this theory (English translation by Professor Henryk Minc of the University of California).

Presenting "Hypernumbers" in the volume of Studia Mathematica dedicated to Professor Jan Mikusiński, we would like to thank him for all the theories and theorems with which he has enriched Polish and world mathematics and for his constant pursuit of beauty, simplicity and elegance in mathematics.

We wish you, Dear Professor, good health and long years of activity to come.

Pupils

STUDIA MATHEMATICA, T. LXXVII. (1983)

Hypernumbers
Part I. Algebra

by

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Introduction. In this article we introduce an analytic element composed of a complex number and a vector; we call it a "hypernumber". The fundamental difference between a hypernumber and a quaternion lies in differently defined multiplication; moreover, the vector space we consider has somewhat different properties from the one that plays a part in the theory of quaternions.

Hypernumbers find applications in the theory of integral and differential equations reducing certain problems of particular character to purely algebraic problems. By operating with a suitably chosen unit, computations have a very clear form, and some theorems, e.g., in the theory of Fredholm's equation become nearly obvious.

In an application to linear differential equations with constant coefficients computations appear almost identical to Heaviside calculus(1). Hypernumbers thus form a new algebraic basis for this calculus. It is conceptually simpler than theory of Laplace transformation in which Heaviside's calculus found for the first time a rigorous justification (2).

As a particular interpretation of hypernumbers one could approximately regard also "commutative functions" (fonctions permutables), the theory of which was developed by Volterra and Péres. However, these authors have not defined accurately the concept of an element composed of a number and a vector; their functions play the part of vectors, and in

(1) Doetsch, p. 344. Heaviside calculus in its original form serves only for finding the integral satisfying initia "zero" conditions; whereas hypernumbers give the general solution.

(2) Doetsch.

place of numbers they introduced "symbolic functions" ([3]) which acquire a real meaning only when certain operations are performed on them.

We divide this paper into two parts: in the first (algebra) we give the theory and certain applications of hypernumbers depending solely on the four elementary operations. In the second part we shall introduce the concept of the sum of an infinite series of hypernumbers. We shall also discuss there an application to Volterra's integral equation.

I sincerely thank Messrs. A. T., T. W., J. W. and A. B. for kind and valuable remarks concerning the method of approach and of editing this paper.

Author

Contents

§ 1. Abstract space W. We shall consider an abstract space W for which we assume that

I. For every pair of its elements f, g there exists a sum $f+g \in W$ and a product $fg \in W$;

II. For every element $f \in W$ and every complex number there exists a product $\alpha f \in W$, as well as $f\alpha \in W$.

We assume that these operations satisfy the following postulates (f, g, h are elements of the space W, whereas α, β are complex numbers):

$$(1)\quad \left.\begin{aligned} f+g &= g+f, \\ \alpha f &= f\alpha; \end{aligned}\right\} \text{(commutativity)}$$

$$(2)\quad \left.\begin{aligned} f+(g+h) &= (f+g)+h, & & \\ f(gh) &= (fg)h, & &(**) \\ \alpha(\beta f) &= (\alpha\beta)f, & & \\ \alpha(fg) &= (\alpha f)g, & &(*) \\ (fg)\alpha &= f(g\alpha); & &(*) \end{aligned}\right\} \text{(associativity)}$$

$$(3)\quad \left.\begin{aligned} \alpha(f+g) &= \alpha f+\alpha g, & & \\ (\alpha+\beta)f &= \alpha f+\beta f, & & \\ (f+g)h &= fh+gh, & &(*) \\ h(f+g) &= hf+hg; & &(*) \end{aligned}\right\} \text{(distributivity)}$$

$$(4)\quad f+g = f+h \supset g = h;$$

$$(5)\quad 1f = f.$$

The above system of postulates contains all the postulates adopted for vector spaces([4]); in addition it contains postulates relating to the product of a vector by a vector (marked by stars). Thus the space W is a special type of a vector space; for this reason its elements will be called *vectors*. A product of two vectors of the space W differs from the "vector product" in the classical vector algebra meaning only in that that the associative law (**) is postulated for it.

It follows from the above adopted postulates that there exists exactly one zero element Θ possessing the following properties:

$$f+\Theta = f,\quad 0\cdot f = \Theta,\quad \Theta f = f\Theta = \Theta.$$

We shall denote the zero element by the same symbol as the number 0, that is, we shall take $\Theta = 0$, which in the face of the above relations does not lead to incorrect conclusions.

Further we adopt the usual definitions:

$$-f = (-1)f;\quad f-g = f+(-g);\quad f^n = ff\ldots f \quad (n \text{ factors}).$$

§ 2. Hypernumbers. A pair of elements (α, f), composed of a complex number α and a vector f, belonging to space W, is called a *hypernumber*([5]); it is written in the form of a sum

$$(\alpha, f) = \alpha+f.$$

Two hypernumbers are considered equal

$$\alpha+f = \beta+g$$

([3]) Volterra–Péres, p. 8.

([4]) Banach, p. 25; Neumann, p. 19; Neumann's system is somewhat different, but it is equivalent.

([5]) A hypernumber is not a quaternion, because the product of two hypernumbers will be defined differently.

if and only if $\alpha = \beta$ and $f = g$. In particular, we adopt the abbreviations

$$0+f = f,$$
$$\alpha+0 = \alpha;$$

in that way a vector and a number can be regarded as special cases of hypernumbers.

The set of all possible pairs of complex numbers and vectors of space W is called "the hypernumber space" and is denoted by $[W]$. We define in this space addition and multiplication (f, g denote vectors, α, β — numbers):

$$(\alpha+f)+(\beta+g) = \underbrace{\alpha+\beta}_{\text{number}}+\underbrace{f+g}_{\text{vector}},$$
$$(\alpha+f)(\beta+g) = \underbrace{\alpha\beta}_{\text{number}}+\underbrace{\alpha g+\beta f+fg}_{\text{vector}}.$$

Rules of addition and multiplication are therefore formally quite the same as in ordinary algebra, except that, in multiplication, the order of vectors must be preserved, for the law of commutativity does not apply in general. If it is established which letters denote numbers and which denote vectors, then the number part in the final result can be easily separated from the vector part; it has only to be kept in mind that multiplication of a number by a number gives a number, whereas multiplication of a number by a vector or a vector by a vector gives always a vector.

§ 3. Invertible and commuting hypernumbers. We can also denote hypernumbers $\alpha+f, \beta+g, \ldots$ by one letter, e.g., $A, B, \ldots$ If two hypernumbers A, B satisfy the relation

$$AB = BA = 1$$

then we shall call them *mutually inverse*, and we shall write

$$B = \frac{1}{A}, \quad A = \frac{1}{B}.$$

It follows from the above definition that for a given hypernumber there can exist at most one inverse hypernumber. However, not every hypernumber has an inverse element; for example, a hypernumber reducible to a vector cannot have an inverse element, because if it is multiplied by any hypernumber it always yields a vector. A hypernumber for which there exists an inverse is called "invertible".

If hypernumbers A and B are invertible then their product AB is invertible as well, and we have

$$\frac{1}{AB} = \frac{1}{B}\frac{1}{A}.$$

Two hypernumbers A, B are called "commuting" if

$$AB = BA.$$

For the commutativity of hypernumbers $A = \alpha+f$, $B = \beta+g$ it is necessary and sufficient that their vector parts commute: $fg = gf$.

For commuting hypernumbers A, B it is easy to derive the following formulas

$$A\cdot\frac{1}{B} = \frac{1}{B}\cdot A \qquad \left(\text{provided that } \frac{1}{B} \text{ exists}\right),$$

$$\frac{1}{A}\cdot\frac{1}{B} = \frac{1}{B}\cdot\frac{1}{A} = \frac{1}{AB} \qquad \left(\text{provided that } \frac{1}{A} \text{ and } \frac{1}{B} \text{ exist}\right).$$

It follows from these formulas that operations on commuting hypernumbers can be performed as an ordinary algebraic fractions provided that the denominators are products of invertible hypernumbers; moreover instead of $A\cdot\frac{1}{B}$ we can write simply $\frac{A}{B}$.

§ 4. Spaces D and $[D]$. These spaces are a particular interpretation of spaces W and $[W]$.

We regard as elements (vectors) of space D all complex functions $f(t)$ of real variable t, continuous in a certain stipulated interval $a \leqslant t < b$, where a is a (real) finite number and b is a (real) number, finite or infinite. We adopt the usual definitions of addition and of multiplication of a function by a complex number, but we shall define the product of a vector by a vector differently. To avoid misunderstandings in multiplication, we shall denote the vector corresponding to function $f(t)$ by $\{f(t)\}$. The product of vectors $\{f(t)\}$ and $\{g(t)\}$ is defined by the equality

$$(1) \qquad \{f(t)\}\cdot\{g(t)\} = \left\{\int_a^t f(a+t-\tau)g(\tau)\,d\tau\right\}.$$

It is easy to verify that all the postulates in § 1 are satisfied for the above definition of multiplication. Moreover, all vectors in the space D commute (to prove commutativity it suffices to make in integral (1) the substitution $\sigma+\tau = a+t$).

Set $l = \{1\}$; then we have

$$l\cdot\{g(t)\} = \left\{\int_a^t g(\tau)\,d\tau\right\};$$

the multiplication by l means integrating the function. We also have

$$l^2 = l\cdot l = \left\{\int_a^t d\tau\right\} = \{t-a\},$$

and in general

$$l^n = \left\{\frac{(t-a)^{n-1}}{(n-1)!}\right\}.$$

If λ is an arbitrary complex number, then

$$(1-\lambda l)\left(1+\lambda\{e^{\lambda(t-a)}\}\right) = 1,$$

which is easy to verify by performing the multiplication according to the definition. It means (in view of the commutativity of all vectors in space D) that

$$\frac{1}{1-\lambda l} = 1+\lambda\{e^{\lambda(t-a)}\}\text{(6)}.$$

By means of this formula we can find the inverse of every hypernumber of the form

$$A = 1+a_1l+a_2l^2+\ldots+a_nl^n,$$

where $a_1, \ldots, a_n$ are arbitrary complex numbers ($a_n \neq 0$); it suffices for this purpose to write it in the form of a product

$$A = (1-\lambda_1 l)(1-\lambda_2 l)\ldots(1-\lambda_n l),$$

which is always feasible according to theorems in algebra. We have then

$$\frac{1}{A} = \frac{1}{1-\lambda_1 l}\,\frac{1}{1-\lambda_2 l}\cdots\frac{1}{1-\lambda_n l}.$$

The best way of effective computations is realized by performing first the decomposition into partial fractions

$$\frac{1}{A} = \frac{\beta_1}{1-\lambda_1 l}+\frac{\beta_2}{1-\lambda_2 l}+\ldots+\frac{\beta_n}{1-\lambda_n l},$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are complex numbers.

In this way we can find effectively the form (i.e., the number part and the vector part) of a hypernumber given by the formula

(2) $$Q = \frac{\gamma_0+\gamma_1 l+\ldots+\gamma_m l^m}{1+a_1 l+\ldots+a_n l^n}.$$

(6) In the second part of the paper we shall prove the commutativity of every element $a+f$ in space $[D]$ for which $a \neq 0$.

Example:

$$\frac{l}{1+\lambda^2 l^2} = \frac{1}{2\lambda i}\left(\frac{1}{1-i\lambda l}-\frac{1}{1+i\lambda l}\right)$$

$$= \frac{1}{2\lambda i}\left[1+i\lambda\{e^{i\lambda(t-a)}\}-1+i\lambda\{e^{-i\lambda(t-a)}\}\right]$$

$$= \left\{\frac{1}{2}\left(e^{i\lambda(t-a)}+e^{-i\lambda(t-a)}\right)\right\} = \{\cos\lambda(t-a)\}.$$

By means of this result the number part and the vector part of an arbitrary hypernumber Q of the form (2) can be computed without introducing imaginary units. It suffices for this purpose to decompose the fraction Q into a sum of terms of the following types

$$a, \quad al^p, \quad a\left(\frac{1}{1-\lambda l}\right)^p, \; a\left(\frac{l}{1+\lambda^2 l^2}\right), \; a\left(\frac{l^2}{1+\lambda^2 l^2}\right),$$

where p is natural; it is easy to show that such a decomposition is always feasible. We have already examined the first four types; as for the fifth, it is immediately reduced to the above by writing

$$\frac{l^2}{1+\lambda^2 l^2} = l\frac{l}{1+\lambda^2 l^2}.$$

§ 5. Application to linear differential equations with constant coefficients. Let

(3) $$x^{(n)}+a_1x^{(n-1)}+\ldots+a_nx = f(t)$$

be a given differential equation with constant coefficients $a_1, \ldots, a_n$. We look for a solution $x(t)$ satisfying at the point $t = a$ the initial conditions

$$x(a) = \gamma_0,\; x'(a) = \gamma_1, \ldots, x^{(n-1)}(a) = \gamma_{n-1}.$$

Integrating (3) n times, we obtain

$$x+a_1lx+\ldots+a_nl^nx = \gamma_0+\gamma_1 l+\ldots+\gamma_{n-1}l^{n-1}+l^nf,$$

where the symbols are already treated as vectors:

$$x = \{x(t)\}, \quad f = \{f(t)\}.$$

Hence we have

$$(1+a_1l+\ldots+a_nl^n)x = \gamma_0+\gamma_1 l+\ldots+\gamma_{n-1}l^{n-1}+l^nf,$$

and multiplying both sides by $\dfrac{1}{1+a_1 l+\ldots+a_n l^n}$,

$$x=\frac{\gamma_0+\gamma_1 l+\ldots+\gamma_{n-1}l^{n-1}}{1+a_1 l+\ldots+a_n l^n}+\frac{l^n}{1+a_1 l+\ldots+a_n l^n}\cdot f.$$

This is at the same time a proof of existence and uniqueness of the solution of the given equation.

The above method gives the general solution in contrast to the Laplace transformation method (7) in which the computations are formally almost the same, but a condition is introduced that the function $f(t)$ be of type $\mathscr{L}$(8).

By means of hypernumbers of space $[D]$ it is also possible to solve easily *systems* of equations with constant coefficients; it suffices to introduce the concept of a determinant with hypernumbers as elements, whose definition, in the case of commuting hypernumbers suggests itself in a quite natural way.

§ 6. Connection with Laplace transformation(9). Assume now in particular (with the notation of § 4) that $a=0$. In this case we have

$$l^n=\left\{\frac{t^{n-1}}{(n-1)!}\right\}\quad \text{and}$$

$$a_1 l+a_2 l^2+\ldots+a_n l^n=\left\{a_1+\frac{a_2 t}{1!}+\ldots+\frac{a_n t^{n-1}}{(n-1)!}\right\}.$$

The expression on the left-hand side can be regarded formally as a polynomial in l; the equality expresses a certain connection between two polynomials

$$w(l),\qquad W(t).$$

If one of these polynomials is known, the other can be easily determined and vice versa. We are dealing therefore with a certain transformation, transforming a polynomial into a polynomial. If an arbitrary positive number is substituted for l, then it is easy to verify by performing the integration that

$$w(l)=\int_0^\infty e^{-t/l}W(t)\,dt.$$

(7) Doetsch, pp. 321–329.
(8) Doetsch, p. 13.
(9) It is not necessary to read this chapter in order to understand the next part of the article.

Hence polynomial $w(l)$ is the resultant of Laplace transformation(10).

It is easy to verify that regarding l as an ordinary parameter, we have for $l>0$

$$\frac{l}{1+\lambda^2 l^2}=\int_0^\infty e^{-t/l}\cos(\lambda t)\,dt,\qquad \frac{l^2}{1+\lambda^2 l^2}=\frac{1}{\lambda}\int_0^\infty e^{-t/l}\sin(\lambda t)\,dt.$$

On the other hand, if l has the meaning as in § 4 (with $a=0$), then

$$\frac{l}{1+\lambda^2 l^2}=\{\cos(\lambda t)\},\qquad \frac{l^2}{1+\lambda^2 l^2}=\frac{1}{\lambda}\{\sin(\lambda t)\};$$

again we have a connection with Laplace transformation.

Now if $f(t)$ is a real function of a real variable t, expansible into a Fourier series in given interval $[0,b]$:

$$f(t)\doteq\sum_{n=0}^{\infty}\left(a_n\cos\frac{2n\pi t}{b}+\beta_n\sin\frac{2n\pi t}{b}\right),$$

then we can write formally

$$\{f(t)\}=F(l)=\sum_{n=0}^{\infty}\left(\frac{a_n l}{1+\left(\frac{2n\pi l}{b}\right)^2}+\frac{2n\pi}{b}\,\frac{\beta_n l^2}{1+\left(\frac{2n\pi l}{b}\right)^2}\right).$$

If in place of l we substitute an arbitrary real number, then due to the convergence of $\sum\limits_{n=0}^{\infty}(a_n^2+\beta_n^2)$ the series $F(l)$ will be also convergent. On the other hand, replacing the expression in the parentheses by a suitable Laplace integral we arrive for $l>0$ at the relation

$$F(l)=\int_0^\infty e^{-t/l}f(t)\,dt,$$

where $f(t)$ should be understood as a periodic function with period b.

§ 7. Space F. This space is another particular interpretation of space W. We shall regard as its elements (vectors) every complex function $f(x,y)$ of two real variables x,y, possessing the following properties:

1) $f(x,y)$ is defined in a square

$$Q\,(a<x\leqslant b,\ a\leqslant y<b);$$

2) it is integrable in Q separately with respect to each variable;

(10) Doetsch, p. 13.

3) it is possible to choose such positive numbers m and k that

$$|f(x,y)| < m(x-a)^{k-1}(b-y)^{k-1} \qquad (\text{in } Q).$$

Addition of two vectors $\{f(x,y)\}$, $\{g(x,y)\}$ of space F and multiplication of a vector by a complex number are defined by analogous operations on functions $f(x,y)$, $g(x,y)$ (similarly as in § 4), multiplication of a vector by a vector is defined by the equality

$$\{f(x,y)\}\{g(x,y)\} = \Big\{\int_a^b f(x,s)g(s,y)\,ds\Big\}^{(11)}.$$

The verification that the space satisfies the postulates in § 1 (i.e. that it is of type W) we leave for the reader.

By applying vectors of space F we can give a particularly simple form to Fredholm's integral equation

$$\varphi(x) = f(x) + \lambda\int_a^b K(x,s)\varphi(s)\,ds;$$

namely, regarding the functions appearing in it as vectors of space F we can write

$$\{\varphi(x)\} = \{f(x)\} + \lambda\{K(x,y)\}\{\varphi(x)\},$$

or simply

$$\varphi = f + \lambda K\varphi.$$

The fact that $f(x)$ and $\varphi(x)$ are functions of only one variable x is no obstacle here, because we can always treat them as functions of two variables which are constant with respect to y.

§ 8. Eigenvalues and the spectrum of a vector. In the next sections we shall see that proofs of some theorems on Fredholm's equation take a very clear form by the introduction of vectors of space F and connected with it hypernumbers. Of course, we shall state these theorems in a more general form relating to an arbitrary space W; besides we shall keep the terminology accepted in the theory of integral equations.

If K is a vector of space W, then the hypernumber

$$1-\lambda K$$

may be either invertible or not invertible, depending on the value of the complex number λ. The values λ for which the hypernumber $1-\lambda K$ is

(11) Such an operation is called by Volterra "composition de deuxième espèce"; Volterra–Pères, p. 5; Doetsch, p. 157.

not invertible we shall call the "eigenvalues" of the vector K, and the set of all eigenvalues — the "spectrum" of the vector K.

Let us consider now vector equations

$$\varphi = f + \lambda K\varphi,$$

$$\varphi = f + \lambda\varphi K,$$

where φ is an unknown, whereas f and K are given vectors.

If λ does not belong to the spectrum of K, then both equations have exactly one solution.

Indeed, we can write the first equation successively in the form

$$\varphi - \lambda K\varphi = f,$$

$$(1-\lambda K)\varphi = f;$$

then multiplying both sides of the last equality by $\dfrac{1}{1-\lambda K}$ we obtain the solution

$$\varphi = \frac{1}{1-\lambda K}f.$$

In an analogous manner we obtain the solution for the second equation

$$\varphi = f\frac{1}{1-\lambda K}.$$

These equations are given in the classical theory of integral equations in a somewhat different form. In order to obtain it we note that

$$\frac{1}{1-\lambda K} = 1 + \lambda\frac{K}{1-\lambda K}.$$

The vector

$$K_\lambda = \frac{K}{1-\lambda K} \tag{4}$$

is called a *resolvent* of the vector K.

Its introduction enables us to write the solutions in a form recalling the given equations, namely

$$\varphi = f + \lambda K_\lambda f$$

for the first, and

$$\varphi = f + \lambda f K_\lambda$$

for the second of the equations.

A resolvent can also be written, as it can be easily verified, in the form

$$K_\lambda = \frac{1}{\lambda}\left(\frac{1}{1-\lambda K} - 1\right). \tag{5}$$

§ 9. Properties of a resolvent. From formula (4) we can easily obtain certain properties of a resolvent.

By performing the computations we can easily verify the equality

$$\frac{K}{1-\lambda K} - \frac{K}{1-\mu K} = (\lambda-\mu)\frac{K}{1-\lambda K}\,\frac{K}{1-\mu K},$$

$$K_\lambda - K_\mu = (\lambda-\mu)K_\lambda K_\mu;$$

this equality is known in the theory of integral equations under the name of the resolvent equation and was given by Hilbert([12]).

If $KN = NK = 0$, a simple computation yields

$$\frac{K}{1-\lambda K} + \frac{N}{1-\lambda N} = \frac{K+N}{1-\lambda(K+N)},$$

or

$$K_\lambda + N_\lambda = (K+N)_\lambda;$$

with reference to integral equations this equality expresses a theorem of Goursat on orthogonal kernels([13]).

§ 10. Vectors of finite rank. A vector K is called "of rank p" if the infinite sequence $K, K^2, K^3, \ldots$ contains exactly p linearly independent vectors.

THEOREM. *If vector K is of rank p, then the vectors*

$$K, K^2, \ldots, K^p$$

are linearly independent.

Proof. Suppose on the contrary that

$$\alpha_1 K + \alpha_2 K^2 + \ldots + \alpha_p K^p = 0$$

for a certain system of complex numbers

$$\alpha_1, \alpha_2, \ldots, \alpha_p, \tag{6}$$

([12]) Lalesco, p. 43; Kowalewski, p. 132.

([13]) Goursat, T. 2, Lalesco, p. 41.

not all of which are zero. If α_q is the last number of system (6) different from zero, then we have $\alpha_1 K + \alpha_2 K^2 + \ldots + \alpha_q K^q = 0$ $(\alpha_q \neq 0)$; this means that K^q can be expressed as a linear combination of $K, K^2, \ldots, K^{q-1}$. Multiplying this equality by K we obtain

$$\alpha_1 K^2 + \alpha_2 K^3 + \ldots + \alpha_q K^{q+1} = 0;$$

then K^{q+1} can be expressed as a linear combination of $K^2, K^3, \ldots, K^q$; thus also of $K, K^2, \ldots, K^{q-1}$. It follows by induction that all K^i, where $i \geqslant q$, are linear combinations of $K, K^2, \ldots, K^{q-1}$, which contradicts the assumption that the infinite sequence $K, K^2, K^3, \ldots$ has p linearly independent elements.

We shall determine now the inverse element for the hypernumber $1-\lambda K$, where K is a vector of finite rank p.

We shall endeavour to determine complex functions $\varphi_1, \varphi_2, \ldots, \varphi_p$ of complex variable λ so that $(1-\lambda K)(1+\varphi_1 K + \varphi_2 K^2 + \ldots + \varphi_p K^p) = 1$. After multiplying out the factors in parentheses and cancelling the numbers 1, we have

$$(\varphi_1-\lambda)K + (\varphi_2-\lambda\varphi_1)K^2 + \ldots + (\varphi_p - \lambda\varphi_{p-1})K^p - \lambda\varphi_p K^{p+1} = 0. \tag{7}$$

On the other hand, since K is of rank p, there exists a system of $p+1$ complex numbers $\alpha_1, \ldots, \alpha_{p+1}$ such that

$$\alpha_1 K + \alpha_2 K^2 + \ldots + \alpha_p K^p + \alpha_{p+1} K^{p+1} = 0 \qquad (\alpha_{p+1} \neq 0); \tag{8}$$

from (7) and (8) we have, due to the independence of $K, K^2, \ldots, K^p$,

$$\frac{\varphi_1-\lambda}{\alpha_1} = \frac{\varphi_2-\lambda\varphi_1}{\alpha_2} = \ldots = \frac{\varphi_p-\lambda\varphi_{p-1}}{\alpha_p} = \frac{-\lambda\varphi_p}{\alpha_{p+1}}.$$

Solving this system of equations we find that

$$\varphi_i = \frac{\lambda\Phi_i(\lambda)}{\varphi(\lambda)} \qquad (i = 1, \ldots, p),$$

where

$$\varphi(\lambda) = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & \alpha_1 \\ -\lambda & 1 & 0 & \ldots & 0 & 0 & \alpha_2 \\ 0 & -\lambda & 1 & \ldots & 0 & 0 & \alpha_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & -\lambda & 1 & \alpha_p \\ 0 & 0 & 0 & \ldots & 0 & -\lambda & \alpha_{p+1} \end{vmatrix},$$

and $\Phi_i(\lambda)$ is the minor corresponding to the ith element of the first row in the above determinant.

Hence we have the following formula

$$\frac{1}{1-\lambda K} = 1+\lambda\frac{\Phi(K,\lambda)}{\varphi(\lambda)}, \tag{9}$$

where $\Phi(K,\lambda) = K\Phi_1(\lambda)+\ldots+K^p\Phi_p(\lambda)$.

Since $\varphi(\lambda)$ is a polynomial of degree $\leqslant p$ and $\frac{1}{1-\lambda K}$ is not defined only at the points at which $\varphi(\lambda) = 0$, then

The spectrum of a vector of rank p consists of p points at most.

We also find from formula (9), in view of (5), the following form of the resolvent

$$K_\lambda = \frac{\Phi(K,\lambda)}{\varphi(\lambda)}.$$

In the application to Fredholm's equation the above way represents a convenient algorithm for finding the resolvent for the kernel with separate variables(14):

$$K(x,y) = \sum_{r=1}^{n} a_r(x)b_r(y);$$

for vector $K = \{K(x,y)\}$ always is of finite rank.

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(14) Goursat, T. 1; Kowalewski, pp. 139-174; Lalesco, p. 49.

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On type II convergence in the Mikusiński operational calculus

by

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Dedicatad to Professor Jan Mikusiński

Abstract. In the paper it is proved that type II convergence in the field $\mathscr{F}$ of Mikusiński's operators is not topological (Theorem 2.1), which is a solution of the problem posed in [1]. It is given a characterization of type II convergence and, defined in the paper, type II′ convergence. A description of compactness and boundedness in $\mathscr{F}$ with type II′ convergence is given and a sequential completeness of $\mathscr{F}$ is proved.

1. In the field of Mikusiński operators three types of convergence: type I, type I′ and type II are introduced (see [5], p. 144, 147 and [2]). Properties of type I and type I′ convergences are described in [2], [3].

In the paper we shall describe properties of type II convergence. In particular, it will be proved that type II convergence is not topological. This is the negative answer to the problem posed in [1]. Moreover, we shall give some facts about type II convergence, similar to that given in [3] for type I′ convergence.

We shall use terminology and notation from [3].

2. We say that a sequence $\{x_n\}$ of operators is *type* II *convergent to* x (and we write $x_n \xrightarrow{\text{II}} x$) if there exist continuous functions f, g, f_n, g_n $(n = 1, 2, \ldots)$ such that $x_n = f_n/g_n$, $x = f/g$ and $f_n \to f$, $g_n \to g$ almost uniformly.

In the above definition continuous functions can be replaced by locally integrable functions (as in [3] L denotes the set of all such functions) and the almost uniform convergence by the convergence with respect to the following family of pseudonorms:

$$\|f\|_T = \int_0^T |f(t)|\,dt \quad \text{for any } f\in L \text{ and } T > 0.$$

The above convergence will be denoted by $\xrightarrow{L}$.