} \leqslant C''\delta^m \Big[\|f\|_p(\boldsymbol{R}^d) + \int_\delta^\infty t^{-m-1}\Big(\sum_{j=1}^d \omega_{m,p}^{(j)}(f;\ t)_{\boldsymbol{R}^d}\Big) dt\Big] \tag{2.26}$$

holds for $1 \leqslant p \leqslant \infty$.

Now, using (2.25), (2.16), (2.24), (2.19) and (2.20), we get

$$\begin{aligned}\omega_{m,p}(E_0 f;\ \delta)_{\boldsymbol{R}^d} &\leqslant C_p \sum_{j=1}^d \omega_{m,p}^{(j)}(E_0 f;\ \delta)_{\boldsymbol{R}^d} \\ &\leqslant C_p\Big[\delta^m \|Ef\|_p(Q) + \sum_{j=1}^d \omega_{m,p}^{(j)}(Ef;\ \delta)_Q\Big] \\ &\leqslant dC_p[\delta^m \|Ef\|_p(Q) + \omega_{m,p}(Ef;\ \delta)_Q] \\ &\leqslant C_p[\delta^m \|f\|_p(F) + \omega_{m,p}(f;\ \delta)_F]\end{aligned}$$

which proves (2.22). In order to prove (2.23) we use (2.26) instead of (2.25).

Using now Hardy's inequality (2.5) and Lemmas 2.14, 2.21, we obtain

COROLLARY 2.27. *Let $Q \subset \boldsymbol{R}^d$ be a parallelepiped and let F be a compact proper set in $\boldsymbol{R}^d$. Then for any $m \in Z_+$ there is $C < \infty$ such that, if $1 \leqslant p, q \leqslant \infty$, $j = 1, \ldots, d$; $k = 0, \ldots, m$; $0 < s < m$, one has for each f*

$$\|T_0 f\|_{p,j}^{(k)}(\boldsymbol{R}^d) \leqslant C\|f\|_{p,j}^{(k)}(Q),$$

$$\|T_0 f\|_{p,q,j}^{(s)}(\boldsymbol{R}^d) \leqslant C\|f\|_{p,q,j}^{(s)}(Q),$$

$$\|E_0 f\|_{p}^{(k)}(\boldsymbol{R}^d) \leqslant C\|f\|_{p}^{(k)}(F),$$

$$\|E_0 f\|_{p,q}^{(s)}(\boldsymbol{R}^d) \leqslant C\|f\|_{p,q}^{(s)}(F).$$

Here T_0 and E_0 are given as in Lemmas 2.14 and 2.21, respectively.

Consequently, if $\mathscr{F}$ is one of the symbols $W_{p,j}^k$, $B_{p,q,j}^s$, then $\mathscr{F}(Q)$ is naturally isomorphic to the quotient Banach space of equivalence classes in $\mathscr{F}(\boldsymbol{R}^d)$ where two functions are in the same class if they are equal a.e. on Q. Similarly, if $\mathscr{F}$ is one of the symbols W_p^k, $B_{p,q}^s$, then $\mathscr{F}(F)$ is naturally isomorphic to the quotient space of equivalence classes in $\mathscr{F}(\boldsymbol{R}^d)$ where two functions are identified if they are equal a.e. on F.

THEOREM 2.28. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $d \geqslant 1$, $0 < s < m$, and let $Q \subset \boldsymbol{R}^d$ be a parallelepided. Then*

$$B_{p,q}^s(Q) = B_{p,q}^{(s,\ldots,s)}(Q),$$

$$B_{p,q}^s(\boldsymbol{R}^d) = B_{p,q}^{(s,\ldots,s)}(\boldsymbol{R}^d)$$

and the norms in these spaces are equivalent.

Proof. In the case of $\boldsymbol{R}^d$ the theorem follows by (2.24) and by Boman's inequality (2.26) in combination with Hardy's inequality (2.5). Let us consider the case of Q. Then according to (2.24) the inclusion $B_{p,q}^s(Q) \subseteq B_{p,q}^{(s,\ldots,s)}(Q)$ and the corresponding inequality for the norms follow immediately. To get the opposite inequality for the norms we use the part of the theorem just proved and Corollary 2.27:

$$\|f\|_{p,q}^{(s)}(Q) = \|T_0 f\|_{p,q}^{(s)}(Q)$$

$$\leqslant \|T_0 f\|_{p,q}^{(s)}(\boldsymbol{R}^d) \leqslant C\sum_{j=1}^{d}\|T_0 f\|_{p,q,j}^{(s)}(\boldsymbol{R}^d)$$

$$\leqslant C'\sum_{j=1}^{d}\|f\|_{p,q,j}^{(s)}(Q) = C'\|f\|_{p,q}^{(s,\ldots,s)}(Q).$$

Remark. Theorem 2.28 in the case of $\boldsymbol{R}^d$ was established earlier in a different way by V. A. Solonnikov [34]. In what follows we describe the spaces $B_{p,q}^s(F)$, $B_{p,q,j}^s(Q)$, $j = 1, \ldots, d$, as interpolation spaces, with F being a compact proper set in $\boldsymbol{R}^d$ and $Q = I_1 \times \ldots \times I_d$ being a parallelepiped. Fix $m > s$. Note that in the spaces $W_p^m(F)$ and $W_{p,j}^m(Q)$, $j = 1, \ldots, d$, one can use, respectively, the following norms

$$\|f\|_p^{(m)}(F)^* = \|f\|_p(F) + \sum_{|\alpha|=m}\|D^\alpha f\|_p,$$

$$\|f\|_{p,j}^{(m)}(Q)^* = \|f\|_p(Q) + \|D^{me_j} f\|_p(Q).$$

The equivalence of the first norm with $\|\ \|_p^{(m)}(F)$ follows e.g. from Theorem 4.14 in [1].

The equivalence of the second norm with $\|\ \|_{p,j}^{(m)}(Q)$ follows from [32] (cf. p. 164, inequalities (7), where the constants are uniformly bounded in p: $1 \leqslant p \leqslant \infty$, and therefore they can be taken as independent of p). To describe the real interpolation spaces (cf. [3]) for the pairs $\big(W_p^0(F), W_p^m(F)\big)$ and $\big(W_p^0(Q), W_{p,j}^m(Q)\big)$ we use the modified Peetre functionals:

$$K_p'(t,f) \equiv K_p'\big(t,f;\, W_p^0(F), W_p^m(F)\big) = \inf\Big\{\|f-g\|_p(F) + t\sum_{|\alpha|=m}\|D^\alpha g\|_p(F):\ g \in W_p^m(F)\Big\},$$

$$K_{p,j}'(t,f) \equiv K_{p,j}'\big(t,f;\, W_p^0(Q), W_{p,j}^{me_j}(Q)\big) = \inf\{\|f-g\|_p(Q) + t\|D^{me_j} g\|_p(Q):\ g \in W_{p,j}^m(Q)\}.$$

We need the Peetre functionals as well:

$$K_p(t,f) \equiv K_p\big(t,f;\, W_p^0(F), W_p^m(F)\big) = \inf\{\|f-g\|_p(F) + t\|g\|_p^{(m)}(F):\ g \in W_p^m(F)\},$$

$$K_{p,j}(t,f) \equiv K_p\big(t,f;\, W_p^0(Q), W_{p,j}^m(Q)\big) = \inf\{\|f-g\|_p(Q) + t\|g\|_{p,j}^{(m)}(Q):\ g \in W_{p,j}^m(Q)\}.$$

PROPOSITION 2.29. *Let $Q = I_1 \times \ldots \times I_d$, $m \in Z_+$ and let a compact proper set F in $\boldsymbol{R}^d$ be given. Then there are constants $C_j = C(m, |I_j|)$ and $C_0 = C(d, m, F)$ such that we have: for $j = 1, \ldots, d$ and $0 < \delta < |I_j|/m$*

$$C_j^{-1}\omega_{m,p}^{(j)}(f;\delta)_Q \leqslant K_{p,j}'(\delta^m, f) \leqslant C_j\,\omega_{m,p}^{(j)}(f;\delta)_Q, \tag{2.30}$$

and for $0 < \delta \leqslant \operatorname{diam} F/m$

$$C_0^{-1}\omega_{m,p}(f;\delta)_F \leqslant K_p'(\delta^m, f) \leqslant C_0\,\omega_{m,p}(f;\delta)_F. \tag{2.31}$$

Both cases hold uniformly in p, $1 \leqslant p \leqslant \infty$.

Proof. Let $Q = Q_{(j)} \times I_j$ and $X_j = W_p^0(Q_{(j)})$. We know that

$$\omega_{m,p}^{(j)}(f;\delta)_Q = \omega_{m,p}(f; X_j; \delta)_{I_j}.$$

Using now (2.12) we can adjust to the vector-valued case the proof of (2.30) as given in Johnen [27] for the real-valued case. Inequalities (2.31) are proved in Johnen and Scherer [28].

PROPOSITION 2.32. *Using the notation of Proposition* 2.29 *we have for the Peetre functionals the following inequalities with the same ranges of parameters*:

$$C_j^{-1}[\delta^m\|f\|_p(Q)+\omega_{m,p}^{(j)}(f;\delta)_Q] \leqslant K_{p,j}(\delta^m,f)\leqslant C_j[\delta^m\|f\|_p(Q)+\omega_{m,p}^{(j)}(f;\delta)_Q], \tag{2.33}$$

$$C_0^{-1}[\delta^m\|f\|_p(F)+\omega_{m,p}(f;\delta)_F] \leqslant K_p(\delta^m;f)\leqslant C_0[\delta^m\|f\|_p(F)+\omega_{m,p}(f;\delta)_F]. \tag{2.34}$$

Proof. The equivalence of the norms $\|\ \|_{p,j}^{(m)}(Q)$ and $\|\ \|_{p,j}^{(m)}(Q)^*$, and (2.30) imply inequalities (2.33). Similarly, the equivalence of $\|\ \|_p^{(m)}(F)$ and $\|\ \|_p^{(m)}(F)^*$, and (2.31) give (2.34).

For given parameters p, q, j and θ $(1\leqslant p, q\leqslant\infty, 0<\theta<1)$ we introduce the norms

$$\begin{aligned}\|f\|_{p,q,j}^{\langle\theta\rangle}(Q) &= \left(\int_0^\infty\left(\frac{K_{p,j}(t,f)}{t^\theta}\right)^q\frac{dt}{t}\right)^{1/q},\\ \|f\|_{p,q}^{\langle\theta\rangle}(F) &= \left(\int_0^\infty\left(\frac{K_p(t,f)}{t^\theta}\right)^q\frac{dt}{t}\right)^{1/q}\end{aligned} \tag{2.35}$$

with the usual convention in the case $q=\infty$. The corresponding real interpolation spaces are now defined for finite q as follows:

$$\big(W_p^0(Q), W_{p,j}^m(Q)\big)_{\theta,q} = \{f\in W_p^0(Q)\colon \|f\|_{p,q,j}^{\langle\theta\rangle}(Q)<\infty\},$$
$$\big(W_p^0(F), W_p^m(F)\big)_{\theta,q} = \{f\in W_p^0(F)\colon \|f\|_{p,q}^{\langle\theta\rangle}(F)<\infty\}.$$

Moreover, for $q=\infty$ we define

$$\big(W_p^0(Q), W_{p,j}^m(Q)\big)_{\theta,q} = \{f\in W_p^0(Q)\colon K_{p,j}(t,f) = o(t^\theta)\ \text{ as } t\to 0_+\},$$
$$\big(W_p^0(F), W_p^m(F)\big)_{\theta,q} = \{f\in W_p^0(F)\colon K_p(t,f) = o(t^\theta)\ \text{ as } t\to 0_+\}$$

(this corresponds to the space $(\cdot,\cdot)^\circ_{\theta,q}$ in [3] and [17]).

THEOREM 2.36. *Assume that* $1\leqslant p\leqslant\infty$, $1\leqslant q\leqslant\infty$, $m-1\in Z_+$, $d\in Z_+$, $0<s<m$, $\theta=s/m$, $j=1,\ldots,d$. *Moreover, let F be a compact proper set in* $\boldsymbol{R}^d$ *and let Q be a parallelepiped in* $\boldsymbol{R}^d$. *Then the following sets are equal:*

$$B_{p,q,j}^s(Q) = \big(W_p^0(Q), W_{p,j}^m(Q)\big)_{\theta,q},$$
$$B_{p,q}^s(F) = \big(W_p^0(F), W_p^m(F)\big)_{\theta,q}.$$

Moreover, the norms given in (2.35) *and* $\|\ \|_{p,q,j}^{(s)}(Q)$, $\|\ \|_{p,q}^{(s)}(F)$ *are equivalent, respectively, uniformly in p and q.*

Proof. Use the definitions of norms in Besov spaces and apply Proposition 2.32.

Let $F\subset\boldsymbol{R}^d$ be a compact proper set. Denote, for $k\in Z_+$ and $1\leqslant p\leqslant\infty$, by $\mathring{W}_p^k(F)$ the closure in $W_p^k(F)$ of those f such that $\overline{\operatorname{supp} f}\subset \operatorname{Int} F$. If k is a negative integer and $1\leqslant p\leqslant\infty$, put for $g\in W_p^0(F)$

$$\|g\|_p^{(k)}(F) = \sup_f\Big|\int_F fg\,dx\Big|,$$

where the sup is extended over all $f\in\mathring{W}_{p'}^{-k}(F)$ with $\|f\|_{p'}^{(-k)}(F)\leqslant 1$ (here and in the sequel we put $p'=p/(p-1)$ for $1<p<\infty$, $1'=\infty$, $\infty'=1$).

The completion of $W_p^0(F)$ in the norm $\|\ \|_p^{(k)}(F)$, where k is a negative integer and $1\leqslant p\leqslant\infty$, will be denoted by $W_p^k(F)$.

Now we shall define the spaces $\mathscr{F}(Q)_Z$, where $\mathscr{F}$ denotes either W_p^k or $B_{p,q}^s$. (In [16] these spaces were denoted by $\mathscr{F}(C,Z)$.)

Fix $d\geqslant 1$, let $I=\langle 0,1\rangle$ and let $Q=I^d$ be the standard d-dimensional cube. Let $Z\subset Q$ be a union of closed $(d-1)$-dimensional faces of Q, i.e. Z is of the form

$$Z = Q\setminus(I\setminus Z_1)\times\ldots\times(I\setminus Z_d), \tag{2.37}$$

where $Z_j\subseteq\{0,1\}$ for $j=1,\ldots,d$. Put

$$\begin{aligned} I_\varnothing &= \langle 0,1\rangle, & I_{\{0\}} &= \langle -1,1\rangle,\\ I_{\{1\}} &= \langle 0,2\rangle, & I_{\{0,1\}} &= \langle -1,2\rangle \end{aligned} \tag{2.38}$$

and consider the parallelepiped

$$Q_Z = I_{Z_1}\times\ldots\times I_{Z_d}.$$

For any measurable set E, let $L_0(E)$ denote the space of (equivalence classes of) measurable functions on E equipped with the topology of convergence in measure on any compact subset of E.

If $f\in L_0(Q)$, we denote by f_Z the element of $L_0(Q_Z)$ such that $f_Z|_Q = f$ and $f_Z=0$ on $Q_Z\setminus Q$.

Define for $f\in W_p^0(Q)$, $1\leqslant p\leqslant\infty$, and all integers k

$$\|f\|_p^{(k)}(Q)_Z = \|f_Z\|_p^{(k)}(Q_Z).$$

Put for $k\geqslant 0$, $1\leqslant p\leqslant\infty$,

$$W_p^k(Q)_Z = \{f\in W_p^0(Q)\colon f_Z\in W_p^k(Q_Z)\}.$$

Observe that $W_p^k(Q)_Z$ is complete, being isometric to a closed subspace of $W_p^k(Q_Z)$ (via the map $f\mapsto f_Z$). Since for $f\in W_p^k(Q)_Z$ one has

$$\|f\|_p^{(k)}(Q)_Z = \|f\|_p^{(k)}(Q),$$

$W_p^k(Q)_Z$ is a closed subspace of $W_p^k(Q)$.

Now let $k<0$. Since $\|\ \|_p^{(k)}(Q)_Z$ is weaker than $\|\ \|_p(Q)$, we let $W_p^k(Q)_Z$ be the completion of $W_p^0(Q)$ with respect to the norm $\|\ \|_p^{(k)}(Q)_Z$. Clearly, the map $f\mapsto f_Z$ extends again to an isometry of $W_p^k(Q)_Z$ into $W_p^k(Q_Z)$.

A projection onto this subspace can be constructed using the following:

PROPOSITION 2.39. *Let $m\geqslant 1$. Then there exist a continuous linear operator P in $L_0(Q_Z)$ and $C<\infty$ such that*

(2.40) *P projects $L_0(Q_Z)$ down onto the subspace $\{f\colon f=0$ a.e. on $Q_Z\setminus Q\}$,*

(2.41) *for each $f\in W_p^k(Q_Z)$, $0\leqslant k\leqslant m$, $1\leqslant p\leqslant\infty$, one has*

$$\|Pf\|_p^{(k)}(Q_Z)\leqslant C\|f\|_p^{(k)}(Q_Z),$$

(2.42) *the (formally) adjoint operator P^* is continuous in $L_0(Q_Z)$ and for each $f\in W_p^k(Q_Z)$, $0\leqslant k\leqslant m$, $1\leqslant p\leqslant\infty$, one has*

$$\|P^*f\|_p^{(k)}(Q_Z)\leqslant C\|f\|_p^{(k)}(Q_Z),$$

$$P^*f|_Q=f|_Q \quad \text{for} \quad f\in L_0(Q_Z), \tag{2.43}$$

(2.44) *if $f\in L_0(Q_Z)$ and $f=0$ a.e. on $U\cap Q$, where U is an open set containing ∂Q_Z, then $P^*f=0$ a.e. in a neighborhood of ∂Q_Z.*

This proposition will be proved in Section 6. Now we can deduce:

COROLLARY 2.45. *Let P be the operator of Proposition 2.39. Then also for $-m\leqslant k<0$, if $g\in W_p^0(Q_Z)$, $1\leqslant p\leqslant\infty$, one has*

$$\|Pg\|_p^{(k)}(Q_Z)\leqslant C\|g\|_p^{(k)}(Q_Z).$$

Consequently, for $-m\leqslant k\leqslant m$, $1\leqslant p\leqslant\infty$, P induces a continuous linear projection from $W_p^k(Q_Z)$ onto its subspace naturally isomorphic to $W_p^k(Q)_Z$ (via the map $f\mapsto f_Z$).

Proof. The estimate follows from (2.42) and (2.44). The second statement follows from the first one, (2.41) and (2.40).

Remark 2.46. It is not difficult to produce another operator, say T, which, for each $k\geqslant 0$ and $1\leqslant p\leqslant\infty$, is a continuous projection from $W_p^k(Q_Z)$ onto $\{f\colon f=0$ a.e. on $Q_Z\setminus Q\}$. It suffices to put

$$Tf=f-E(f|_{Q_Z\setminus Q})|_{Q_Z},$$

where E is a Stein extension operator corresponding to the set $\operatorname{Int}Q_Z\setminus Q$.

If Z is given by (2.37), we put

$$Z'=\overline{\partial Q\setminus Z}. \tag{2.47}$$

Observe that Z' is of the form (2.37) (with $Z_1,\ldots,Z_d$ replaced by $Z_1',\ldots,Z_d'$).

Clearly, if $k\leqslant 0$, $1\leqslant p\leqslant\infty$, then each element $g\in W_p^0(Q)$ defines a functional $g^*\in\big(W_{p'}^{-k}(Q)_{Z'}\big)^*$ by the formula

$$g^*(f)=\int_Q fg\,dx.$$

LEMMA 2.48. *For each $k\leqslant 0$, $1\leqslant p\leqslant\infty$, the map $g\mapsto g^*$ extends to a linear topological isomorphism of $W_p^k(Q)_Z$ onto a subspace of $\big(W_{p'}^{-k}(Q)_{Z'}\big)^*$.*

Proof. It suffices to prove that there is a $C<\infty$ such that for $g\in W_p^0(Q)$ one has

$$\|g\|_p^{(k)}(Q)_Z\leqslant\|g^*\|\leqslant C\|g\|_p^{(k)}(Q)_Z. \tag{2.49}$$

The lower estimate follows from the fact that the operation $Rf=f|_Q$ maps $\mathring{W}_{p'}^{-k}(Q_Z)$ into $W_{p'}^{-k}(Q)_{Z'}$ and has norm $\leqslant 1$.

To prove the upper estimate it suffices to know that R is onto. Proposition 2.39 yields a stronger fact, namely there exists a continuous linear map $E\colon W_{p'}^{-k}(Q)_{Z'}\to\mathring{W}_{p'}^{-k}(Q_Z)$ such that $REf=f$ for all f. This implies that (2.49) holds with $C\leqslant\|E\|$.

Put $Ef=P^*(Tf)$, where T is a continuous linear extension from $W_{p'}^{-k}(Q)$ into $W_{p'}^{-k}(Q_Z)$. By (2.43) and (2.44), E has the same property as T. Let X be the set of those $f\in W_{p'}^{-k}(Q)$ such that $f=0$ a.e. on $U\cap Q$, where U is a neighborhood of ∂Q_Z. Using (2.44) we obtain that $E(X)\subseteq\mathring{W}_{p'}^{-k}(Q_Z)$. Since X is dense in $W_{p'}^{-k}(Q)_{Z'}$, we conclude that $E\big(W_{p'}^{-k}(Q)_{Z'}\big)\subseteq\mathring{W}_{p'}^{-k}(Q_Z)$. This completes the proof of Lemma 2.48.

Now let $s>0$ and $1\leqslant p,q\leqslant\infty$. We put

$$B_{p,q}^s(Q)_Z=\{f\in W_p^0(Q)\colon f_Z\in B_{p,q}^s(Q_Z)\}$$

and let for $f\in B_{p,q}^s(Q)_Z$

$$\|f\|_{p,q}^{(s)}(Q)_Z=\|f_Z\|_{p,q}^{(s)}(Q_Z).$$

Note that $B_{p,q}^s(Q)_Z\subseteq B_{p,q}^s(Q)$, but now (for some values of s) $B_{p,q}^s(Q)_Z$ may not be closed in $B_{p,q}^s(Q)$ (cf., e.g., [29]).

PROPOSITION 2.50. *Let Z be as in (2.37), $1\leqslant p,q\leqslant\infty$, $0<\theta<1$ and let l, r be integers, $l<r$. Put $s=(1-\theta)l+\theta r$, $\mathscr{F}_0=W_p^l$, $\mathscr{F}_1=W_p^r$ and let*

$$\mathscr{F}(Q)_Z=\big(\mathscr{F}_0(Q)_Z,\mathscr{F}_1(Q)_Z\big)_{\theta,q}.$$

Then, if $s>0$, $\mathscr{F}(Q)_Z=B_{p,q}^s(Q)_Z$ (the respective norms being equivalent), and if $s<0$, then $\mathscr{F}(Q)_Z$ can be naturally identified with the closure of $W_p^0(Q)$ in $\big(B_{p',q'}^{-s}(Q)_{Z'}\big)^*$.

Remark. In particular we see that the space $\mathscr{F}(Q)_Z$ in Proposition 2.50 depends only on s, p, q, d and Z (and not on l, r). In the sequel this space will be denoted by $B_{p,q}^s(Q)_Z$ (s can be an arbitrary real number). Hence we can write

$$B_{p,q}^s(Q)_Z\underset{\mathbb{A}}{=}\big(W_p^l(Q)_Z,W_p^r(Q)_Z\big)_{\theta,q}.$$

Proof. In the case where $Z = \emptyset$ this proposition was proved in [17]. (If $l = 0$, then it also follows from Theorem 2.36.) This yields, for the parallelepiped Q_Z, if $s > 0$,

$$\mathscr{F}(Q_Z) = \big(\mathscr{F}_0(Q_Z), \mathscr{F}_1(Q_Z)\big)_{\theta,q} = B^s_{p,q}(Q_Z).$$

The projection P of Corollary 2.45 acts in $\mathscr{F}_0(Q_Z)$ and $\mathscr{F}_1(Q_Z)$, hence also in $\mathscr{F}(Q_Z)$, and one has

$$\Big(P\big(\mathscr{F}_0(Q_Z)\big), P\big(\mathscr{F}_1(Q_Z)\big)\Big)_{\theta,q} = P\big(\mathscr{F}(Q_Z)\big)$$

(cf. Theorem 6.4.2 in [3]). Since $P\big(\mathscr{F}_i(Q_Z)\big)$ is naturally isomorphic to $\mathscr{F}_i(Q)_Z$ for $i = 0, 1$, we obtain that

$$\mathscr{F}(Q)_Z = P\big(\mathscr{F}(Q_Z)\big).$$

It is easy to see that if $s > 0$, then

$$P\big(B^s_{p,q}(Q_Z)\big) = \{f \in B^s_{p,q}(Q_Z) : f = 0 \text{ a.e. on } Q_Z \setminus Q\}$$

which gives $\mathscr{F}(Q)_Z = B^s_{p,q}(Q)_Z$.

Now, if $s < 0$, then we have by the first part

$$\big(W^{-r}_{p'}(Q)_{Z'}, W^{-l}_{p'}(Q)_{Z'}\big)_{1-\theta,q'} = B^{-s}_{p',q'}(Q)_{Z'}$$

and hence we can apply a general result on duality of interpolation spaces (cf. Theorem 3.7.1 in [3]) and Lemma 2.48. This completes the proof of Proposition 2.50.

Remark 2.51. As we have already mentioned in the special case where $l = 0$ one can avoid using Proposition 2.39 in the proof of Proposition 2.50 (thanks to Remark 2.46).

Let $\mathscr{F} = W^k_p$ or $\mathscr{F} = B^s_{p,q}$. The fact that one can define the spaces $\mathscr{F}(M), \mathscr{F}(Q)$ and $\mathscr{F}(Q)_Z$, where M is a compact manifold, $Q \subset M$ is diffeomorphic to the d-cube I^d and Z is a union of $(d-1)$-dimensional faces of Q, depends on the following well-known property of the spaces on subsets of $\boldsymbol{R}^d$.

LEMMA 2.52. *Let U_1, U_2 be open subsets of $\boldsymbol{R}^d$ and let Φ be a C^∞ diffeomorphism of U_1 onto U_2. Let $\mathscr{F} = W^k_p$, $\mathring{W}^k_p$ or $B^s_{p,q}$, where $k \geq 0$, $s > 0$, $1 \leq p, q \leq \infty$. If $F_i \subset U_i$ are proper sets such that $\Phi(F_1) = F_2$, then the formula $f \mapsto f\circ\Phi$ defines a linear topological isomorphism of $\mathscr{F}(F_2)$ onto $\mathscr{F}(F_1)$.*

Proof. The case of W^k_p and $\mathring{W}^k_p$ can be checked directly and the case of $B^s_{p,q}$ follows by interpolation, using Theorem 2.36.

Let $U \subseteq \boldsymbol{R}^d$ be an open set such that $\operatorname{Int} Q \subseteq U$ and let $\mathscr{F} = W^k_p$, $k \geq 0$, $1 \leq p \leq \infty$. U is said to *determine* $\mathscr{F}(Q)_Z$ if

(2.53) $$\mathscr{F}(Q)_Z = \{g|_Q : g \in \mathscr{F}(\boldsymbol{R}^d),\ g|_{U\setminus Q} = 0\}.$$

Clearly, $\operatorname{Int} Q_Z$ determines $\mathscr{F}(Q)_Z$ in this sense. This follows from the extension theorem for parallelepipeds (cf. e.g. Lemma 2.14).

LEMMA 2.54. *Let $\mathscr{F} = W^k_p$, where $k \geq 0$, $1 \leq p \leq \infty$. Let $U \subseteq \boldsymbol{R}^d$ be an open set such that* (cf. (2.47))

(2.55) $$U \cap Z' = \emptyset,$$

(2.56) $$Q \subseteq Z' \cup U,$$

(2.57) *if $h \in \mathscr{F}(U)$ and $h = 0$ a.e. on $U \setminus Q$, then $h = g|_U$ for some $g \in \mathscr{F}(\boldsymbol{R}^d)$.*

Then U determines $\mathscr{F}(Q)_Z$.

Proof. Let us check (2.53). Suppose first that $f = g|_Q$ where $g \in \mathscr{F}(\boldsymbol{R}^d)$ and $g|_{U\setminus Q} = 0$. By (2.56), the sets $U' = U \cap \operatorname{Int} Q_Z$ and $U'' = \operatorname{Int} Q_Z \setminus Q$ form an open covering of $\operatorname{Int} Q_Z$. Since $f_Z|_{U'} = g|_{U'} \in \mathscr{F}(U')$ and $f_Z|_{U''} = 0$, using standard arguments involving partition of unity we obtain that $f_Z \in \mathscr{F}(\operatorname{Int} Q_Z)$, i.e. $f \in \mathscr{F}(Q)_Z$.

Conversely, if $f \in \mathscr{F}(Q)_Z$, put $h(x) = f(x)$ for $x \in Q \cap U$ and $h(x) = 0$ for $x \in U \setminus Q$. To prove that $h \in \mathscr{F}(U)$, note that, by (2.55), U' and $U \setminus Q$ form an open covering of U and $h|_{U'} = f_Z|_{U'} \in \mathscr{F}(U')$, $h|_{U\setminus Q} = 0 \in \mathscr{F}(U \setminus Q)$. Using (2.57) we obtain $h = g|_U$ for some $g \in \mathscr{F}(\boldsymbol{R}^d)$. Since $g|_Q = h|_Q = f$ and $g|_{U\setminus Q} = 0$, the proof is complete.

Remark 2.58. Conditions (2.55), (2.56), (2.57) are satisfied by an open set U iff they are satisfied by $U \cap \Omega$, where Ω is some (resp. any) open neighborhood of Q. Hence it follows from Lemma 2.52 that there is a version of definition (2.53) and Lemma 2.54 in which $\boldsymbol{R}^d$ is replaced by a d-dimensional C^∞ manifold and Q by a subset diffeomorphic to I^d. The proof is almost identical. (This is the version we shall use in Section 4.)

3. Decomposition of compact C^∞ manifolds.

The main result of this section, Theorem 3.3, is essentially the same as Lemma 2.1 formulated without proof in [16]. (Also the proof is the same.) Let us mention that some arguments in Section 6 depend on specific properties of the constructions used in the proof of Theorem 3.3. The situation in [16] was simpler because we were proving less.

In this section we use some terminology and elementary facts from differential topology which can be found in [26].

By a *d-manifold* we mean a d-dimensional C^∞ manifold. The boundary of a d-manifold M, which may be either a $(d-1)$-manifold or the empty set, is denoted by ∂M.

DEFINITION 3.1. Let M be a d-manifold and let $Z \subseteq M$. If $\partial M = \emptyset$, then Z is said to be *proper* if for each $x \in Z$ there exist a chart Φ: $U \to \boldsymbol{R}^d$, a Lipschitzian function φ: $\boldsymbol{R}^{d-1} \to \boldsymbol{R}$ and a $\delta > 0$ such that $x \in U$ and, if

$$V = K\big(\Phi(x), \delta\big) = \{y \in \boldsymbol{R}^d : \|y - \Phi(x)\| < \delta\},$$

where $\| \ \|$ is the Euclidean norm, then

$$\Phi(Z \cap U) \cap V = \{y \in V:\ y_d \geqslant \varphi(y_1, \ldots, y_{d-1})\}.$$

If $\partial M \neq \emptyset$, then Z is said to be *proper* if there exists a d-manifold without boundary, say $\tilde{M}$, such that M is a closed submanifold of $\tilde{M}$ and Z is proper as a subset of $\tilde{M}$.

In that case the interior of Z (considered as a subset of a d-manifold $\tilde{M} \supseteq M$) is locally a set with minimally smooth boundary in the sense of E. M. Stein ([35], Chapter VI). Thus the analogue of the Stein extension theorem (Th. 6.5 in [35]) will hold for Z, if Z is compact.

Recall a well-known fact that a Lipschitzian real-valued function defined on a subset of $\boldsymbol{R}^d$ admits an extension to $\boldsymbol{R}^d$ with the same Lipschitz constant. This allows us to make the simplifying assumption about the domain of φ in Definition 3.1 and also on other occasions.

Now let K be a compact subset of a d-manifold M_1.

A map Φ: $K \to M$ is said to be a *diffeomorphism* if there exist d-manifolds $\tilde{M}$, $\tilde{M}_1$, containing M, M_1 as submanifolds, respectively, an open set $U \subseteq \tilde{M}_1$ and a C^∞ diffeomorphism Ψ: $U \to \tilde{M}$ such that $\partial\tilde{M} = \emptyset$, $\partial\tilde{M}_1 = \emptyset$, $U \supseteq K$ and $\Psi|_K = \Phi$.

A subset $Q \subseteq M$ is said to be a *d-cube* if there is a diffeomorphism Φ: $\langle 0, 1\rangle^d \to M$ such that $\Phi(\langle 0, 1\rangle^d) = Q$.

DEFINITION 3.2. Given a d-manifold M and closed sets $A, B \subseteq M$, the pair (A, B) is said to *admit a decomposition* (*into d-cubes*) if for some $N \geqslant 0$ there exist d-cubes $Q_1, \ldots, Q_N \subseteq A$ such that

$$A \subseteq B \cup \bigcup_{j \leqslant N} Q_j$$

and, if Φ_j is a diffeomorphism of $\langle 0, 1\rangle^d$ onto Q_j, $1 \leqslant j \leqslant N$, then the set

$$\Phi_j^{-1}\big(B \cup \bigcup_{i<j} Q_i\big)$$

is the union of a family of $(d-1)$-dimensional faces of $\langle 0, 1\rangle^d$. The decomposition $Q_1, \ldots, Q_N$ of (A, B) is said to be *proper* if the sets $B \cup \bigcup_{i \leqslant j} Q_i$ are proper for $j = 0, 1, \ldots, N$.

THEOREM 3.3. *Let M be a compact d-manifold. Then $(M, \emptyset)$ admits a proper decomposition which is a decomposition of $(M, \partial M)$.*

This theorem will be obtained using the following technical lemmas. (Lemma 3.4 follows directly from the definitions, the others will be proved later.)

LEMMA 3.4. *Suppose that $A_1, A_2, B \subseteq M$ and $Q_1, \ldots, Q_N$ (resp. $Q'_1, \ldots, Q'_{N'}$) is a (proper) decomposition of (A_1, B) (resp. of $(A_2, A_1 \cup B)$). Then $Q_1, \ldots, Q_N, Q'_1, \ldots, Q'_{N'}$ is a (proper) decomposition of $(A_1 \cup A_2, B)$.*

LEMMA 3.5. *Let φ_1, φ_2: $\boldsymbol{R}^d \to \boldsymbol{R}$ be Lipschitzian functions, let $A \subseteq \boldsymbol{R}^d$ and*

$$\tilde{A} = \{(x, z) \in \boldsymbol{R}^d \times \boldsymbol{R}:\ z \leqslant \varphi_1(x) \ \text{or}\ x \in A,\ z \leqslant \varphi_2(x)\}.$$

If A is proper (as a subset of $\boldsymbol{R}^d$), then $\tilde{A}$ is proper as a subset of $\boldsymbol{R}^d \times \boldsymbol{R}$.

LEMMA 3.6. *Let $D_d = \{x \in \boldsymbol{R}^d:\ \|x\| \leqslant 1\}$, $S^{d-1} = \partial D_d$. There exists a decomposition of (D_d, S^{d-1}) which is a proper decomposition of $(D_d, \emptyset)$ and of $(D_d, \boldsymbol{R}^d \setminus \operatorname{Int} D_d)$.*

LEMMA 3.7. *Let $1 \leqslant k \leqslant d-1$, $m = d-k$,*

$$H = \{(x, y) \in \boldsymbol{R}^k \times \boldsymbol{R}^m:\ \|y\|^2 \leqslant \|x\|^2 + 1\},$$
$$B = \{(x, y) \in H:\ \|x\|^2 \geqslant \min\{5/4, \|y\|^2 + 1\}\}.$$

Then (H, B) admits a proper decomposition.

Proof of Theorem 3.3. The assertion is obvious if $d = \dim M \leqslant 1$. Hence we may assume that $d > 1$ and that the theorem has already been proved for all compact $(d-1)$-manifolds (with or without boundary).

We shall use some results from Morse theory which can be found in Chapter 6 of [26].

We fix a non-negative Morse function on M, say τ, such that $\partial M = \tau^{-1}(0)$ and the critical points $z_1, \ldots, z_\nu$ of τ satisfy $0 \leqslant \tau(z_1) < \tau(z_2) < \ldots < \tau(z_\nu)$. Let ε be a small positive number. Put $a_0 = 0$, and

$$a_{2j-1} = \tau(z_j) - \varepsilon, \quad a_{2j} = \tau(z_j) + \varepsilon,$$
$$A_s = \tau^{-1}\big((-\infty, a_s\rangle\big), \quad M_s = \tau^{-1}(\langle a_{s-1}, a_s\rangle), \quad B_s = \tau^{-1}(a_{s-1})$$

for $j = 1, \ldots, \nu$, $s = 1, \ldots, 2\nu$.

We prove that if ε is sufficiently small, then for $s = 2, 3, \ldots, 2\nu$ the pair (A_s, A_{s-1}) admits a proper decomposition, whereas $(A_1, \partial M)$ admits a decomposition which is a proper decomposition of $(A_1, \emptyset)$. The assertion of the theorem will then follow from Lemma 3.4.

Assume first that $s = 2j-1$ is odd. (If $s = 1$, then we may also assume that $A_1 \neq \emptyset$.) Let Φ be the diffeomorphism from $B_s \times \langle a_{s-1}, a_s\rangle$ onto M_s constructed in Theorem 6.2.2 of [26]. By the inductive assumption there is a proper decomposition into $(d-1)$-cubes for the pair $(B_s, \emptyset)$. (Indeed, B_s is a compact $(d-1)$-manifold without boundary.) Using Φ and this decomposition we obtain a decomposition of (M_s, B_s). Lemma 3.5 can then be used to prove that this decomposition is proper for (A_s, A_{s-1}) if $s > 1$ and for $(A_1, \emptyset)$ if $s = 1$. Here we need only $0 < \varepsilon < \tau(z_1)$ and $2\varepsilon < \tau(z_j) - \tau(z_{j-1})$ for $j = 2, 3, \ldots, \nu$.

Now let $s = 2j$ be even. By Morse's Lemma (cf. [26], Lemma 6.1.1), there exist $\delta > 0$, an integer k and a chart Φ: $U \to \boldsymbol{R}^d$ such that $0 \leqslant k \leqslant d$,

$z_j \in U$, $\Phi(z_j) = 0$, $\Phi(U) = K(0, \delta) = \{v \in \boldsymbol{R}^d\colon \|v\| < \delta\}$ and for $v \in \Phi(U)$ one has

$$\tau(\Phi^{-1}(v)) = \tau(z_j) - \sum_{i=1}^{k} v_i^2 + \sum_{i=k+1}^{d} v_i^2.$$

Suppose first that $k \in \{0, d\}$. If $0 < \varepsilon < \delta^2$, then $M' = M_{2j} \cap U$ is open in M_{2j} and diffeomorphic to the closed disk D_d, while its complement $M_{2j} \setminus M'$ is diffeomorphic to $B' \times \langle a_{s-1}, a_s \rangle$, where $B' = B_{2j} \setminus U$ is either a $(d-1)$-manifold without boundary or the empty set. Using the inductive assumption and Lemmas 3.6, 3.5 and 3.4 we can complete the discussion as in the previous case where s was odd.

Now we consider the the case where $1 \leqslant k \leqslant d-1$. We assume that $0 < \varepsilon < \delta^2/36$ and set $m = d-k$,

$$M' = \{z \in M_s\colon z \in U \text{ or } \Phi(z) = (x, y),\ \|x\|^2 \geqslant (5/4)\varepsilon\}.$$

The method of proof of Theorem 6.3.1 in [26] (cf. Figure 6-5) yields that M' is diffeomorphic to $B' \times \langle a_{s-1}, a_s \rangle$, where $B' = B_{2j} \cap M'$ is a $(d-1)$-manifold with boundary. Again using the inductive assumption and Lemma 3.5 we construct a proper decomposition of (M', A_{s-1}).

It remains to show that $(A_s, M' \cup A_{s-1})$ admits a proper decomposition. This, however, follows readily from Lemma 3.7, because $z \mapsto \varepsilon^{-1/2}\Phi(z)$ defines a suitable diffeomorphism. This completes the proof of Theorem 3.3.

Now we pass to the proofs of the lemmas. We shall need the following technical concept.

Definition 3.8. Let $Z \subseteq \boldsymbol{R}^d$, x, $a \in \boldsymbol{R}^d$, $a \neq 0$. We say that Z is *a-directed at x* if there exist $\eta > 0$ and a Lipschitzian function ψ on $\boldsymbol{R}^d$ such that for $y \in K(x, \eta)$ one has

$$\{t \in \langle -\eta, \eta \rangle\colon y + ta \in Z\} = \langle -\eta, \eta \rangle \cap \langle \psi(y), \infty).$$

Lemma 3.9. *A subset Z of $\boldsymbol{R}^d$ is proper (in the sense of 3.1) if and only if for each $x \in \boldsymbol{R}^d$ there is an $a \in \boldsymbol{R}^d \setminus \{0\}$ such that Z is a-directed at x.*

Proof. To show the sufficiency assume first that Z is $(0, \ldots, 0, a)$-directed at $x = (x_1, \ldots, x_d)$, where $a > 0$. Let η, ψ be as in Definition 3.8. Putting $U = V = K(x, \eta)$, Φ = identity, we have

$$\Phi(Z \cap U) \cap V = \{y \in V\colon y_d \geqslant x_d + a\psi(y_1, \ldots, y_{d-1}, x_d)\},$$

i.e. the conditions of Definition 3.1 are satisfied. The general case where Z is a-directed at x can be reduced to that where $a = (0, \ldots, 0, a)$ by a suitable rotation of $\boldsymbol{R}^d$.

Now we prove the necessity. We may assume that $d > 1$. Suppose Φ is a diffeomorphism of an open set $U \subseteq \boldsymbol{R}^d$ into $\boldsymbol{R}^d$ such that $x \in U$ and for $V = K(\Phi(x), \delta)$, where $\delta > 0$, one has

$$\Phi(Z \cap U) \cap V = \{y \in V\colon y_d \geqslant \varphi(y_1, \ldots, y_{d-1})\},$$

where $\varphi\colon \boldsymbol{R}^{d-1} \to \boldsymbol{R}$ satisfies the Lipschitz condition with a constant $\varkappa$.

Assume first that $D\Phi(x)$ = identity. In this case we shall prove that Z is e_d-directed at x, where $e_d = (0, \ldots, 0, 1)$. Pick $\eta > 0$ so that $W = K(x, 2\eta) \subset U$ and for $w \in W$ one has

$$\|D\Phi(w) - \mathrm{Id}\| < \beta = (2 + 2\varkappa)^{-1}. \tag{3.10}$$

Write $\Phi = (\phi_1, \ldots, \phi_d)$ and, if $y = (y_1, \ldots, y_d) \in \boldsymbol{R}^d$, let $y' = (y_1, \ldots, y_{d-1})$. Observe that

$$W \cap \partial Z = S = \{y \in W\colon \phi_d(y) = \varphi(\Phi(y)')\}.$$

It suffices to prove that if $y, z \in S$, then

$$|y_d - z_d| \leqslant (2\varkappa + 1)\|y' - z'\|. \tag{3.11}$$

For, this will show that $W \cap Z = \{y \in W\colon y_d \geqslant \chi(y')\}$, where χ is a function whose Lipschitz constant is $\leqslant 2\varkappa + 1$, and hence for $y \in K(x, \eta)$ one has

$$\{t \in \langle -\eta, \eta \rangle\colon y + te_d \in Z\} = \langle -\eta, \eta \rangle \cap \langle \chi(y') - y_d, \infty)$$

which means that Z is e_d-directed at x.

To prove (3.11) observe that, by (3.10) and the mean value theorem,

$$\begin{aligned} |y_d - z_d| - |\phi_d(y) - \phi_d(z)| &\leqslant |y_d - \phi_d(y) - (z_d - \phi_d(z))| \\ &\leqslant \beta\|y - z\| \leqslant \beta|y_d - z_d| + \beta\|y' - z'\| \end{aligned}$$

for $y, z \in W$, and similarly

$$\|\Phi(y)' - \Phi(z)'\| - \|y' - z'\| \leqslant \beta|y_d - z_d| + \beta\|y' - z'\|.$$

The estimate (3.11) follows from these inequalities because, if $y, z \in S$, then

$$|\phi_d(y) - \phi_d(z)| = |\varphi(\Phi(y)') - \varphi(\Phi(z)')| \leqslant \varkappa\|\Phi(y)' - \Phi(z)'\|.$$

Now, in the general case where $\Delta = D\Phi(x) \neq \mathrm{Id}$, we infer that the set $\Delta(Z)$ is e_d-directed at $\Delta(x)$. (Use the chart $\Phi \circ (\Delta^{-1}|_{\Delta(U)})$.) Hence, for some $\gamma > 0$ and y close to x, we have

$$\{t \in \langle -\gamma, \gamma \rangle\colon \Delta(y) + te_d \in \Delta(Z)\} = \langle -\gamma, \gamma \rangle \cap \langle \psi(\Delta(y)), \infty).$$

This, however, means that

$$\{t \in \langle -\gamma, \gamma \rangle\colon y + t\Delta^{-1}(e_d) \in Z\} = \langle -\eta, \eta \rangle \cap \langle (\psi \circ \Delta)(y), \infty)$$

and $\psi \circ \Delta$ is Lipschitzian if so is ψ. This proves that Z is $(\Delta^{-1}e_d)$-directed at x and completes the proof of the necessity.

Proof of Lemma 3.5. Let L be a Lipschitz constant for φ_1 and φ_2 By Lemma 3.9, it suffices to prove that if A is a-directed at $x \in \boldsymbol{R}^d$ and $a > L|\alpha|$, then $\tilde{A}$ is $(\alpha, -a)$-directed at each point $(x, x_{d+1}) \in \boldsymbol{R}^d \times \boldsymbol{R}$.

To this end observe that there are functions $\bar{\varphi}_1$, $\bar{\varphi}_2$ on $\boldsymbol{R}^d \times \boldsymbol{R}$ such that, for $y \in \boldsymbol{R}^d$, $z \in \boldsymbol{R}$ and $i = 1, 2$,

$$\{t \in \boldsymbol{R}:\ z - ta \leqslant \varphi_i(y + t\alpha)\} = \langle \bar{\varphi}_i(y, z),\ \infty)$$

and $\bar{\varphi}_i$ has the Lipschitz constant $\leqslant \sqrt{L^2+1}/(a - L|\alpha|)$.

For, if $u_j = \bar{\varphi}(y_j, z_j)$, where $\varphi \in \{\varphi_1, \varphi_2\}$ and $j = 1, 2$, then $z_j - u_j a = \varphi(y_j + u_j \alpha)$. Hence

$$a|u_1 - u_2| \leqslant |z_1 - z_2| + L(\|y_1 - y_2\| + \|\alpha\|\,|u_1 - u_2|)$$

and the required estimate of the Lipschitz constant of $\bar{\varphi}$ follows from Schwarz's inequality.

Now, since A is α-directed at x, there is a $\delta > 0$ such that for y close to x one has

$$\{t \in \langle -\delta, \delta\rangle:\ y + t\alpha \in A\} = \{t \in \langle -\delta, \delta\rangle:\ t \geqslant \chi(y)\},$$

where χ satisfies the Lipschitz condition. Clearly, if $|t| < \delta$ then

$$(y, z) + t(\alpha, -a) \in \tilde{A}$$

is equivalent to

$$t \geqslant \min\{\bar{\varphi}_1(y, z),\ \max\{\chi(y), \bar{\varphi}_2(y, z)\}\}.$$

The right-hand side being a Lipschitzian function of (y, z), the proof of Lemma 3.5 is complete.

LEMMA 3.12. *Let A_i, $B_i \subseteq \boldsymbol{R}^{d_i}$ for $i = 1, 2$ and let*

$$A = A_1 \times B_2 \cup B_1 \times A_2 \subseteq \boldsymbol{R}^{d_1} \times \boldsymbol{R}^{d_2}.$$

Suppose that both A_i and B_i are α_i-directed at $x_i \in \boldsymbol{R}^{d_i}$ for $i = 1, 2$. Then A is (α_1, α_2)-directed at (x_1, x_2).

Proof. We have $y_i + t\alpha_i \in A_i$ equivalent to $t \geqslant \varphi_i(y_i)$ for y_i close to x_i and $|t| \leqslant \delta_i$. Similarly, $y_i + t\alpha_i \in B_i$ is equivalent to $t \geqslant \psi_i(y_i)$ for $i = 1, 2$. It follows that

$$(y, z) + t(\alpha_1, \alpha_2) \in A$$

is equivalent to

$$t \geqslant \min\{\max\{\varphi_1(y), \psi_2(z)\}, \max\{\varphi_2(y), \psi_1(z)\}\}.$$

This proves the lemma.

DEFINITION 3.13. Given an integer $d \geqslant 1$ and $c \in (0, d^{-1/2})$, we let for $j = 1, \ldots, d$

$$Q_j = \{x \in D^d:\ x_j = \|x\|_\infty \geqslant c\},$$
$$Q_{j+d} = \{x \in D^d:\ -x \in Q_j\},$$
$$Q_{2d+1} = \langle -c, c\rangle^d.$$

The sequence $Q_1, \ldots, Q_{2d+1}$ is said to be a *standard decomposition of the disk D_d into d-cubes.*

It is easy to check that $Q_1, \ldots, Q_{2d+1}$ is a decomposition of (D_d, S^{d-1}). For instance, the formula

$$\Phi(y_1, \ldots, y_d) = \Big(c + y_1\Big(-c + \Big(1 + \sum_{j=2}^{d} y_j^2\Big)^{-1/2}\Big)\Big)(1, y_2, \ldots, y_d)$$

defines a diffeomorphism of the d-cube $\langle 0, 1\rangle \times \langle -1, 1\rangle^{d-1}$ onto Q_1. The remaining assertions of Lemma 3.6 will be checked with the use of the following lemma.

LEMMA 3.14. *Suppose that $x \in [0, \infty)^d$, $\alpha \in \boldsymbol{R}^d$, $I \subseteq \{1, \ldots, d\}$ and*

$$\min_{i \in I} \alpha_i > \max_{i \notin I} \alpha_i.$$

The set $Z = \bigcup_{i \in I} Q_i$ is α-directed at x in each of the following cases:

$$x \notin S^{d-1} \quad \text{and} \quad \|x\|_\infty \neq c, \tag{3.15}$$

$$\|x\|_\infty = c \quad \text{and} \quad \min_{i \in I} \alpha_i > 0, \tag{3.16}$$

$$x \in S^{d-1} \quad \text{and} \quad \sum_{j \leqslant d} \alpha_j x_j < 0. \tag{3.17}$$

Proof. We have $Z = \{z \in D_d:\ \|z\|_\infty = \max_{i \in I} z_i \geqslant c\}$. If $y \in \boldsymbol{R}^d$ satisfies $\|y - x\|_\infty \leqslant (1/2)\|x\|_\infty$, then for small $t \in \boldsymbol{R}$ the condition $y + t\alpha \in Z$ is equivalent to the following inequalities:

$$\max_{i \in I}(y_i + \alpha_i t) \geqslant \max_{i \notin I}(y_i + \alpha_i t), \tag{3.18}$$

$$\max_{i \in I}(y_i + \alpha_i t) \geqslant c, \tag{3.19}$$

$$\sum_{i \leqslant d}(y_i + \alpha_i t)^2 \leqslant 1. \tag{3.20}$$

Moreover, inequality (3.20) may be dropped in cases (3.15) and (3.16) and inequality (3.19) may be dropped in cases (3.15) and (3.17) provided that y is sufficiently close to x and $|t|$ is small enough.

It is easy to check that our assumptions (in each of the cases (3.15), (3.16) and (3.17)) allow one to replace the resulting system of inequalities by a single condition of the form $t \geqslant \varphi(y)$, where φ satisfies the Lipschitz condition near x. This completes the proof of Lemma 3.14, because it is well known that φ can be extended to a Lipschitzian function on $\boldsymbol{R}^d$.

Using Lemma 3.14 we can complete the proof of Lemma 3.6. In fact, we can check a stronger fact that if $1 \leqslant j \leqslant k \leqslant 2d+1$, then, at each point $x \in \boldsymbol{R}^d$, the sets $\bigcup_{i\leqslant j} Q_i$, $\bigcup_{i\leqslant k} Q_i$ are α-directed for some common $\alpha \neq 0$. This will be essential when we consider product decompositions.

To prove this we may assume that $k \leqslant 2d$ and $x \neq 0$. Then we apply the isometry T of $\boldsymbol{R}^d$ given by

$$T(y) = (\varepsilon_1 y_1, \ldots, \varepsilon_d y_d),$$

where $\varepsilon_i = -1$ if $x_i < 0$ and $\varepsilon_i = 1$ otherwise. This reduces the problem to the case considered in Lemma 3.14. In a neighborhood of $T(x)$ the sets $T(\bigcup_{i\leqslant j} Q_i)$, $T(\bigcup_{i\leqslant k} Q_i)$ are of the form $\bigcup_{i\in J} Q_i$, $\bigcup_{i\in K} Q_i$, where $J \subseteq K \subseteq \{1, \ldots, d\}$, and one of the cases of Lemma 3.14 can be applied.

Obviously, $Q_1, \ldots, Q_{2d+1}$ remains a proper decomposition of $(D_d, \emptyset)$ after any permutation of the sequence that leaves Q_{2d+1} fixed. It follows that this sequence is also a proper decomposition of $(D_d, \boldsymbol{R}^d \setminus \operatorname{Int} D_d)$. To see this observe that if $Z = \overline{U}$, where $U \subset \boldsymbol{R}^d$ is open, then Z is α-directed at $x \in \boldsymbol{R}^d$ if and only if $\boldsymbol{R}^d \setminus \operatorname{Int} Z$ is $(-\alpha)$-directed at x. This completes the proof of Lemma 3.6.

Before we pass to the proof of Lemma 3.7, observe the following fact. Suppose $A \subset \boldsymbol{R}^{d-1}$, and $A \times \langle -1, 1\rangle \subset \boldsymbol{R}^d$ is (a, v)-directed at $(x, 0)$, where $x, a \in \boldsymbol{R}^{d-1}$, $v \in \boldsymbol{R}$. Then A is a-directed at x if $a \neq 0$, and A is β-directed at x for any $\beta \in \boldsymbol{R}^{d-1} \setminus \{0\}$ otherwise.

This follows directly from the definitions. The case $a = 0$ is easy. If $a \neq 0$, (y, s) is sufficiently close to $(x, 0)$ and $|t|$ is small enough, then

$$(y, s) + t(a, v) \in A \times \langle -1, 1\rangle$$

is equivalent to $y + ta \in A$. Since the former condition is also equivalent to $t \geqslant \psi(y, s)$, we infer that in a neighborhood of $(x, 0)$ the function ψ does not depend on s.

Using this fact we obtain easily that the sequence $(Q_i \cap S^{d-1})$, $i = 1, 2, \ldots, 2d$, is a proper decomposition of $(S^{d-1}, \emptyset)$ which has the property similar to that we have proved for the Q_i's.

Now suppose that $Q_1', \ldots, Q_p'$ is a decomposition of (A', B') into k-cubes and $Q_1'', \ldots, Q_q''$ is a decomposition of (A'', B'') into m-cubes. Set

$$Q_i = Q_j' \times Q_l'' \quad \text{if} \quad i = (j-1)q + l, \ 1 \leqslant l \leqslant q, \ 1 \leqslant j \leqslant p. \tag{3.21}$$

It is easy to see that $Q_1, \ldots, Q_{pq}$ is a decomposition of $(A' \times A'', A' \times B'' \cup B' \times A'')$ into $(k+m)$-cubes. We call (3.21) a *product decomposition*.

If $B' = \emptyset = B''$, then a sufficient condition for $Q_1, \ldots, Q_{pq}$ to be a proper decomposition for $(A' \times A'', \emptyset)$ can be obtained from Lemma 3.12 and the identity in which $s = (j-1)q + l$:

$$\bigcup_{i\leqslant s} Q_i = \Big(\bigcup_{i\leqslant j-1} Q_i'\Big) \times \Big(\bigcup_{i\leqslant q} Q_i''\Big) \cup \Big(\bigcup_{i\leqslant j} Q_i'\Big) \times \Big(\bigcup_{i\leqslant l} Q_i''\Big).$$

This sufficient condition has already been checked for our standard decompositions of disks and spheres.

Now we can prove Lemma 3.7. Set

$$\begin{aligned} G &= \{(x, y) \in H \colon \|x\|^2 \leqslant \min\{5/4, \|y\|^2+1\}\}, \\ G_1 &= \{(x, y) \in G \colon \|y\| \leqslant 1/2\}, \\ G_2 &= \{(x, y) \in G \colon \|y\| \geqslant 1/2\}. \end{aligned} \tag{3.22}$$

It suffices to prove that (G_1, B) and $(G_2, B \cup G_1)$ admit proper decompositions. Indeed, since $H = G \cup B$, then the proper decomposition of $(G_1 \cup G_2, B)$ obtained via Lemma 3.4 will also be a proper decomposition of (H, B). (This follows from the definitions we have adopted.)

The diffeomorphism $\Psi_1(x, y) = \big(x(1+\|y\|^2)^{-1/2}, 2y\big)$ of $\boldsymbol{R}^k \times \boldsymbol{R}^m$ onto itself satisfies $\Psi_1(G_1) = D_k \times D_m$ and $\Psi_1(B \cap G_1) = S^{k-1} \times D_m$. It follows that the product decomposition $Q_1, \ldots, Q_p$, where $p = (2k+1)(2m+1)$, made of those we have given for (D_k, S^{k-1}) and $(D_m, \emptyset)$ induces a decomposition of $(G_1, B \cap G_1)$.

Since $Q_1, \ldots, Q_p$ is a proper decomposition of $(D_k \times D_m, \emptyset)$, in order to prove that $\Psi_1^{-1}(Q_1), \ldots, \Psi_1^{-1}(Q_p)$ is a proper decomposition of (G_1, B) it is sufficient to investigate the sets

$$\Psi_1(B) \cup \bigcup_{i\leqslant j} Q_i,$$

where $j = 0, 1, \ldots, p$, at the points of the set

$$\Psi_1(B) \cap \bigcup_{i\leqslant p} Q_i = S^{k-1} \times D_m.$$

Let $\bar{z} = (\bar{x}, \bar{y}) \in S^{k-1} \times D_m$. Suppose first that $\|\bar{y}\| > (1/2)c$ (c being the number used in Definition 3.13). Passing to "polar coordinates" $r = \|x\|$, $\varrho = \|y\|$, $\xi = x/r$, $\eta = y/\varrho$, we obtain that near $\bar{z}$ the set $\Psi_1(B) \cup \bigcup_{i\leqslant j} Q_i$ is of the form

$$\{(r, \varrho, \xi, \eta) \in \boldsymbol{R} \times \boldsymbol{R} \times S^{k-1} \times S^{m-1} \colon r \geqslant \varphi(\varrho) \text{ or } (\varrho, \xi, \eta) \in A'\},$$

where

$$\varphi(\varrho) = \min\{1, \sqrt{5/(4+\varrho^2)}\}$$

is a Lipschitzian function and $A' = (-\infty, 1\rangle \times A_j$, where

$$A_j = (S^{k-1} \times S^{m-1}) \cap \bigcup_{i\leqslant j} Q_i.$$

It follows from our preliminary discussion that A_j is proper (as a subset of $S^{k-1}\times S^{m-1}$). Hence, using twice Lemma 3.5, we infer that A' is proper in $\boldsymbol{R}\times S^{k-1}\times S^{m-1}$, and therefore the set $\Psi_1(B)\cup\bigcup_{i\leqslant j} Q_i$ is proper in a neighborhood of $\bar{z}$.

The case where $\|\bar{y}\|\leqslant(1/2)c$ is similar. Since $\bar{y}$ is in the interior of a cube from the decomposition of D_m, we can use the "cylindrical coordinates", i.e. $r=\|x\|$, $\xi=x/r$, $y=y$. Then, in a neighborhood of $\bar{z}$, the set $\Psi_1(B)\cup\bigcup_{i\leqslant j} Q_i$ is of the form

$$\{(r,\xi,y)\in\boldsymbol{R}\times S^{k-1}\times\boldsymbol{R}^m:\ r\leqslant 1 \text{ or } \xi\in A\},$$

where A is the union of some $(k-1)$-cubes from our standard decomposition of S^{k-1}. The proof that $\Psi_1(B)\cup\bigcup_{i\leqslant j} Q_i$ is proper in a neighborhood of $\bar{z}$ is analogous to that in the previous case.

In order to construct a proper decomposition of $(G_2, B\cup G_1)$ we set for $(x,y)\in\boldsymbol{R}^k\times\boldsymbol{R}^m$ with $\varrho=\|y\|>0$

$$\Psi_2(x,y)=\big(xg(\varrho,\|x\|),\ y/\varrho\big),$$

where

$$g(t,r)=\big(t+\sqrt{1+r^2}-1\big)/\big(2\sqrt{1+r^2}-1\big).$$

Then Ψ_2 is a diffeomorphism of $\boldsymbol{R}^k\times(\boldsymbol{R}^m\setminus\{0\})$ into $\boldsymbol{R}^k\times\boldsymbol{R}\times S^{m-1}$ such that

$$\Psi_2(G_2)=D_k\times\langle 1/2,1\rangle\times S^{m-1},$$
$$\Psi_2\big(\partial(G_1\cup B)\big)=D_k\times\{1/2\}\times S^{m-1}\cup S^{k-1}\times\langle 1/2,1\rangle\times S^{m-1}\cup$$
$$\cup(\boldsymbol{R}^k\setminus D_k)\times\{1\}\times S^{m-1}.$$

Again the product decomposition $Q_1,\ldots,Q_q$, $q=(2k+1)2m$, of the decompositions for (D_k, S^{k-1}), $(S^{m-1},\emptyset)$ and $\langle 1/2,1\rangle$ induces a proper decomposition of $(G_2, B\cup G_1)$. This follows easily from Lemma 3.5.

This completes the proof of Lemma 3.7, and hence that of Theorem 3.3.

4. Decomposition of function spaces. Let M be a compact d-dimensional C^∞ manifold. In [16] we gave a simple scheme for constructing Schauder bases for several classes of function spaces on M. Given a function space $\mathscr{F}(M)$, that scheme can be applied if there exist extension operators in $\mathscr{F}(M)$ corresponding to some sets defined in terms of the decomposition $Q_1,\ldots,Q_N$ given by Theorem 3.3. The operators we needed in [16] are obtained easily from the Stein extension theorem (Theorem 6.5 in [35]). All details which were skipped in [16] can be found in the present paper, cf. e.g. Remark 4.10.

In this paper we improve the result of [16]. To do this, however, we need extension operators with some additional properties, cf. Proposition 4.3. Those operators are constructed in Section 6. Now we want only to state some essential properties of those operators so that we can explain how they are used in order to solve our problem.

Let μ be a smooth measure on M equivalent to the Lebesgue measure (by this we mean that, whenever $\Phi\colon U\to\boldsymbol{R}^d$ is a chart for M, then on the open set U one has $d\mu=h\,dx$, where h is a positive C^∞ function and dx is the measure transported from the Lebesgue measure on $\Phi(U)$ by means of Φ). By $L_0(M)$ we denote the space of (equivalence classes of) measurable functions on M with the topology of convergence in measure.

The Sobolev spaces $W_p^k(M)$, where $1\leqslant p<\infty$ and $k\in Z_+$, are defined in the usual way (cf. [4], [17]). (If $f\in L_0(M)$, then $f\in W_p^k(M)$ provided that "it is locally in W_p^k". The latter condition makes sense by Lemma 2.52.) The symbol $W_\infty^k(M)$ stands, as in Section 2, for the space usually denoted by $C^k(M)$. Thus, for each $k\in Z_+$ and $1\leqslant p\leqslant\infty$, the space $C^\infty(M)$ of smooth functions is dense in $W_p^k(M)$. We set

$$\mathring{C}^\infty(M)=\{f\in C^\infty(M):\ \operatorname{supp} f\cap\partial M=\emptyset\}.$$

The closure of $\mathring{C}^\infty(M)$ in $W_p^k(M)$, where $k\in Z^+$ and $1\leqslant p\leqslant\infty$, will be denoted by $\mathring{W}_p^k(M)$.

Put as usual, for $1\leqslant p\leqslant\infty$, $p'=p/(p-1)$, with $1'=\infty$ and $\infty'=1$. Define, for $k<0$ and $1<p<\infty$, the space $W_p^k(M)$ to be the completion of $C^\infty(M)$ in the norm

$$\|f\|_p^{(k)}(M)=\sup\Big\{\Big|\int_M fg\,d\mu\Big|:\ g\in\mathring{W}_{p'}^{-k}(M),\ \|g\|_{p'}^{(-k)}(M)\leqslant 1\Big\}$$

and let $\mathring{W}_p^k(M)$ be the completion of $\mathring{C}^\infty(M)$ in the norm

$$^\circ\|f\|_p^{(k)}(M)=\sup\Big\{\Big|\int_M fg\,d\mu\Big|:\ g\in W_{p'}^{-k}(M),\ \|g\|_{p'}^{(-k)}(M)\leqslant 1\Big\}.$$

(These spaces coincide with the spaces $\mathscr{C}W_p^k(M)$ and $\mathscr{C}\mathring{W}_p^k(M)$ discussed in [17].) Finally, let $W_p^k(Q)_Z$ denote the closure of the smooth functions, e.g. of $\mathring{C}^\infty(Q)$, in the dual space of $W_{p'}^{-k}(Q)_{Z'}$, where $Z'=\overline{\partial Q\setminus Z}$ (cf. Lemma 2.48). Let us remark that the spaces we have just defined do not depend on the choice of μ (choosing another smooth measure one obtains an equivalent norm).

The Besov spaces on M can be defined by real interpolation between Sobolev spaces (cf. [17]). Namely, if s is a real number and $l<s$, $r>s$ are integers, then, letting $s=(1-\theta)l+\theta r$, one has the formulae

$$B_{p,q}^s(M)=\big(W_p^l(M),\ W_p^r(M)\big)_{\theta,q},\qquad (4.1)$$
$$\mathring{B}_{p,q}^s(M)=\big(\mathring{W}_p^l(M),\ \mathring{W}_p^r(M)\big)_{\theta,q}.$$

The proof of analogous formula

$$B^s_{p,q}(Q)_Z = \big(W^l_p(Q)_Z,\, W^r_p(Q)_Z\big)_{\theta,q}, \tag{4.2}$$

which appears after Proposition 2.50, will be completed when we prove Proposition 2.39 in Section 6.

In the following, if A is a subset of a given set, χ_A may denote either the characteristic function of A or the operation of multiplication by this function.

PROPOSITION 4.3. *Let M be a compact d-dimensional C^∞ manifold and let $Q_1, \ldots, Q_N$ be the proper decomposition of M into d-cubes constructed in Theorem 3.3. Let μ be a smooth measure on M. Then, for any $m \geqslant 1$, one can construct continuous linear operators $P_1, \ldots, P_N$ in the space $L_0(M)$ which have the following properties for $f \in L_0(M)$:*

$$\sum_{i\leqslant N} P_i f = f, \tag{4.4}$$

$$P_iP_jf = 0 \quad \text{if} \quad 1 \leqslant i \neq j \leqslant N, \tag{4.5}$$

$$\chi_{Q_i}P_jf = P_i\chi_{Q_j}f = 0 \quad \text{if} \quad 1 \leqslant i < j \leqslant N, \tag{4.6}$$

(4.7) *the P_i's act in all the spaces $W^k_p(M)$ for $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$; in fact, there is $C < \infty$ so that if $g \in W^k_p(M)$ and $1 \leqslant i \leqslant N$, then*

$$\|P_i g\|^{(k)}_p(M) \leqslant C\|g\|^{(k)}_p(M),$$

(4.7*) *the adjoint operators (in the sense of Hilbert space $L_2(\mu)$) $P^*_1, \ldots, P^*_N$ satisfy the analog of (4.7) and are continuous in $L_0(M)$,*

(4.8) *if $f = 0$ a.e. on a neighborhood of ∂M, then also P_if and P^*_if, $i = 1, \ldots, N$, vanish on a neighborhood of ∂M.*

This proposition will be obtained in Section 6. Now we wish to formulate and deduce from Proposition 4.3 our basic result on decomposition of function spaces on M, i.e. Theorem 4.9. We need some notation.

If $Q_1, \ldots, Q_N$ is the sequence constructed in Theorem 3.3, we put for $i = 1, \ldots, N$

$$Z_i = Q_i \cap \bigcup_{j<i} Q_j, \qquad Z^\circ_i = Z_i \cup (Q_i \cap \partial M),$$

$$Z'_i = Q_i \cap \Big(\bigcup_{j>i} Q_j \cup \partial M\Big), \qquad Z^{\circ\prime}_i = Q_i \cap \bigcup_{j>i} Q_j.$$

If $\mathscr{F}$ denotes W^k_p or $B^s_{p,q}$, $k \geqslant 0$, $s \geqslant 0$, $1 \leqslant p, q \leqslant \infty$, Φ is a diffeomorphism of I^d onto Q_i, $1 \leqslant i \leqslant N$, and $Z \in \{Z_i, Z^\circ_i, Z'_i, Z^{\circ\prime}_i\}$, we let

$$\mathscr{F}(Q_i)_Z = \{f \in L_0(Q_i):\ f\circ\Phi \in \mathscr{F}(I^d)_{\Phi^{-1}(Z)}\}.$$

This set is well defined (i.e. it does not depend on the choice of Φ), by Lemmas 2.52 and 2.54. Different choices of Φ lead to equivalent norms. Clearly, $W^k_p(Q_i)_Z$ is closed in $W^k_p(Q_i)$, and hence it is complete.

THEOREM 4.9. *Let $M, \mu, Q_1, \ldots, Q_N, m, P_1, \ldots, P_N$ be as in Proposition 4.3. Then the formulae*

$$T_0f = \sum_{i\leqslant N} \chi_{Q_i}P_if, \qquad V_0f = \sum_{i\leqslant N}\chi_{Q_i}P^*_if,$$

define linear isomorphisms of $L_0(M)$ onto itself, the inverse maps being, respectively,

$$S_0f = \sum_{i\leqslant N} P_i\chi_{Q_i}f, \qquad U_0f = \sum_{i\leqslant N} P^*_i\chi_{Q_i}f.$$

Moreover, if $\mathscr{F}$ denotes W^k_p, $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$, then T_0, V_0 induce linear topological isomorphisms

$$T\colon \mathscr{F}(M) \to \sum_{i\leqslant N} \oplus \mathscr{F}(Q_i)_{Z_i},$$

$$\mathring{T}\colon \mathring{\mathscr{F}}(M) \to \sum_{i\leqslant N} \oplus \mathscr{F}(Q_i)_{Z^\circ_i},$$

$$V\colon \mathscr{F}(M) \to \sum_{i\leqslant N} \oplus \mathscr{F}(Q_i)_{Z^{\circ\prime}_i},$$

$$\mathring{V}\colon \mathring{\mathscr{F}}(M) \to \sum_{i\leqslant N} \oplus \mathscr{F}(Q_i)_{Z'_i}.$$

Remark 4.10. The proof that T is an isomorphism depends only on those properties of $P_1, \ldots, P_N$ which are satisfied by the operators defined in [16] with the help of Stein's theorem, i.e. (4.4), (4.5), (4.6), (4.7).

Theorem 4.9 has been formulated so that its assertion can be easily extended to the case where $\mathscr{F} = W^k_p$ with $-m \leqslant k \leqslant 0$, $1 \leqslant p \leqslant \infty$. This is a bit simpler if $1 < p < \infty$, because then the respective Sobolev spaces are reflexive and one has for all integers k

$$\big(W^k_p(M)\big)^* = \mathring{W}^{-k}_{p'}(M), \qquad \big(W^k_p(Q)_Z\big)^* = W^{-k}_{p'}(Q)_{Z'},$$

where $p' = p/(p-1)$ and $Z' = \overline{\partial Q \setminus Z}$. Let us show that the operator T of Theorem 4.9 is an isomorphism in the $\|\ \|^{(k)}_p$ norm for $-m \leqslant k < 0$, $1 < p < \infty$. By this we mean the following.

Let $\mathring{U}$ denote the inverse to the isomorphism

$$\mathring{V}\colon \mathring{W}^{-k}_{p'}(M) \to \sum_{i\leqslant N} \oplus W^{-k}_{p'}(Q_i)_{Z'_i}$$

from Theorem 4.9. (Observe that $\mathring{U}$ is a restriction of the operator U_0 which is the inverse of V_0.) Then the isomorphism

$$(\mathring{U})^*\colon W^k_p(M) \to \sum_{i\leqslant N} \oplus W^k_p(Q_i)_{Z_i}$$

is easily seen to be an extension of T from the (dense) subspace $W^0_p(M)$ of $W^k_p(M)$.

If $p=1$ or $p=\infty$, one can argue similarly. (We refer to [17] for a systematic exposition of some problems which arise in this case.) One obtains first that T has a weak* continuous extension to an isomorphism between the spaces dual to $\mathring{W}_p^{-k}$, $-m\leqslant k\leqslant 0$ (we denoted them by $\mathscr{M}W_p^k$ in [17]). Then it follows that T is also an isomorphism between the closures of smooth functions in the latter spaces (denoted by $\mathscr{C}W_p^k$ in [17]) which we have denoted here by W_p^k.

Analogous assertions are true, their proofs being similar, for the operators $\mathring{T}$, V, $\mathring{V}$. Thus we have obtained the first part of the following corollary.

Corollary 4.11. *The assertion of Theorem 4.9 concerning the operators T, $\mathring{T}$, V, $\mathring{V}$ remains true if $\mathscr{F}$ denotes W_p^k, where $-m\leqslant k\leqslant m$, $1\leqslant p\leqslant\infty$, or $B_{p,q}^s$, where $-m<s<m$, $1\leqslant p, q\leqslant\infty$.*

The second part will follow by real interpolation between the W_p^{-m} and W_p^m cases of Theorem 4.9, using (4.1) and (4.2) when we have proved the latter formula. At this moment we can interpolate between W_p^0 and W_p^m using Remark 2.51, which yields the result for $0<s<m$, $1\leqslant p, q\leqslant\infty$.

Proof of Theorem 4.9. Observe that (4.4) and (4.5) yield for $f\in L_0(M)$, $i=1,\ldots,N$, that $P_if=P_iP_if$, i.e. P_i is a projection. From this and (4.6) we obtain

$$P_if=P_i\Big(\sum_{j=1}^{N}\chi_{Q_j}\Big)P_if=P_i\chi_{Q_i}P_if \tag{4.12}$$

and, similarly, applying (4.4) and (4.6),

$$\chi_{Q_i}f=\chi_{Q_i}\Big(\sum_{j=1}^{N}P_j\Big)\chi_{Q_i}f=\chi_{Q_i}P_i\chi_{Q_i}f. \tag{4.13}$$

Using these facts and (4.4), (4.5), (4.6) we obtain easily that $T_0S_0=S_0T_0=\text{identity}$.

The proof in the case of V_0, U_0 is similar (or it can be deduced from the case of T_0, S_0 using duality in $L_2(\mu)$ and the density of $L_2(\mu)$ in $L_0(M)$).

Now, by (4.4) and (4.7), the projections $P_1,\ldots,P_N$ define a decomposition of $\mathscr{F}(M)$ into the direct sum

$$\mathscr{F}(M)\leftrightarrow\sum_{i\leqslant N}\oplus P_i\big(\mathscr{F}(M)\big).$$

Since $\mathring{\mathscr{F}}(M)$ is the closure in $\mathscr{F}(M)$ of those $f\in\mathscr{F}(M)$ which vanish in a neighborhood of ∂M, (4.7) and (4.8) imply that $P_i\big(\mathring{\mathscr{F}}(M)\big)\subseteq\mathring{\mathscr{F}}(M)$ for $i=1,\ldots,N$, and hence one has the decomposition

$$\mathring{\mathscr{F}}(M)\leftrightarrow\sum_{i\leqslant N}\oplus P_i\big(\mathring{\mathscr{F}}(M)\big).$$

Analogously, using (4.7*) and (4.8) we obtain the decompositions

$$\mathscr{F}(M)\leftrightarrow\sum_{i\leqslant N}\oplus P_i^*\big(\mathscr{F}(M)\big),\quad \mathring{\mathscr{F}}(M)\leftrightarrow\sum_{i\leqslant N}\oplus P_i^*\big(\mathring{\mathscr{F}}(M)\big).$$

In Lemma 4.14 we describe explicit isomorphisms between the ranges of the P_i's and P_i^*'s and the suitable spaces $\mathscr{F}(Q_i)_Z$. It is easy to see that this leads exactly to our assertion that T, $\mathring{T}$, V, $\mathring{V}$ are isomorphisms between the respective spaces.

Lemma 4.14. *Given i, $1\leqslant i\leqslant N$, let $Rf=f|_{Q_i}$ be the restriction map. Then, if $\mathscr{F}=W_p^k$, where $0\leqslant k\leqslant m$, $1\leqslant p\leqslant\infty$, R defines linear topological isomorphisms*

$$R_1\colon P_i\big(\mathscr{F}(M)\big)\to\mathscr{F}(Q_i)_{Z_i},$$

$$R_2\colon P_i\big(\mathring{\mathscr{F}}(M)\big)\to\mathscr{F}(Q_i)_{Z_i^\circ},$$

$$R_3\colon P_i^*\big(\mathscr{F}(M)\big)\to\mathscr{F}(Q_i)_{Z_i^{\circ\prime}},$$

$$R_4\colon P_i^*\big(\mathring{\mathscr{F}}(M)\big)\to\mathscr{F}(Q)_{Z_i'},$$

all the spaces being equipped with the W_p^k norm.

Proof. Since the R_j's are obviously continuous and all the above spaces are complete, it suffices, by the open mapping theorem, to check that the R_j's are algebraic isomorphisms between the respective spaces. (A direct proof that P_i induces the (continuous) inverse maps to R_1 and R_2, cf. (4.13) (resp. P_i^* induces the inverse maps to R_3 and R_4) can be obtained using Remark 2.58 and Corollary 5.39 (b)).

Suppose that $f\in P_i\big(\mathscr{F}(M)\big)$ and $Rf=0$. Since $f=P_if$, using (4.12) we obtain $f=P_i\chi_{Q_i}f=P_i(0)=0$, i.e. R_1, and hence also R_2, is one-to-one.

Let us check that

$$R\Big(P_i\big(\mathscr{F}(M)\big)\Big)\subseteq\mathscr{F}(Q_i)_{Z_i},\quad R\Big(P_i\big(\mathring{\mathscr{F}}(M)\big)\Big)\subseteq\mathscr{F}(Q_i)_{Z_i^\circ}. \tag{4.15}$$

Observe that if $Y=\bigcup_{j\leqslant i}Q_j$, then $\operatorname{Int}Y$ determines $\mathscr{F}(Q_j)_{Z_j}$ in the sense of Lemma 2.54 and Remark 2.58. The assumptions of that lemma are satisfied thanks to Theorem 3.3 and the Stein theorem.

Now, if $f\in P_i\big(\mathscr{F}(M)\big)$, then $f=P_if$, and hence $f|_{Y\setminus Q_i}=0$ because by (4.6) one gets

$$\sum_{j<i}\chi_{Q_j}f=\sum_{j<i}\chi_{Q_j}P_if=0.$$

Since $\operatorname{Int}Y$ determines $\mathscr{F}(Q_j)_{Z_j}$, this implies $Rf\in\mathscr{F}(Q_j)_{Z_j}$.

To prove the second inclusion in (4.15), consider a compact d-dimensional C^∞ manifold $\tilde{M}$ which contains M and satisfies $\partial\tilde{M} = \emptyset$. If $f \in P_i(\mathring{\mathscr{F}}(M)) \subseteq \mathring{\mathscr{F}}(M)$, then $f|_{Y\setminus Q_i} = 0$ and there is an $\tilde{f} \in \mathscr{F}(\tilde{M})$ with $\tilde{f}|_M = f$ and $\tilde{f}|_{\tilde{M}\setminus M} = 0$. Again, since $\operatorname{Int}(Y \cup (\tilde{M}\setminus M))$ determines $\mathscr{F}(Q_i)_{Z_i^\circ}$, we obtain that $Rf = R\tilde{f} \in \mathscr{F}(Q_i)_{Z_i^\circ}$.

We have shown (4.15). The converse inclusions are proved as follows. Let $f \in \mathscr{F}(Q_i)_{Z_i}$. Since $\operatorname{Int} Y$ determines $\mathscr{F}(Q_i)_{Z_i}$, we can find a $g \in \mathscr{F}(M)$ so that $Rg = f$ and $g|_{Y\setminus Q_i} = 0$. Moreover, if $f \in \mathscr{F}(Q_i)_{Z_i^\circ}$, then g can be chosen in $\mathring{\mathscr{F}}(M)$, viz. take $g = \tilde{g}|M$, where $\tilde{g} \in \mathscr{F}(\tilde{M})$ satisfies $R = f$ and $\tilde{g}|_X = 0$, where $X = (\tilde{M}\setminus M) \cup (Y\setminus Q_i)$. (Such a $\tilde{g}$ exists because $\operatorname{Int}(Y \cup (\tilde{M}\setminus M))$ determines $\mathscr{F}(Q_i)_{Z_i^\circ}$.)

Since $f = Rg$, it will suffice if we show that $Rg = RP_i g$. To do this observe that, for $j > i$, (4.6) yields $RP_j g = R\chi_{Q_i} P_j g = 0$. On the other hand, if $j < i$, then by (4.6)

$$P_j g = P_j \sum_{k\leq j} \chi_{Q_k} g = P_j(0) = 0.$$

Hence, by (4.4), we get $f = Rg = RP_i g$. This proves that in (4.15) equalities hold. Applying the open mapping theorem we conclude that R_1 and R_2 are topological isomorphisms.

The proof in the case of R_3 and R_4 is analogous and can be omitted. This completes the proof of Lemma 4.14 and of Theorem 4.9.

Remark. In this proof it is not necessary to know that Y is a proper set. Namely, e.g. using the extension from Y constructed in Section 6 and Corollary 5.39 (b), one can check (the analog of) condition (2.57) knowing only that $\operatorname{Int} Y$ has the so-called segment property. The same comment applies in other places as well.

Theorem 4.9 and Corollary 4.11 show that certain problems concerning the spaces $\mathscr{F}(M)$ and $\mathring{\mathscr{F}}(M)$ (where $\mathscr{F} = W_p^k$ or $B_{p,q}^s$ and M is a compact d-dimensional C^∞ manifold) can be completely reduced to the study of finitely many standard spaces of the type $\mathscr{F}(I^d)_Z$. We are interested in constructing Schauder bases in $\mathscr{F}(M)$ with properties specified in Theorems A and B of the Introduction. This will be done for the spaces $\mathscr{F}(Q)_Z$ in Sections 7–10 (Part II). The discussion will be completed in Section 11.

5. H-operators. Let M be a d-dimensional σ-compact C^∞ manifold (perhaps with boundary). Let μ be a fixed smooth measure on M equivalent to the Lebesgue measure (this notion is defined as in Section 4). Let $L_0(M)$ denote the linear space of all (equivalence classes of) measurable functions on M equipped with the topology of convergence in measure on each compact subset of M. Finally, let $\tilde{M}$ be a d-dimensional C^∞ manifold without boundary so that M is a closed subset of $\tilde{M}$ and the two C^∞ structures on M coincide (if $\partial M = \emptyset$, we simply take $\tilde{M} = M$).

We shall consider a special class, $\mathscr{H}(M)$, of linear operators acting in $L_0(M)$ which we shall call *H-operators* (cf. [25]). The properties of this class which we study in this section will enable us to construct the sequence $P_1, \ldots, P_N$ described in Proposition 4.3 and to prove that it satisfies the conditions we need.

Let us recall that a subset of $\mathbf{R}^d$ is said to be *Jordan measurable* (*J-measurable*) if its boundary has d-dimensional Lebesgue measure zero. An analogous notion can be introduced on M.

Suppose we are given a function $\varphi \in C^\infty(\tilde{M})$, a C^∞ map $\Phi\colon \tilde{M}\to\tilde{M}$ and a measurable set $V \subseteq M$ such that $\Phi(V) \subseteq M$. If f is a function on M, we define $H_{\varphi,\Phi,V} f$ by the formula

$$(5.1)\qquad (H_{\varphi,\Phi,V} f)(x) = \begin{cases} \varphi(x) f(\Phi(x)), & x \in V, \\ 0, & x \in M\setminus V. \end{cases}$$

The operation $H_{\varphi,\Phi,V}$ will be called a *simple H-operator* if V is J-measurable and there is an open set $W \subseteq \tilde{M}$ such that $W \supset \bar{V}$ and $\Phi|_W$ is a diffeomorphism. (Clearly, the operation $H_{\varphi,\Phi,V}$ depends only on $\varphi|_V$, $\Phi|_V$, and so does the property of being a simple H-operator, but it is convenient for us to have the notation that displays both φ and Φ.)

Obviously, a simple H-operator induces a continuous linear map of the space $L_0(M)$ into itself. We define $\mathscr{H}(M)$ to be the set of all operators acting in $L_0(M)$ that are finite sums of such maps.

Observe that $\mathscr{H}(M)$ is an algebra of operators. This follows from the simple formula

$$(5.2)\qquad H_{\varphi_2,\Phi_2,V_2} \circ H_{\varphi_1,\Phi_1,V_1} = H_{\varphi,\Phi,V},$$

where $\varphi = \varphi_2\cdot(\varphi_1\circ\Phi_2)$, $\Phi = \Phi_1\circ\Phi_2$ and $V = V_2 \cap \Phi_2^{-1}(V_1)$.

Before we introduce further definitions let us explain that in our notation we identify (scalar) functions on a set with the multiplication operators determined by the functions, e.g. χ_V may also denote the operator $f \mapsto \chi_V f$.

Let $V \subseteq M$ be a measurable subset and let $A \in \mathscr{H}(M)$. We say that A is the *identity on* V, in symbols $A = 1$ on V, if for $f \in L_0(M)$ one has $Af = f$ μ-a.e. on V.

LEMMA 5.3. *$A = 1$ on V iff $\chi_V A = \chi_V$.*

Proof. This is obvious because, for $f \in L_0(M)$, $Af - f = 0$ a.e. on V if and only if $\chi_V(Af - f) = 0$ in $L_0(M)$.

Let $A\colon X \to Y$ be a linear operator, where $X \subseteq L_0(M)$ and Y is a linear space, and let $V \subseteq M$ be measurable. We say that A is *supported on* V, in symbols $\operatorname{supp} A \subseteq V$, if $Af = 0$ for each $f \in X$ such that $f = 0$ a.e. on V.

LEMMA 5.4. *If $\chi_V f \in X$ for $f \in X$, then* $\operatorname{supp} A \subseteq V$ *is equivalent to* $A = A\chi_V$.

Proof. Suppose $A = A\chi_V$. Then, if $f \in X$ and $f = 0$ a.e. on V, we have $Af = A(\chi_V f) = A(0) = 0$. Conversely, if $\operatorname{supp} A \subseteq V$ and $f \in X$, then $f - \chi_V f = 0$ on V, and hence $Af - A\chi_V f = A(f - \chi_V f) = 0$.

Remark 5.5. If $A\colon X \to Y$ is a linear operator, $X \subseteq L_0(M)$ and $\operatorname{supp} A \subseteq V \subseteq M$, then there is a natural way to extend the domain where A is defined. Namely, if g is a measurable function defined on a set V' such that $V \subseteq V' \subseteq M$ and there is $f \in X$ such that $f - g = 0$ a.e. on V, then we set

$$\tilde{A} g = Af.$$

LEMMA 5.6. *If $A \in \mathscr{H}(M)$, $\operatorname{supp} A \subseteq V$ and $A = 1$ on V, then A is a projection.*

Proof. We have, by Lemmas 5.3 and 5.4,

$$AA = (A\chi_V)A = A(\chi_V A) = A\chi_V = A.$$

LEMMA 5.7. *Suppose $A \in \mathscr{H}(M)$ and $\operatorname{supp} A \subseteq V$, where $V \subseteq M$ is J-measurable. If $A = \sum_i H_{\varphi_i, \phi_i, V_i}$, then A has also the H-representation*

$$A = \sum_i H_{\varphi_i, \phi_i, V_i'},$$

where $V_i' = V_i \cap \phi_i^{-1}(V)$.

Proof. Use Lemma 5.4 and formula (5.2).

If $f, g \in L_0(M)$ and $fg \in L_1(M, \mu)$, then we set

$$(f, g) = (f, g)_\mu = \int_M f(x)\overline{g(x)}\,d\mu(x).$$

It is convenient to introduce the subspace $L_2^c(M) \subset L_0(M)$ which consists of all $f \in L_2(M, d\mu)$ that vanish off a compact subset of M. Obviously, $L_2^c(M)$ is a linear subspace, and, if $A \in \mathscr{H}(M)$, then $A\big(L_2^c(M)\big) \subseteq L_2^c(M)$. Also $L_2^c(M)$ does not depend on the choice of the smooth measure μ.

LEMMA 5.8. *If $A \in \mathscr{H}(M)$, then there is a unique $B \in \mathscr{H}(M)$ such that for $f, g \in L_2^c(M)$ one has*

$$(Af, g)_\mu = (f, Bg)_\mu.$$

Proof. The uniqueness of B is obvious since H-operators map $L_2^c(M)$ into $L_2^c(M)$ and this subspace is dense in $L_2(M, \mu)$ as well as in $L_0(M)$.

To prove the existence it is enough to consider the case where A is a simple H-operator, say $A = H_{\varphi, \phi, V}$. Observe that, letting $V_1 = \varphi(V)$, we have

$$(Af, g)_\mu = \int_V \varphi(x) f\big(\phi(x)\big)\overline{g(x)}\,d\mu(x)$$
$$= \int_{V_1} f(y)\varphi\big(\phi^{-1}(y)\big)\overline{g\big(\phi^{-1}(y)\big)}\,\psi(y)\,d\mu(y),$$

where ψ is the Radon–Nikodym derivative $d\nu/d\mu$ on V_1, ν being the measure transported from the open set $W \supset \overline{V}$ by the diffeomorphism $\phi|_W$. Our assumptions give that ψ is the restriction of a function $\psi_1 \in C^\infty(\tilde{M})$ to V_1. Thus we can write the right-hand side in the form $(f, Bg)_\mu$, where $B = H_{\varphi_1, \phi_1, V_1}$, if we let ϕ_1 be a C^∞ map equal to $(\phi|_W)^{-1}$ in a neighborhood of $\overline{V}_1$ and $\varphi_1 \in C^\infty(\tilde{M})$ satisfies

$$\varphi_1(y) = \overline{\varphi\big(\phi^{-1}(y)\big)}\,\psi_1(y)$$

in a neighborhood of $\overline{V}_1$.

DEFINITION 5.9. The operator $B \in \mathscr{H}(M)$ described in Lemma 5.8 is said to be the *H-operator adjoint to* A and is denoted by A^* or $A^{*\mu}$.

Remark 5.10. All the notions previously introduced in this section did not depend on our choice of μ, i.e. μ could be replaced by another smooth measure ν, where $d\nu = h\,d\mu$, $h \in C^\infty(M)$, $h(x) > 0$ for $x \in M$.

This is not the case with the operation $A \to A^{*\mu}$. We have, however, the simple relationship for $A \in \mathscr{H}(M)$:

$$A^{*\nu} = (1/h)A^{*\mu}h. \tag{5.11}$$

(Indeed, it suffices to check that, for $f, g \in L_2^c(M)$,

$$\int_M (A^{*\nu}f)\bar{g}h\,d\mu = (A^{*\nu}f, g)_\nu = (f, Ag)_\nu$$
$$= \int_M f(\overline{Ag})h\,d\mu = (fh, Ag)_\mu = \big(A^{*\mu}(fh), g\big)_\mu$$
$$= \int_M A^{*\mu}(fh)\bar{g}\,d\mu.)$$

It follows from (5.11) that, although in general $A^{*\nu} \neq A^{*\mu}$, the property "A^* acts continuously in a space $\mathscr{F}(M)$ of smooth functions on M" may be independent of the choice of a smooth measure on M, e.g., this is the case if M is compact.

LEMMA 5.12. *Let $A \in \mathscr{H}(M)$ and let V be a measurable subset of M. Then $A = 1$ on V is equivalent to* $\operatorname{supp}(1 - A^*) \subseteq M \setminus V$.

Proof. Since $\chi_V A = \chi_V$ is equivalent to $A^*\chi_V = \chi_V$, which is the same as $(1 - A^*)\chi_V = 0$, this lemma follows from Lemmas 5.3 and 5.4.

DEFINITION. Let $A \in \mathscr{H}(M)$ and let $V \subseteq M$ be measurable. We say that A is an *H-extension from* V if $\operatorname{supp} A \subseteq V$ and $A = 1$ on V.

COROLLARY 5.13. *If $A \in \mathscr{H}(M)$, then A is an H-extension from V if and only if $1 - A^*$ is an H-extension from $M \setminus V$.*

LEMMA 5.14. *Let $A \in \mathscr{H}(M)$ and $m \geqslant 0$. Put*

$$C_A^m = \{f \in C^m(M) \colon Af = g \text{ a.e. for some } g \in C^m(M)\}.$$

Then C_A^m is closed in $C^m(M)$ (in the topology of uniform C^m convergence on compact subsets of M) and the induced map $A\colon C_A^m \to C^m(M)$ is continuous.

DEFINITION. The operator $A \in \mathscr{H}(M)$ is said to be *of class* C^m, $m \geqslant 0$, if C_A^m is dense in $C^m(M)$ (by Lemma 5.14 this is equivalent to $C_A^m = C^m(M)$). A is said to be *of class* C^{-m}, $m \geqslant 1$, if A^* is of class C^m (cf. the remark before Lemma 5.12).

We shall prove Lemma 5.14 at the end of this section.

Now we shall introduce tensor products of H-operators. We let M' be another C^∞ manifold of dimension d'. M' is equipped with a σ-finite smooth measure μ'. We consider the manifold $M \times M'$ with the smooth measure $\mu \otimes \mu'$. It is not important in this context whether or not M and (or) M' have a boundary. Finally, if $f \in L_0(M)$, $g \in L_0(M')$, then $f \otimes g \in L_0(M \times M')$ is given by $(f \otimes g)(x, y) = f(x) g(y)$.

LEMMA 5.15. *If $A \in \mathscr{H}(M)$ and $B \in \mathscr{H}(M')$, then there is a unique continuous linear operator T in the space $L_0(M \times M')$ such that*

$$T(f \otimes g) = (Af) \otimes (Bg) \tag{5.16}$$

for $f \in L_0(M)$, $g \in L_0(M')$. Moreover, $T \in \mathscr{H}(M \times M')$.

Proof. The uniqueness of such an operator is obvious, because the subspace of $L_0(M \times M')$ spanned by the functions of the form $f \otimes g$ is dense in $L_0(M \times M')$. To prove the existence of T it is enough to consider the case where A and B are simple H-operators, say $A = H_{\varphi, \Phi, V}$ and $B = H_{\psi, \Psi, W}$. Clearly, the operator

$$T = H_{\varphi \otimes \psi, \Phi \times \Psi, V \times W},$$

where $(\Phi \times \Psi)(x, y) = (\Phi(x), \Psi(y))$, has the required property.

DEFINITION. The operator T given by Lemma 5.15 is denoted by $A \otimes B$ and is said to be the *tensor product* of the H-operators A and B.

COROLLARY 5.17. *Let M, M' be manifolds, $A, A_1 \in \mathscr{H}(M)$, $B, B_1 \in \mathscr{H}(M')$, $V \subseteq M$ and $V' \subseteq M'$ be measurable sets. Then*

$$(A_1 \otimes B_1) \circ (A \otimes B) = (A_1 \circ A) \otimes (B_1 \circ B), \tag{5.18}$$

(5.19) *if* $\operatorname{supp} A \subseteq V$, $\operatorname{supp} B \subseteq V'$, *then* $\operatorname{supp}(A \otimes B) \subseteq V \times V'$,

(5.20) *if $A = 1$ on V, $B = 1$ on V', then $A \otimes B = 1$ on $V \times V'$,*

$$(A \otimes B)^* = A^* \otimes B^*, \tag{5.21}$$

(5.22) *if A, B are of class C^m for some m, then so is $A \otimes B$.*

Proof. Property (5.18) follows easily from Lemma 5.15. Similarly, (5.21) follows from Lemmas 5.15 and 5.8. Properties (5.19) and (5.20) follow from (5.18) and Lemmas 5.4 and 5.3, respectively. Finally, if $m \geqslant 0$, then (5.22) is obvious, because by (5.16) we have

$$C_{A \otimes B}^m \supseteq C_A^m \otimes C_B^m$$

and, if A and B are of class C^m, then the subspace $C_A^m \otimes C_B^m$ is dense in $C^m(M \times M')$. The case where $m < 0$ follows immediately from (5.22) for the positive parameter $-m$, thanks to (5.21).

LEMMA 5.23. *Suppose M_1, M_2 are manifolds, $A_i \in \mathscr{H}(M_i)$ and $V_i \subseteq M_i$ is measurable for $i = 1, 2$. If $\operatorname{supp} A_1 \subseteq V_1$ and $A_2 = 1$ on V_2, then, for $f \in L_0(M_1 \times M_2)$, $f = 0$ a.e. on $V_1 \times V_2$ implies that*

$$(A_1 \otimes A_2) f = 0 \text{ a.e. on } M_1 \times V_2.$$

Proof. This follows from the identity

$$(1 \otimes \chi_{V_2})(A_1 \otimes A_2) f = (A_1 \otimes \chi_{V_2} A_2) f = (A_1 \otimes 1)(\chi_{V_1} \otimes \chi_{V_2}) f = 0$$

(we used (5.18) and Lemmas 5.3 and 5.4).

DEFINITION. Let $A \in \mathscr{H}(M)$ and let $U \subseteq M$ be a measurable set. We say that A *preserves vanishing on* U if, for $f \in L_0(M)$, $f = 0$ a.e. on U implies that $Af = 0$ a.e. on U.

LEMMA 5.24. *Let $U \subseteq M$ be a measurable set and $A \in \mathscr{H}(M)$. Then A preserves vanishing on U iff A^* preserves vanishing on $M \setminus U$.*

Proof. The lemma follows from the identity $(A^* h, g) = \overline{(Ag, h)}$ for $g, h \in L_2^c(M)$. Indeed, A (resp. A^*) preserves vanishing on U (resp. U^*) iff $(Ag, h) = 0$ (resp. $(A^* h, g) = 0$) whenever $g = 0$ a.e. on U and $h = 0$ a.e. on $M \setminus U$.

LEMMA 5.25. *Suppose $A \in \mathscr{H}(M)$ and $U \subseteq M \times M'$ is a measurable set such that, for μ'-a.e. $t \in M'$, the operator A preserves vanishing on $U_t = \{s \in M \colon (s, t) \in U\}$. Then the operator $A \otimes 1 \in \mathscr{H}(M \times M')$ preserves vanishing on U.*

Proof. Observe that if $f \in L_0(M \times M')$ and $f_t(s) = f(s, t)$, then one has

$$\big((A \otimes 1) f\big)(s, t) = (Af_t)(s) \text{ a.e. on } M \times M'.$$

(This formula is obvious if A is a simple H-operator, cf. the proof of Lemma 5.15.) If $\chi_U f = 0$ a.e. on $M \times M'$, then $\chi_{U_t} f_t = 0 \in L_0(M)$ for a.e. $t \in M'$, and hence $\chi_{U_t} A f_t = 0 \in L_0(M)$ for a.e. $t \in M'$, which yields that

$$\chi_U(A \otimes 1) f = 0 \in L_0(M \times M').$$

LEMMA 5.26. *Let $E', E'' \in \mathscr{H}(M)$ and let $V', V'' \subseteq M$ be measurable sets. Suppose that E' (resp. E'') is an H-extension from V' (resp. V'') and, in addition, E'' preserves vanishing on V'. Then the operator*

$$E = E' + E'' - E''E' \tag{5.27}$$

is an H-extension from the set $V = V' \cup V''$.

Proof. Write $\chi = \chi_V$, $\chi' = \chi_{V'}$, $\chi'' = \chi_{V''}$. Since E'' preserves vanishing on V', $\chi'(1-E') = 0$ implies that $\chi' E''(1-E') = 0$, and hence

$$\chi'(1-E) = \chi'(1-E') - \chi' E''(1-E') = 0.$$

On the other hand, since $\chi - \chi' = (1-\chi')\chi''$, we have

$$(\chi - \chi')(1-E) = (1-\chi')\big(\chi''(1-E'')\big)(1-E') = 0.$$

It follows that $\chi(1-E) = 0$ and, by Lemma 5.3, $E = 1$ on V.

Now, if $\chi f = 0$, then $E'f = 0$ and $E''f = 0$, and hence $Ef = 0$. This shows that $\operatorname{supp} E \subseteq V$.

LEMMA 5.28. *Suppose that M_1, M_2 are manifolds and, for $i = 1, 2$, $E_{A_i}, E_{B_i} \in \mathscr{H}(M_i)$ are H-extensions from subsets $A_i, B_i \subseteq M_i$ such that $A_i \subseteq B_i$. Then*

$$E = E_{A_1} \otimes E_{B_2} + E_{B_1} \otimes E_{A_2} - (E_{B_1} E_{A_1}) \otimes (E_{A_2} E_{B_2}) \tag{5.29}$$

is an H-extension from the set

$$A_1 \times B_2 \cup B_1 \times A_2 \subseteq M_1 \times M_2.$$

Moreover, E is of class C^m if so are $E_{A_1}, E_{B_1}, E_{A_2}, E_{B_2}$.

Proof. We use Lemma 5.27 with $E' = E_{A_1} \otimes E_{B_2}$ and $E'' = E_{B_1} \otimes E_{A_2}$. The fact that E' (resp. E'') is an H-extension from $A_1 \times B_2$ (resp. from $B_1 \times A_2$) follows from (5.19) and (5.20). To check that E'' preserves vanishing on the set $A_1 \times B_2$ we can use Lemma 5.23, because $E_{B_1} = 1$ on A_1 and $\operatorname{supp} E_{A_2} \subseteq B_2$. Therefore, the first statement follows from Lemma 5.27.

The final assertion almost follows from (5.22). Only the last term in (5.29) presents a difficulty which will disappear when we have proved Lemma 5.14.

Remark 5.30. Note that if, in the setting of Lemma 5.28, $A_1 \cup B = B_1$, then

$$A_1 \times B_2 \cup B \times A_2 = A_1 \times B_2 \cup B_1 \times A_2,$$

and hence (5.29) gives an H-extension from the set $A_1 \times B_2 \cup B \times A_2$.

DEFINITION. A finite family $\mathscr{U}$ of open subsets of M will be called a *J-partition* if

$$\sum_{U \in \mathscr{U}} \chi_U = 1 \quad \mu\text{-a.e.}$$

Observe that the sets in $\mathscr{U}$ must be mutually disjoint and J-measurable.

LEMMA 5.31. *For every H-operator $A = \sum_{i=1}^{n} H_{\varphi_i, \phi_i, V_i} \in \mathscr{H}(M)$ there is a J-partition $\mathscr{U}$ of M such that*

$$A = \sum_{U \in \mathscr{U}} \sum_{V_i \supseteq U} H_{\varphi_i, \phi_i, U}. \tag{5.32}$$

Proof. Write $V_{i,1} = \operatorname{Int} V_i$, $V_{i,-1} = M \setminus \overline{V}_i$ for $i = 1, \ldots, n$. Set, for $a = \big(a(1), \ldots, a(n)\big) \in \{-1, 1\}^n$,

$$U_a = \bigcap_{i \leqslant n} V_{i, a(i)}.$$

It is easy to see that the family $\mathscr{U} = \{U_a \colon a \in \{-1, 1\}^n\}$ has the required property.

It is convenient to use the following scheme in order to prove continuity of an H-operator $A \in \mathscr{H}(M)$. We write A as in (5.32). Let $|||\ |||$ be a seminorm on a linear subspace $X_0 \subseteq L_0(M)$ and let $\|\ \|_{\mathscr{F}}$ be a function on $L_0(M)$. Suppose that there exist seminorms $\|\ \|_{\mathscr{F}(U)}$, $U \in \mathscr{U}$, such that:

(5.33) there is $C < \infty$ such that for $g \in A(X_0)$

$$\|g\|_{\mathscr{F}} \leqslant C \max_{U \in \mathscr{U}} \|g\|_{\mathscr{F}(U)},$$

(5.34) there exist constants $C_{U,i} < \infty$ such that if $f \in X_0$ and $U \subseteq V_i$, then

$$\|H_{\varphi_i, \phi_i, U} f\|_{\mathscr{F}(U)} \leqslant C_{U,i} |||f|||.$$

Then, for $f \in X_0$, one has

$$\|Af\|_{\mathscr{F}} \leqslant C' |||f|||,$$

where $C' = C \max_U \sum_i C_{U,i} < \infty$ does not depend on f.

Recall that the topology of the space $C^m(M)$ can be determined by a system of seminorms

$$\|f\|^{(m)}(K) = \max_{0 \leqslant k \leqslant m} \sup_{x \in \operatorname{Int} K} \|D^k f(x)\|, \tag{5.35}$$

where K ranges over relatively compact subsets of M. (Here $D^k f(y)$ denotes the appropriate k-linear map. It is not essential to specify what we mean because all reasonable descriptions lead to equivalent seminorms.)

Proof of Lemma 5.14. Given $A \in \mathscr{H}(M)$, we may assume that A is written as in (5.32). Put $X_0 = C_A^m$. Fix a compact set $K \subseteq M$ and set for $g \in C^m(M)$, $U \in \mathscr{U}$ ($\mathscr{U}$ being that of Lemma 5.31)

$$\|g\|_{\mathscr{F}} = \|g\|^{(m)}(K), \qquad \|g\|_{\mathscr{F}(U)} = \|g\|^{(m)}(K \cap U)$$

(cf. (5.35)). Condition (5.33) will be satisfied because $\|g\|_{\mathscr{F}} = \max_U \|g\|_{\mathscr{F}(U)}$. Recall the easy fact that if $H_{\varphi,\Phi,U} \in \mathscr{H}(M)$ is a simple H-operator and $V \subseteq U$ is J-measurable and relatively compact in M, then there is $c < \infty$ such that for $f \in C^m(M)$ one has

$$\|H_{\varphi,\Phi,V} f\|^{(m)}(V) \leqslant c\|f\|^{(m)}\big(\Phi(V)\big). \tag{5.36}$$

Thus condition (5.34) of our scheme will also be satisfied if we let

$$|||f||| = \max_{U \in \mathscr{U}} \max_{V_i \supseteq U} \|f\|^{(m)}\big(\Phi_i(U \cap K)\big).$$

This shows the continuity of $A\colon X_0 \to C^m(M)$, because $|||\ |||$ is a continuous seminorm on $C^m(M)$, and hence on the subspace X_0. In fact, $|||\ ||| = |||\ |||^{(m)}(L)$, where $L \subseteq \bigcup_i \phi_i(V_i \cap K)$ is a relatively compact set.

Finally, the fact that $X_0 = C_A^m$ is closed in $C^m(M)$ follows from the next lemma, if we let X denote the closure of X_0 in $C^m(M)$ and set $Y = L_0(M)$, $Y_0 = C^m(M)$ letting $j\colon Y_0 \to Y$ be the natural embedding.

LEMMA 5.37. *Let $A\colon X \to Y$ be a continuous linear operator. Suppose that Y_0 is a complete topological vector space, $j\colon Y_0 \to Y$ is a continuous linear one-to-one map and Y is Hausdorff.*

If there exists a dense subspace $X_0 \subseteq X$ such that $A(X_0) \subseteq j(Y_0)$ and $j^{-1} \circ (A|_{X_0})$ is continuous, then $A(X) \subseteq j(Y_0)$ and $j^{-1} \circ A$ is continuous.

Proof. Since Y_0 is complete, the operator $j^{-1} \circ (A|_{X_0})$ has a unique continuous extension $B\colon X \to Y_0$. The operators jB and A are equal, because they are continuous and coincide on a dense subspace. It follows that $A(X) \subseteq j(Y_0)$ and $j^{-1} \circ A = B$ is continuous.

LEMMA 5.38. *Let $A \in \mathscr{H}(M)$ be of class C^m, $m \geqslant 1$. Suppose that $\operatorname{supp} A \subseteq V$, where $V \subseteq M$ is a compact J-measurable set. Then there is $C' < \infty$ such that for all $f \in C^m(M)$, $1 \leqslant p \leqslant \infty$, $0 \leqslant k \leqslant m$, one has*

$$\|Af\|_p^{(k)}(M) \leqslant C'\|f\|_p^{(k)}(V).$$

Proof. This follows from our scheme. We start from an H-representation of A obtained using Lemmas 5.7 and 5.31. Then we set for $g \in C^m(M)$, $h \in C^m(U)$, where $U \in \mathscr{U}$:

$$\|g\|_{\mathscr{F}} = \|g\|_p^{(k)}(M), \qquad |||g||| = \|g\|_p^{(k)}(V), \qquad \|h\|_{\mathscr{F}(U)} = \|h\|_p^{(k)}(U).$$

The easy verification of (5.33) and (5.34), which uses the equality $C_A^m = C^m(M)$ (cf. Lemma 5.14) and an analog of (5.36) for $|||\ |||_p^{(k)}$ norms, can be omitted.

COROLLARY 5.39. *Let M be a compact manifold and let $A \in \mathscr{H}(M)$ be of class C^m, $m \geqslant 1$. If $1 \leqslant p \leqslant \infty$, $0 \leqslant k \leqslant m$, then*

(a) A acts continuously in $W_p^k(M)$,

(b) if A is an H-extension from a closed J-measurable set $V \subseteq M$ and $C^m(V)$ is dense in $W_p^k(V)$ ($= W_p^k(\operatorname{Int} V)$), then A induces a continuous linear extension operator

$$E\colon W_p^k(V) \to W_p^k(M).$$

Proof. Both parts follow from Lemma 5.37. To prove (a) take $X = W_p^k(M)$, $X_0 = C^m(M)$, $Y = L_0(M)$, $Y_0 = W_p^k(M)$ and let j be the natural embedding. Then recall Lemma 5.38 (with $V = M$).

For (b), let $X = W_p^k(V)$, Y, Y_0, j be as above and $X_0 = \{f|_V\colon f \in C^m(M)\}$. The density of X_0 in X follows from our assumption and Whitney's extension theorem. We let $Eg = A\tilde{g}$ where for $g \in L_0(V)$ we set $\tilde{g}(x) = g(x)$ if $x \in V$ and $\tilde{g}(x) = 0$ if $x \in M \setminus V$.

The estimate of Eg for $g \in X_0$ follows from Lemma 5.38. Indeed, if $g = f|_V$, where $f \in C^m(M)$, then we have $Eg = A\tilde{g} = Af$, so that

$$\|Eg\|_p^{(k)}(M) = \|Af\|_p^{(k)}(M) \leqslant C\|f\|_p^{(k)}(V) = C\|g\|_p^{(k)}(V).$$

Finally, $Eg|_V - g = 0$ for $g \in X$, because this is true on a dense subspace, namely X_0.

Remark. $C^m(V)$ is dense in $W_p^k(\operatorname{Int} V)$ for $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$, whenever $\operatorname{Int} V$ satisfies the segment property (cf. [1], Theorem 3.18).

In Section 6 we shall need the following fact.

COROLLARY 5.40. *Let $Q \subset \mathbf{R}^d$ be a compact parallelepiped and let $A \in \mathscr{H}(\mathbf{R}^d)$, $\operatorname{supp} A \subseteq Q$. Suppose that, for some $m \geqslant 1$, the set*

$$\{f \in C^m(Q)\colon \tilde{A}f = g \ \text{a.e. for some}\ g \in C^m(Q)\}$$

(cf. Remark 5.5) *is dense in $C^m(Q)$.*

Then there is $C < \infty$ so that, for $0 \leqslant k \leqslant m$, $1 \leqslant p \leqslant \infty$ and $f \in W_p^k(Q)$, one has

$$\|\tilde{A}f\|_p^{(k)}(Q) \leqslant C\|f\|_p^{(k)}(Q).$$

We omit the proof because it involves no new idea.

6. Construction of H-operators. A subset $U \subseteq M$ is said to *admit arbitrarily smooth H-extensions* provided that for each $m \geqslant 1$ there is an $E \in \mathscr{H}(M)$ that is an H-extension from U of class C^m and C^{-m}.

LEMMA 6.1. *For each $m \geqslant 1$ there is $E \in \mathscr{H}(\mathbf{R})$ which is an H-extension of class C^m and C^{-m} from the set $(-\infty, 0\rangle$.*

Proof. Consider the operator $A \in \mathscr{H}(\boldsymbol{R})$ defined by

$$(Af)(x) = \begin{cases} f(x), & x \leqslant 0, \\ \sum_{j=1}^{s} a_j f(\beta_j x), & x > 0, \end{cases}$$

where $a_1, \ldots, a_s, \beta_1 \ldots, \beta_s$ are real constants such that $0 > \beta_1 > \ldots > \beta_s$. It is clear that A is an H-extension from $(-\infty, 0)$. We shall show that if $s \geqslant 2m+2$ then, for any choice of $\beta_1, \ldots, \beta_s$, one can find $a_1, \ldots, a_s$ so that A is of class C^m and C^{-m}.

Observe that if $f \in C^m(\boldsymbol{R})$, then Af is of class C^m on $(-\infty, 0\rangle$ and on $(0, \infty)$. Since for $x > 0$ and $k = 0, 1, \ldots, m$ one has

$$D^k(Af)(x) = \sum_{j=1}^{s} a_j \beta_j^k D^k f(\beta_j x),$$

we see that A is of class C^m whenever

$$\sum_{j=1}^{s} a_j \beta_j^k = 1 \quad \text{for} \quad k = 0, 1, \ldots, m.$$

Now, if $\boldsymbol{R}$ is given the Lebesgue measure, then the adjoint to A is

$$(A^* g)(y) = \begin{cases} g(x) - \sum_{j=1}^{s} a_j \beta_j^{-1} g(x/\beta_j), & x < 0, \\ 0, & x > 0. \end{cases}$$

Consequently, A^* is of class C^m provided that

$$\sum_{j=1}^{s} a_j \beta_j^{-k-1} = 1 \quad \text{for} \quad k = 0, 1, \ldots, m.$$

This gives a system of $2m+2$ linear equations for $a_1, \ldots, a_s$, which is well known to admit solutions if $s \geqslant 2m+2$.

The next lemma is somewhat more special. We shall use the notation $x \vee y$ for $\max\{x, y\}$, and let for $x \in \boldsymbol{R}^n$

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

Consider the sets $V_{k,n} \subset \boldsymbol{R}^n$, $0 \leqslant k \leqslant n$, given by

$$V_{k,n} = \{x \in \boldsymbol{R}^n \colon 0 \vee x_1 \vee \ldots \vee x_k \geqslant x_{k+1} \vee \ldots \vee x_n\}.$$

LEMMA 6.2. *Let $m \geqslant 1$ be fixed. Then for each $0 \leqslant k < n$ there exists $E = E_{k,n} \in \mathscr{H}(\boldsymbol{R}^n)$ such that*

(6.3) *E is of class C^m and C^{-m},*

(6.4) *E is an H-extension from $V_{k,n}$,*

(6.5) *for each $c \in \boldsymbol{R}$, E preserves vanishing on the set*

$$S_c = \boldsymbol{R}^k \times (-\infty, c\rangle^{n-k}.$$

Proof. In the case $k = 0$, $n = 1$ the operator A constructed in Lemma 6.1 has the required properties. (To prove (6.5), observe that if $c \leqslant 0$, then $A = 1$ on $S_c = (-\infty, c\rangle$, and if $c > 0$, then $\operatorname{supp} A \subseteq S_c$.)

If $k = 0$, $n > 1$, then the n-fold tensor product $E = E_{0,1} \otimes \ldots \otimes E_{0,1}$ has the required properties. This follows from Corollary 5.17.

We prove the general case by induction on k. Suppose that $1 \leqslant k < n$ and the operator $E_{k-1,n-1}$ has already been constructed. Set

$$V' = \{x \in \boldsymbol{R}^n \colon x_{k+1} \vee \ldots \vee x_n \leqslant 0\},$$

$$V'' = \{x \in \boldsymbol{R}^n \colon x_{k+1} \vee \ldots \vee x_n \leqslant x_1 \vee \ldots \vee x_k\}$$

and let $E' = 1_{\boldsymbol{R}^k} \otimes E_{0,n-k}$. We need an H-extension E'' from the set V'' which preserves vanishing on V' so that we can use Lemma 5.26. Consider the isomorphism $\phi \colon \boldsymbol{R}^n \to \boldsymbol{R} \times \boldsymbol{R}^{n-1}$ given by

$$\phi(x) = \big(x_1, (x_2 - x_1, \ldots, x_n - x_1)\big).$$

Clearly,

$$\phi(V') = \{(y_0, y) \in \boldsymbol{R} \times \boldsymbol{R}^{n-1} \colon y_k \vee \ldots \vee y_{n-1} \leqslant -y_0\},$$

$$\phi(V'') = \boldsymbol{R} \times V_{k-1,n-1}.$$

The operator $R = 1_{\boldsymbol{R}} \otimes E_{k-1,n-1}$ is an H-extension from $\phi(V'')$ which preserves vanishing on $\phi(V')$. This follows from Lemma 5.25 and property (6.5) of the operator $E_{k-1,n-1}$. Thus, if $E'' f = R(f \circ \phi^{-1}) \circ \phi$ for $f \in L_0(\boldsymbol{R}^n)$, then, by Lemma 5.26,

$$E = E' + E'' - E'' E'$$

satisfies (6.4). E satisfies (6.3) because so do E' and E''. To check property (6.5) observe that, if $c \leqslant 0$, then $S_c \subseteq V'$, hence $E = 1$ on S_c. If $c > 0$ and $f \in L_0(M)$ satisfies $f|_{S_c} = 0$ a.e., then $E'f = 0$ since $V' \subset S_c$, and hence $Ef = E''f$. Thus we need to check that if $g = f \circ \phi^{-1}$ vanishes on the set

$$\phi(S_c) = \{(y_0, y) \in \boldsymbol{R} \times \boldsymbol{R}^{n-1} \colon y_k \vee \ldots \vee y_{n-1} \leqslant c - y_0\},$$

then so does Rg. This again follows from Lemma 5.25 and property (6.5) of $E_{k-1,n-1}$.

COROLLARY 6.6. *Let $I \subseteq \{1, \ldots, n\}$ and let*

$$Z = \big\{x \in \boldsymbol{R}^n \colon \max_{1 \leqslant i \leqslant n} x_i \leqslant \max\{0, \max_{i \in I} x_i\}\big\}.$$

Then both Z and $\overline{\boldsymbol{R}^n \setminus Z}$ admit arbitrarily smooth H-extensions.

Proof. This follows from Lemma 6.2 and Corollary 5.13.

Consider now the following situation. M and M_1 are d-dimensional manifolds, $U \subseteq M$ and $U_1 \subseteq M_1$ are open sets and ϕ is a diffeomorphism of U onto U_1. Given any complex function g on M we let $g_1(y) = g(\phi^{-1}(y))$ for $y \in U_1$ and $g_1(y) = 0$ for $y \in M_1 \setminus U_1$. Thus, if $\operatorname{supp}\lambda \subseteq U$, then for any k, $\lambda \in C^k(M)$ iff $\lambda_1 \in C^k(M_1)$.

Now, if $R_1 \in \mathscr{H}(M_1)$ and $\lambda \in C_0^\infty(U)$, then we have the induced H-operator $R = R_\lambda \in \mathscr{H}(M)$, where $g = R_\lambda f$ satisfies $\operatorname{supp} g \subseteq U$ and $g_1 = \lambda_1 R_1(\lambda_1 f_1)$.

It is clear that if R_1 is of class C^m, where $m \geqslant 0$, then so is R_λ. The same is true for $m \leqslant -1$. To see this observe that the operators $(R_\lambda)^* = (R_1)^*_\lambda$ are related by the change of density that results from our identification of $(U, \mu|_U)$ with $(U_1, \mu_1|_{U_1})$ by means of ϕ (in a neighborhood of $\operatorname{supp}\lambda$) (cf. (5.11)).

LEMMA 6.7. *Let $V \subseteq M$ be a closed set and let $U_1, \ldots, U_l \subseteq M$ be open sets such that $\partial V \subseteq \bigcup_{i=1}^{l} U_i$. Then there exist $\lambda, \lambda_1, \ldots, \lambda_l \in C^\infty(M)$ with the following property. Suppose that, for $i = 1, 2, \ldots, l$, $E_i \in \mathscr{H}(M)$ is an H-extension from a closed set V_i such that*

$$V_i \cap U_i = V \cap U_i. \tag{6.8}$$

Then the operator E defined by the formula

$$E = \lambda^2 + \sum_{i=1}^{l} \lambda_i E_i \lambda_i$$

is an H-extension from the set V.

Proof. Set $U_0 = M \setminus \partial V$ and $\chi_i = \chi_{U_i}$ for $0 \leqslant i \leqslant l$. Let $\varphi_0, \ldots, \varphi_m \in C^\infty(M)$ be a partition of unity subordinate to the covering $\{U_0, \ldots, U_l\}$ of M. We let $\lambda_i = \varphi_i / (\sum_{j=0}^{l} \varphi_j^2)^{1/2}$ for $i = 0, 1, \ldots, l$ and $\lambda = \lambda_0 \chi$, where $\chi = \chi_V$. Obviously, $\lambda_0, \ldots, \lambda_l \in C^\infty(M)$, $\sum_{i=0}^{l} \lambda_i^2 = 1$ and

$$\lambda_i \chi_i = \lambda_i, \quad i = 1, \ldots, l.$$

Note that $\chi_{V_i} E_i = \chi_{V_i}, E_i \chi_{V_i} = E_i$ and, by (6.8),

$$\chi_{V_i} \chi_i = \chi \chi_i, \quad i = 1, \ldots, l.$$

Using these identities we obtain easily that

$$\lambda_i \chi = \lambda_i \chi_i \chi = \lambda_i \chi_i \chi_{V_i} = \lambda_i \chi_{V_i}, \quad i = 1, \ldots, l,$$

$$\chi E = \chi \lambda^2 + \sum_{i=1}^{l} \lambda_i \chi_{V_i} E_i \lambda_i = \chi \Big(\sum_{i=0}^{l} \lambda_i^2 \Big) = \chi,$$

$$E \chi = \lambda^2 \chi + \sum_{i=1}^{l} \lambda_i E_i \chi_{V_i} \lambda_i = \lambda^2 + \sum_{i=1}^{l} \lambda_i E_i \lambda_i = E.$$

Hence the conclusion follows from Lemmas 5.3 and 5.4.

Remark 6.9. There is an easy generalization of Lemma 6.7 that can be proved in exactly the same way. Namely, let now E_i' be an H-extension from a closed subset V_i of another d-dimensional manifold M_i and suppose we are given a diffeomorphism ϕ_i from U_i into M_i such that

$$V_i \cap \phi_i(U_i) = \phi_i(U_i \cap V), \quad i = 1, \ldots, l.$$

Then the operator E' defined for $f \in L_0(M)$ by the formula (cf. the discussion after Corollary 6.6)

$$E' f = \lambda^2 f + \sum_{i=1}^{l} \lambda_i \big(E_i' ((f\lambda_i) \circ \phi_i^{-1}) \big) \circ \phi_i$$

is again an H-extension (in $\mathscr{H}(M)$) from the set V.

Remark 6.10. If the E_j's in Lemma 6.7 are of class C^m for some integer m, then so is E. The same is true in the situation described in Remark 6.9. This follows from the discussion after Corollary 6.6.

Thus we can conclude that if $V \subset M$ is a compact set, then V admits arbitrarily smooth H-extensions in $\mathscr{H}(M)$ iff V admits such extensions locally, i.e. for each $x \in \partial V$ and $m \geqslant 1$ there are an open set U and a diffeomorphism $\phi\colon U \to M_1$ such that $x \in U$ and $\phi(U \cap V) = \phi(U) \cap V'$, where V' admits an H-extension of class C^m and C^{-m} in $\mathscr{H}(M')$.

In the next two lemmas J is a subset of $\{1, 2, \ldots, d\} \times \{-1, 1\}$ and the sets $K_{j,\varepsilon}$ are defined by

$$K_{j,\varepsilon} = \{x \in \boldsymbol{R}^d \colon x_j = \varepsilon \|x\|_\infty\}.$$

LEMMA 6.11. *Given $c > 0$, let*

$$V = \langle -c, c \rangle^d \cap \bigcup \{K_{j,\varepsilon} \colon (j, \varepsilon) \in J\}.$$

Then V admits arbitrarily smooth H-extensions in $\mathscr{H}(\boldsymbol{R}^d)$.

Proof. Set, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \{-1, 1\}^d$,

$$U_\alpha = \{x \in \boldsymbol{R}^d \colon \alpha_j x_j > -\|x\|_\infty \text{ for } 1 \leqslant j \leqslant d\}.$$

$$V_\alpha = \bigcup \{K_{j,\varepsilon} \colon (j, \varepsilon) \in J,\ \varepsilon = \alpha_j\} \cup \{x \in \boldsymbol{R}^d \colon \alpha_j x_j \leqslant c,\ 1 \leqslant j \leqslant d\}.$$

Observe that $U_\alpha \cap V = U_\alpha \cap V_\alpha$. For, if $x \in U_\alpha \cap V$, then either $\|x\|_\infty \leqslant c$ or, for some $(j, \varepsilon) \in J$, $x \in K_{j,\varepsilon}$, and hence

$$\varepsilon x_j = \|x\|_\infty \geqslant \alpha_j x_j > -\|x\|_\infty$$

which implies $\varepsilon = \alpha_j$. Thus in each case $x \in V_\alpha$. This proves that $U_\alpha \cap V \subseteq U_\alpha \cap V_\alpha$. The inclusion $U_\alpha \cap V_\alpha \subseteq V$ is proved similarly.

It follows easily from Corollary 6.6 that for each $m \geqslant 1$ and $\alpha \in \{-1, 1\}^d$ there is $E_\alpha \in \mathscr{H}(\boldsymbol{R}^d)$ which is an H-extension of class C^m and C^{-m} from V_α. Since $\bigcup_\alpha U_\alpha = \boldsymbol{R}^d \setminus \{0\} \supset \partial V$, the conclusion follows from Lemma 6.7.

LEMMA 6.12. *Suppose $V' \subseteq S^{d-1}$ is of the form*

$$S^{d-1} \cap \bigcup \{K_{j,\varepsilon}: (j, \varepsilon) \in J\},$$

where $J \subseteq \{1, \ldots, d\} \times \{-1, 1\}$. Then V' admits arbitrarily smooth H-extensions in $\mathscr{H}(S^{d-1})$.

Proof. Let $W = \{x \in S^{d-1}: \sum_{j=1}^{n} x_j > 0\}$. Define $\phi: W \to \boldsymbol{R}^{d-1}$ to be the composition of the central projection of W onto the hyperplane $\{x \in \boldsymbol{R}^d: \sum_{j=1}^{d} x_j = 1\}$ (i.e. the map $x \mapsto (\sum_{j=1}^{d} x_j)^{-1} x$) and the affine map

$$y \mapsto (y_1 - y_d, \ldots, y_{d-1} - y_d).$$

Thus, for $x \in W$,

$$\phi(x) = \Big(\sum_{j=1}^{d} x_j\Big)^{-1} (x_1 - x_d, \ldots, x_{d-1} - x_d)$$

and, clearly, ϕ is a diffeomorphism of W onto $\boldsymbol{R}^{d-1}$.

Now, if $\alpha = (\alpha_1, \ldots, \alpha_d) \in \{-1, 1\}^d$, we let

$$U'_\alpha = U_\alpha \cap \Big\{x \in S^{d-1}: \sum_{j=1}^{d} \alpha_j x_j > 1/2\Big\},$$

where U_α is the same as in the proof of Lemma 6.11. Set for $x \in U_\alpha$

$$\phi_\alpha(x) = \phi(\alpha_1 x_1, \ldots, \alpha_d x_d).$$

Note that the U'_α's form an open covering of S^{d-1}. Observe also that for each $\alpha \in \{-1, 1\}^d$ there is a set $V_\alpha \subseteq \boldsymbol{R}^{d-1}$ which is of the form described in Corollary 6.6 and satisfies

$$V_\alpha \cap \phi_\alpha(U'_\alpha) = \phi_\alpha(U'_\alpha \cap V').$$

Thus the lemma follows from Remark 6.9.

COROLLARY 6.13. (a) *If $Q_1, Q_2, \ldots, Q_{2d}$ is the standard decomposition of the sphere S^{d-1}, then the sets $\bigcup_{i \leq j} Q_i$, $1 \leq j \leq 2d$, admit arbitrarily smooth H-extensions in $\mathscr{H}(S^{d-1})$.*

(b) *If $Q_1, Q_2, \ldots, Q_{2d+1}$ is the standard decomposition of the ball $D_d \subset \boldsymbol{R}^d$ and $I \subseteq \{1, \ldots, 2d+1\}$, then the set $C = \bigcup_{i \in I} Q_i$ admits arbitrarily smooth H-extensions in $\mathscr{H}(\boldsymbol{R}^d)$.*

Proof. (a) follows directly from Lemma 6.12.

(b) Given $J \subseteq \{1, \ldots, d\} \times \{-1, 1\}$, set as in Lemma 6.11

$$V' = \langle -c, c \rangle^d \cup \bigcup \{K_{j,\varepsilon}: (j, \varepsilon) \in J\}.$$

Recall that in Section 3 we assumed that $0 < c < d^{-1/2}$. Set $V'' = \{x \in \boldsymbol{R}^d: \sum_{j=1}^{d} x_j^2 \geq 1\}$. Clearly, $\langle -c, c \rangle^d \cap V'' = \emptyset$. Observe that V'' admits arbitrarily smooth H-extensions in $\mathscr{H}(\boldsymbol{R}^d)$ which preserve vanishing on V'. Indeed, pick a $\varphi \in C^\infty(\boldsymbol{R})$ so that $\varphi(t) = 1$ if $t \geq 1$, and $\varphi(t) = 0$ if $t \leq c\sqrt{d}$, and let $A \in \mathscr{H}((0, \infty))$ be an H-extension of class C^m and C^{-m} from $\langle 1, \infty)$. We set

$$(Hf)(x) = \varphi(\|x\|_2)(H_1 f_1)(\|x\|_2, x/\|x\|_2),$$

where f_1 is obtained by expressing f in terms of polar coordinates $(r, \xi) \in (0, \infty) \times S^{d-1}$ and $H_1 = A \otimes 1_{S^{d-1}}$.

Thus using Lemma 6.11 and Lemma 5.26 we obtain that $V' \cup V''$ admits arbitrarily smooth H-extensions in $\mathscr{H}(\boldsymbol{R}^d)$. Now, if $I \subseteq \{1, \ldots, 2d\}$, then $\boldsymbol{R}^d \setminus C$ is of the form $V' \cup V''$ (up to a set of measure 0), so it suffices to recall Lemma 5.12. If $2d+1 \in I$, then we can apply Lemma 6.7 (cf. Remark 6.10), because we have just shown that if $I' = I \setminus \{2d+1\}$, then the sets

$$V_1 = \bigcup_{i \in I'} Q_i, \quad V_2 = V_1 \cup Q_{2d+1} \cup V'',$$

admit arbitrarily smooth H-extensions in $\mathscr{H}(\boldsymbol{R}^d)$. (Take

$$U_1 = \{x \in \boldsymbol{R}^d: \|x\|_2 > c\sqrt{d}\}, \quad U_2 = \{x \in \boldsymbol{R}^d: \|x\|_2 < 1\}).$$

PROPOSITION 6.14. *Let M be a d-dimensional compact C^∞ manifold. Let $Q_1, \ldots, Q_N$ be a proper decomposition of M into d-cubes. Suppose that either $d = 1$ or $d > 1$ and $Q_1, \ldots, Q_N$ is the decomposition constructed in Section 3. Then, for each $n = 1, 2, \ldots, N$ and $r \geq 1$, the set*

$$W_n = \bigcup_{i \leq n} Q_i$$

admits an H-extension of class C^r and C^{-r} in $\mathscr{H}(M)$.

Proof. Observe that if $d = 1$, then the assertion follows easily from Lemma 6.1 and Lemma 6.7. Thus we may assume that $d > 1$ and that the proposition has been proved for all manifolds of dimension $d-1$. We shall also fix $r \geq 1$ and abbreviate "H-extension of class C^r and C^{-r}" to "H-extension". The proof includes several cases all of which are reduced to the following scheme.

We find open sets U, $U_0 \subseteq M$, a diffeomorphism Ψ and an $E_0 \in \mathscr{H}(M)$ so that

$$W_n \subseteq U \cup U_0;$$

$$E_0 \text{ is an } H\text{-extension from a closed set } S_0 \subseteq M;$$

$$U_0 \cap W_n = U_0 \cap S_0;$$

Ψ maps U onto V, where V is an open subset of another manifold, M_1, such that there exists an H-extension $E \in \mathscr{H}(M_1)$ from a closed set $S \subset M_1$ such that

$$\Psi(U \cap W_n) = V \cap S.$$

Having done this, we obtain the desired H-extension from W_n by applying Remark 6.9.

We shall follow the notation we used in the proof of Theorem 3.3. Given n, $1 \leqslant n \leqslant N$, let s be the maximal index such that

$$A_s = \tau^{-1}\big((-\infty, a_s\rangle\big) \subseteq W_n.$$

We consider first the case where either $A_s = W_n$ or the cube Q_n does not correspond to any critical point z_j (we say that Q_i *corresponds to* z_j if Q_i was obtained in the process of the subdividing the neighborhood of z_j specified in the proof of Theorem 3.3). We take $U_0 = \tau^{-1}\big((-\infty, a_s)\big)$, $E_0 = 1_M \in \mathscr{H}(M)$ and, setting $B = \tau^{-1}(a_s)$, we define Ψ in an open neighborhood of $W_n \setminus U_0$ by the formula

$$\Psi(x) = \big(\varrho(x), \tau(x)\big) \in B \times \boldsymbol{R},$$

where ϱ is the retraction onto B along the flow lines of the vector field $-\operatorname{grad}\tau$ (cf. [26], Theorem 6.22 and the proof of Theorem 6.31). (If s is even, in particular if $s = 0$, then Ψ is defined and is a diffeomorphism in a neighborhood of $A_{s+1} \setminus \operatorname{Int} A_s = \tau^{-1}(\langle a_s, a_{s+1}\rangle)$.) By the quoted results in [26], Ψ is a diffeomorphism of an open set $U \supset W_n \setminus U_0$ onto an open subset $V \subset B \times R$ (where $R = \boldsymbol{R}$ if $s > 0$, and $R = \langle 0, \infty)$ if $s = 0$). Clearly,

$$\Psi(U \cap W_n) = V \cap S,$$

where $S = \{(y, a) \in B \times R\colon a \leqslant a_{s+1},\ y \in W \text{ or } a \leqslant a_s > 0\}$ and the set $W \subset B$ admits, by our induction hypothesis, an H-extension in $\mathscr{H}(B)$. Therefore, the H-extension $E \in \mathscr{H}(B \times R)$ from the set S can be constructed by using Lemma 5.28. Thus we have fulfilled all the conditions of our scheme.

Now we consider the case where Q_n corresponds to a critical point, say z_j, whose index is either 0 or d. We take the chart Φ: $U \to \boldsymbol{R}^d$ and the set M' from the proof of Theorem 3.3 and let

$$U_0 = M \setminus M', \quad S_0 = W_n \setminus M', \quad \Psi(u) = \varepsilon^{-1/2}\Phi(u).$$

The H-extension $E_0 \in \mathscr{H}(M)$ from S_0 exists by the previously considered case. In order to produce an H-extension $E \in \mathscr{H}(\boldsymbol{R}^d)$ from $S = \Psi(M' \cap W_n)$ we apply Corollary 6.23 (b).

It remains to consider the case where Q_n corresponds to a critical point z_j with index k, where $1 \leqslant k \leqslant d-1$. This case leads to two possibilities. We have the chart Φ: $U \to \boldsymbol{R}^k \times \boldsymbol{R}^m$, $m = d-k$, in a neighborhood U of z_j and then either $\varepsilon^{-1/2}\Phi(Q_n) \subseteq G_1$ or $\varepsilon^{-1/2}\Phi(Q_n) \subseteq G_2$ (cf. (3.22) and the proof of Lemma 3.7).

In the first sub-case we set $U_0 = M \setminus \Phi^{-1}(\varepsilon^{1/2}G_1)$, $S_0 = A_s \cup M'$ (M' being that from the proof of Theorem 3.3). The H-extension $E_0 \in \mathscr{H}(M)$ from S_0 exists again by the first case.

We let $M_1 = \boldsymbol{R}^k \times \boldsymbol{R}^m$, $\Psi(u) = \varepsilon^{-1/2}\Phi(u)$ and

$$S = \Psi(U \cap W_n) \cap \{(x, y) \in M_1\colon \|x\| \geqslant \sqrt{5/4}\}.$$

Below we prove

LEMMA 6.15. *There exists an $E \in \mathscr{H}(\boldsymbol{R}^k \times \boldsymbol{R}^m)$ which is an H-extension of class C^r and C^{-r} from this set S.*

This will settle the first sub-case. In the second sub-case we let $U_0 = M \setminus \Phi^{-1}(\varepsilon^{1/2}G_2)$, $S_0 = A_{s+1}$. The H-extension $E_0 \in \mathscr{H}(M)$ from S_0 exists again by the first case. The set U has now to be replaced by the set

$$U_1 = \{u \in U\colon \Phi(u) = (x, y) \neq (0, y)\}.$$

We let $M_1 = \boldsymbol{R}^k \times S^{m-1} \times \boldsymbol{R}$ and define Ψ: $U_1 \to M_1$ to be the composition of $\varepsilon^{-1/2}\Phi|_{U_1}$ with the diffeomorphism Ψ_2 which was used in the proof of Lemma 3.7 in order to produce our decomposition of G_2 into d-cubes. This time the H-extension $E \in \mathscr{H}(M_1)$ from a suitable set S is constructed in the technical Lemma 6.16 below. Thus the proof of Proposition 6.14 has been reduced to Lemmas 6.15 and 6.16.

Proof of Lemma 6.15. Suppose that $E' \in \mathscr{H}(\boldsymbol{R}^k \times \boldsymbol{R}^m)$ is an H-extension (of class C^r and C^{-r}) from the set

$$S' = \{(x, y) \in S\colon \|y\| \leqslant 1/2\}.$$

It is easy to construct an H-extension $E'' \in \mathscr{H}(\boldsymbol{R}^k \times \boldsymbol{R}^m)$ from the set $S'' = \{(x, y) \in M_1\colon \|x\| \geqslant \sqrt{5/4}\}$ which preserves vanishing on S'. Namely, if $E_1 \in \mathscr{H}(\boldsymbol{R}^k)$ is an H-extension (of class C^r and C^{-r}) from the set $\{x \in \boldsymbol{R}^k\colon \|x\| \geqslant \sqrt{5/4}\}$ (cf. the proof of Corollary 6.13), then the tensor product $E_1 \otimes 1$, $1 \in \mathscr{H}(\boldsymbol{R}^m)$, has the desired property by Lemma 5.23.

Thus it follows from Lemma 5.26 that $E = E' + E'' - E''E'$ is the desired H-extension from $S' \cup S'' = S$ and we should only prove that E' exists.

To this end we use the diffeomorphism Ψ_1 of $\boldsymbol{R}^k \times \boldsymbol{R}^m$ onto itself (which we already used in the proof of Lemma 3.7) defined by the formula

$$\Psi_1(x, y) = \big((1 + \|y\|^2)^{-1/2}x, 2y\big).$$

This diffeomorphism transforms S' onto a set of the form (cf. the proof of Lemma 3.7)

$$(\boldsymbol{R}^k\setminus \operatorname{Int} D_k)\times D_m \cup \Big(\bigcup_{i\leqslant v-1} Q_i'\Big)\times\Big(\bigcup_{i\leqslant 2m+1} Q_i''\Big)\cup\bigcup_{i\leqslant w} Q_v'\times Q_i'',$$

where $1\leqslant v\leqslant 2k+1$, $1\leqslant w\leqslant 2m+1$. (Here $Q_1',\ldots,Q_{2k+1}'$ (resp. $Q_1'',\ldots$ $\ldots, Q_{2m+1}''$) is the standard decomposition of D_k (resp. D_m) into cubes.)

The set $\Psi_1(S')$ can also be written as

$$\Big((\boldsymbol{R}^k\setminus \operatorname{Int} D_k)\cup\bigcup_{i\leqslant v-1} Q_i'\Big)\times D_m\cup\Big(\bigcup_{i\leqslant v} Q_i'\Big)\times\Big(\bigcup_{i\leqslant w} Q_i''\Big).$$

Therefore the desired H-extension, say E_2, from this set can be obtained by using Remark 5.30. Indeed, Corollary 6.13 (b) and Lemma 5.12 supply all H-extensions which are required in order to use Lemma 5.28.

Now we simply take

$$E'f = \big(E_2(f\circ\Psi_1^{-1})\big)\circ\Psi_1.$$

This completes the proof of Lemma 6.15.

LEMMA 6.16. *Let $k, m\geqslant 1$ and let $Q_1',\ldots,Q_{2k+1}'$ (resp. $Q_1'',\ldots,Q_{2m}''$) be the standard decomposition of D_k (resp. S^{m-1}) into cubes. Fix $1\leqslant v\leqslant 2k+1$, $1\leqslant w\leqslant 2m$ and let*

$$A = (\boldsymbol{R}^k\setminus \operatorname{Int} D_k)\times S^{m-1}\cup\bigcup\{Q_i'\times Q_j'':\ i<v \text{ or } i=v,\ j\leqslant w\}.$$

Then the set $S\subset \boldsymbol{R}^k\times S^{m-1}\times\boldsymbol{R}$ defined by

$$S = A\times\langle 0,1\rangle\cup\boldsymbol{R}^k\times S^{m-1}\times(-\infty,0\rangle$$

admits arbitrarily smooth H-extensions in $\mathscr{H}(\boldsymbol{R}^k\times S^{m-1}\times\boldsymbol{R})$.

Proof. We shall use Remark 5.30 twice. First we write A in the form

$$A = \Big((\boldsymbol{R}^k\setminus \operatorname{Int} D_k)\cup\bigcup_{i\leqslant v-1} Q_i'\Big)\times S^{m-1}\cup\Big(\bigcup_{i\leqslant v} Q_i'\Big)\times\Big(\bigcup_{i\leqslant w} Q_i''\Big)$$

and obtain as before that A admits arbitrarily smooth H-extensions in $\mathscr{H}(\boldsymbol{R}^k\times S^{m-1})$.

Using this fact (and Lemma 6.1) we again apply Remark 5.30 to conclude that S admits arbitrarily smooth H-extensions in $\mathscr{H}(\boldsymbol{R}^k\times S^{m-1}\times\boldsymbol{R})$. This completes the proof of Lemma 6.16, and hence also Proposition 6.14.

We are ready for the proof of Proposition 4.3. Fix $r\geqslant 1$ and let $H_1,\ldots,H_N\in\mathscr{H}(M)$ be a sequence given by Proposition 6.14. Thus for $n=1,\ldots,N$, H_n is an H-extension of class C^r and C^{-r} from the set W_n. Put $E_0=0$ and

$$E_i = H_N\circ\ldots\circ H_i,\qquad P_i = E_i - E_{i-1} \tag{6.17}$$

for $i=1,\ldots,N$. We shall show that the P_i's satisfy conditions (4.4), (4.5), (4.6), (4.7) and (4.7*).

(This is sufficient if $\partial M=\emptyset$ or if $d=1$ or if one wants to obtain only the isomorphisms T, V in Theorem 4.9, cf. Remark 4.10. Later we shall construct a modified sequence $H_1^0,\ldots,H_N^0$. Using that sequence in (6.17) instead of the H_i's, we do not affect the proof of (4.4)–(4.7*), while (4.8) becomes obvious.)

It is clear that the P_i's defined in (6.17) are H-operators, and so are their adjoints P_i^* (hence they are continuous in $L_0(M)$). Moreover, they are of class C^r, hence (4.7) and (4.7*) follow from Corollary 5.39 (a).

Property (4.4) is obvious, because $E_N=H_N$ is the identity operator and $E_0=0$.

Now, $\operatorname{supp} H_i\subseteq W_i$ implies that $\operatorname{supp} E_i\subseteq W_i$ for $i=1,\ldots,N$. Similarly, since $H_j=1$ on W_i for $j\geqslant i$, we have $E_i=1$ on W_i. By Lemmas 5.3 and 5.4 we see that if $1\leqslant i\leqslant j\leqslant N$, then

$$E_i\chi_{W_j} = E_i,\qquad \chi_{W_i}E_j = \chi_{W_i}. \tag{6.18}$$

It follows that if $1\leqslant i\leqslant j\leqslant N$, then

$$E_iE_j = E_i\chi_{W_i}E_j = E_i\chi_{W_i} = E_i,$$

and hence if $j<i$ and $E=H_{i-1}\circ\ldots\circ H_j$, then

$$E_iE_j = E_iE_iE = E_iE = E_j.$$

Using these relations, we obtain (4.5). Finally, if $1\leqslant i<j\leqslant N$, then using (6.18) we obtain

$$\chi_{Q_i}P_j = \chi_{W_i}E_j-\chi_{W_{i-1}}E_j-\chi_{W_i}E_{j-1}+\chi_{W_{i-1}}E_{j-1} = 0,$$

$$P_i\chi_{Q_j} = E_i\chi_{W_j}-E_{i-1}\chi_{W_j}-E_i\chi_{W_{j-1}}+E_{i-1}\chi_{W_{j-1}} = 0,$$

i.e. we have verified properties (4.4)–(4.7*).

Now suppose that $\partial M\neq\emptyset$ and $d>1$. Recall that the first layer of the cubes Q_i in the decomposition of M constructed in Theorem 3.3 is simply the set $A_1=\tau^{-1}(\langle 0,a_1\rangle)$.

Similarly as in Lemma 6.7, we can pick $\lambda_0,\lambda_1\in C^\infty(\boldsymbol{R})$ so that $\lambda_0(s)=0$ for $s\geqslant(2/3)a_1$, $\lambda_1(s)=0$ for $s\leqslant(1/3)a_1$ and $\lambda_0(s)^2+\lambda_1(s)^2=1$ for $s\in\boldsymbol{R}$. Set $g_0=\lambda_0\circ\tau$, $g_1=\lambda_1\circ\tau$.

Given $H\in\mathscr{H}(M)$, define $H^0\in\mathscr{H}(M)$ by the formula

$$H^0 = g_0Hg_0+g_1Hg_1.$$

Observe that $(H^0)^* = (H^*)^0$. In particular, for all integers r, if H is of class C^r, then so is H^0. Also if H is an H-extension from $V\subseteq M$, then so is H^0 (use Lemmas 5.3 and 5.4). This shows that the operators $H_1^0,\ldots,H_N^0$ have all the properties of $H_1,\ldots,H_N$ which were needed in order to verify (4.4)–(4.7*).

Let $\mathscr{A}$ denote the set of all $B \in \mathscr{H}(M)$ such that if $V = \tau^{-1}(\langle 0, s\rangle)$, where $0 < s < (1/3)a_1$, then B preserves vanishing on V and on $M \setminus V$. Observe that, by Lemma 5.24, $B \in \mathscr{A}$ iff $B^* \in \mathscr{A}$. Moreover, $\mathscr{A}$ is an algebra of operators.

Thus if we prove that, for $i = 1, \ldots, N$,

$$H_i^0 \in \mathscr{A}, \tag{6.19}$$

then it will follow that $P_i \in \mathscr{A}$ and $P_i^* \in \mathscr{A}$ for $i = 1, \ldots, N$. This, of course, will imply (4.8) because, if $f \in L_0(M)$ vanishes on a neighborhood of ∂M, then for some $s \in (0, (1/3)a_1)$ f vanishes on $\tau^{-1}(\langle 0, s\rangle)$.

Now, to prove (6.19) observe that, for any $H \in \mathscr{H}(M)$, $g_1 H g_1 \in \mathscr{A}$. Thus it suffices to show that $g_0 H_i g_0 \in \mathscr{A}$. Fix an i, put $H = \chi_{A_1} H_i \chi_{A_1}$. It follows from our construction of H_i that the study of H can be reduced to that of a tensor product $E \otimes 1$, where $E \in \mathscr{H}(\partial M)$ and $1 \in \mathscr{H}(\langle 0, a_1\rangle)$. Hence Lemma 5.23 yields that $H \in \mathscr{A}$. Since $g_0 \chi_{A_1} = g_0$, we have $g_0 H_i g_0 = H \in \mathscr{A}$. This proves (6.19) and completes the proof of Proposition 4.3.

Now we shall construct the operator P of Proposition 2.39. We start from some constructions on the real line. Fix a function $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(x) = 1$ if $|x - 1/2| \leqslant 3/4$, $\varphi(x) = 0$ if $|x - 1/2| \geqslant 1$. Put $s = 2m+2$ and fix numbers $-2 < \beta_1 < \ldots < \beta_s < -1$. Finally, let $a_1, \ldots, a_s$ be the solution of the following system of equations (cf. Lemma 6.1):

$$\sum_{j=1}^{s} a_j \beta_j^k = 1, \qquad k = -m-1, \ldots, m.$$

Recall that I_Z, where $Z \subseteq \{0, 1\}$, is defined by (2.38). Let $\varphi^Z = \chi_{I_Z} \varphi \in L_0(\mathbf{R})$ for $Z \subseteq \{0, 1\}$, and let $P_Z \in \mathscr{H}(\mathbf{R})$ be defined by the formula

$$(P_Z f)(x) = f(x) - \sum_{j=1}^{s} a_j (\varphi^Z f)(\beta_j x) - \sum_{j=1}^{s} a_j (\varphi^Z f)\big(1 + \beta_j (x-1)\big) \tag{6.20}$$

for $x \in \langle 0, 1\rangle$ and $(P_Z f)(x) = 0$ for $x \in \mathbf{R} \setminus \langle 0, 1\rangle$.

Note that the following elements of $\mathscr{H}(\mathbf{R})$ are equal:

$$P_Z \chi_I = \chi_I, \qquad \chi_I P_Z = P_Z. \tag{6.21}$$

Our choice of the a_j's ensures that if $f\colon \mathbf{R} \to \mathbf{R}$ and $f|_{I_Z} \in C^m(I_Z)$, then

$$(P_Z f)|_{I_Z} \in C^m(I_Z), \qquad \big((P_Z)^* f\big)|_{I_Z} \in C^m(I_Z) \tag{6.22}$$

(this is proved as in Lemma 6.1).

Observe also that there is $A_Z \in \mathscr{H}(\mathbf{R})$ such that

$$(P_Z)^* = \varphi^Z A_Z. \tag{6.23}$$

Now we are ready to prove Proposition 2.39. Let $d \geqslant 1$. Recall that Z is given by (2.37). Define $P_Z \in \mathscr{H}(\mathbf{R}^d)$ by the formula

$$P_Z = P_{Z_1} \otimes \ldots \otimes P_{Z_d},$$

where $P_{Z_i} \in \mathscr{H}(\mathbf{R}_{(i)})$ (the i-th copy of $\mathbf{R}$) is the H-operator defined by (6.20). Since $\operatorname{supp} P_Z \subseteq I_{Z_1} \times \ldots \times I_{Z_d} = Q_Z$, then P_Z induces a continuous operator P in the space $L_0(Q_Z)$ (cf. Remark 5.5).

Let us check that P has the required properties. Property (2.41) of P follows from Corollary 5.40 and (6.22). Similarly, since the formal adjoint P^* is induced by the operator

$$(P_Z)^* = (P_{Z_1})^* \otimes \ldots \otimes (P_{Z_d})^*, \tag{6.24}$$

we obtain (2.42). (2.40) is a consequence of the identities

$$P\chi_Q = \chi_Q, \qquad \chi_Q P = P, \tag{6.25}$$

which follow from (6.21) by (5.18). Clearly, (2.43) follows from the dual form of (6.25).

It remains to check (2.44). Observe first that for any $f \in L_0(Q_Z)$ one has $P^* f = 0$ a.e. on V, where $V = Q_Z \setminus \langle 1/2, 3/2\rangle^d$ is an open subset of Q_Z. This follows from (6.23), (6.24) and (5.18).

Now suppose that $f \in L_0(Q_Z)$ vanishes on a set $U \cap Q$, where U is an open neighborhood of ∂Q_Z. Let S be a $(d-1)$-dimensional face of ∂Q_Z, say $S = \{x \in Q_Z\colon x_i = a\}$ for some $i \in \{1, \ldots, d\}$, $a \in \{-1, 0, 1, 2\}$. We assume, without loss of generality, that $i = d$.

If $S \cap Q = \emptyset$, then, as we have proved, $P^* f$ vanishes on V which is a neighborhood of S in Q_Z.

If $S \cap Q \neq \emptyset$ and $d = 1$, then $P^* f$ vanishes on a neighborhood of S in $Q_Z = I_Z$ by property (2.43). If $d > 1$, then we use Lemma 5.23.

Observe that there is $\varepsilon > 0$ such that if $y \in Q$ and $|y_d - a| < \varepsilon$, then $y \in U$. Let

$$M_1 = \mathbf{R}^{d-1}, \quad M_2 = \mathbf{R}, \quad V_1 = I^{d-1}, \quad V_2 = I \cap (a - \varepsilon, a + \varepsilon),$$

$$A_1 = (P_{Z_1})^* \otimes \ldots \otimes (P_{Z_{d-1}})^*, \quad A_2 = (P_{Z_d})^*.$$

If $\tilde{f} \in L_0(\mathbf{R}^{d-1} \times \mathbf{R})$, $\tilde{f}|_{I^d} = f$, then using Lemma 5.23 we get

$$(A_1 \otimes A_2)\tilde{f} = 0 \text{ a.e. on } \mathbf{R}^{d-1} \times V_2.$$

This obviously yields that P^*f vanishes on the ε-neighborhood of S in Q_Z.

This completes the proof of Proposition 2.39.

We close this section with a simple example which shows that H-operators can provide linear continuous extensions in some cases where no other known method works. Conceivably, this idea can be used in order to simplify our constructions of extension operators in Proposition 6.14 and also to avoid using the fact that the sets $\operatorname{Int} W_n$, $1 \leqslant n \leqslant N$, have the segment property (so that a weaker and easier version of Theorem 3.3 would be sufficient).

We shall construct a closed set $V \subset \boldsymbol{R}^3$ so that $\operatorname{Int} V$ fails the segment property and yet, if $\mathscr{F} = W_p^k$ or $\mathscr{F} = B_{p,q}^s$, where $0 \leqslant k \leqslant m$, $0 < s < m$, $1 \leqslant p, q \leqslant \infty$, then

$$\mathscr{F}(V) = \{g|_V \colon g \in \mathscr{F}(\boldsymbol{R}^3)\}. \tag{6.26}$$

Moreover, V admits arbitrarily smooth H-extensions in $\mathscr{H}(\boldsymbol{R}^3)$.

The set V is defined as $V' \cup V''$, where

$$V' = \{(x, y) \colon x \vee (-y) \geqslant 0\} \times \langle 0, \infty),$$

$$V'' = \langle 0, \infty) \times \{(y, z) \colon y \vee z \geqslant 0\} = \langle 0, \infty) \times A.$$

It is easy to check that $\operatorname{Int} V$ fails the segment property, the origin being the only singular point.

Now fix an $m \geqslant 1$ and pick three H-extensions of class C^m and C^{-m}: $E' \in \mathscr{H}(\boldsymbol{R}^3)$ from the set V', $E_1 \in \mathscr{H}(\boldsymbol{R})$ from the set $\langle 0, \infty)$ and $E_2 \in \mathscr{H}(\boldsymbol{R}^2)$ from A (use Lemmas 6.1, 6.2 and 5.13). Put $E'' = E_1 \otimes E_2$ and let

$$E = E' + E'' - E''E'.$$

Using (5.22) and Lemma 5.14 we infer that E is of class C^m and C^{-m}.

To show that E is an H-extension from V, observe first that E'' is an H-extension from V''. We would like to use Lemma 5.26. To this end observe that if $B = \{(y, z) \colon y \leqslant 0, z \geqslant 0\}$, then $f \in L_0(\boldsymbol{R}^3)$, $f|_{\langle 0,\infty) \times B} = 0$ implies $E''f|_{\boldsymbol{R} \times B} = 0$. (This follows from Lemma 5.23, because $\operatorname{supp} E_1 \subseteq \langle 0, \infty)$ and $E_2 = 1$ on B.) Since E'' preserves vanishing on $V' \cap V''$ and

$$\langle 0, \infty) \times B \subseteq V \subseteq (\boldsymbol{R} \times B) \cup (V' \cap V''),$$

we see that E'' preserves vanishing on V'. Hence, by Lemma 5.26, E is an H-extension from V, i.e. E has the required properties.

Choosing more carefully E', E_1, E_2 we could get that E defines a bounded map from $\mathscr{F}(V)$ to $\mathscr{F}(\boldsymbol{R}^3)$, where $\mathscr{F} = W_p^k$ or $\mathscr{F} = B_{p,q}^s$, $0 \leqslant k \leqslant m$, $0 < s < m$, $1 \leqslant p, q \leqslant \infty$. We shall not do this because the results of Section 5 cannot be applied directly, since V is not compact. It is not difficult, however, to produce such extension operators from $\mathscr{F}(V \cap D_3)$ to $\mathscr{F}(\boldsymbol{R}^3)$. This fact also depends on (6.26), so let us indicate why (6.26) is true. It suffices to check the following: given $f \in \mathscr{F}(V)$ such that $\operatorname{supp} f$ is bounded in $\boldsymbol{R}^3$, there is $g \in \mathscr{F}(\boldsymbol{R}^3)$ such that $f = g|_V$.

To this end pick $m \geqslant 1$ so that $m \geqslant k$ (or $m > s$) and let E' (resp. E'') $\in \mathscr{F}(\boldsymbol{R}^3)$ be H-extensions from V' (resp. V'') corresponding to this m. Write $\mathscr{F}_0(U)$ for the set of those $f \in \mathscr{F}(U)$ such that $\operatorname{supp} f$ is a bounded subset of $\boldsymbol{R}^3$.

Observe that, since $\operatorname{Int} V'$ has the segment property, it can be deduced from Corollary 5.39 that if $h \in \mathscr{F}_0(V')$, then $E'h \in \mathscr{F}_0(\boldsymbol{R}^3)$. Similarly, $h_1 \in \mathscr{F}_0(V'')$ implies $E''h_1 \in \mathscr{F}_0(\boldsymbol{R}^3)$. Hence, if $f \in \mathscr{F}_0(V)$, then letting $h = f|_{V'}$, $h_1 = f|_{V''}$ we obtain easily that $g = Ef \in \mathscr{F}_0(\boldsymbol{R}^3)$ and $g|_V = f$. This completes the proof of (6.26).

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On weighted norm inequalities for the maximal function

by

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Abstract. We give a refinement of a lemma of C. Fefferman and E. Stein, and we show an application to weighted norm inequalities.

The lemma of C. Fefferman and E. Stein given in [3], p. 111, states that

$$\int_{R^n} Mf(x)^p g(x)\,dx \leqslant C \int_{R^n} f(x)^p Mg(x)\,dx, \tag{1}$$

where $1 < p < \infty$, M is the Hardy–Littlewood maximal function, and f and g are positive measurable functions.

In this note we show that by restricting the radius in the definition of the maximal function a similar inequality holds. This inequality can be used as a substitute for (1) in weighted norm inequalities when the assumption $Mg < \infty$ cannot be made.

Lemma. *Let f be a measurable function and define*

$$\bar{f}(x) = \sup_{r<(|x|+1)/2} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(t)|\,dt,$$

where $B_r(x)$ is the ball of center x and radius r, and

$$\bar{\bar{f}}(x) = \sup_{\substack{x\in B_r\\ r<|x|+1}} \frac{1}{|B_r|} \int_{B_r} |f(t)|\,dt,$$

where B_r is a ball of radius r. If $g > 0$ almost everywhere, then

$$\int_{R^n} \bar{f}(x)^p g(x)\,dx \leqslant C \int_{R^n} |f(x)|^p \bar{\bar{g}}(x)\,dx.$$

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