STUDIA MATHEMATICA, T. LXXVII. (1983)

A characterization of Gaussian measures

by

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Abstract. The main purpose of this paper is to characterize Gaussian probability measures among weakly stable ones.

This paper has its origin in the study of generalized convolutions [5] and [6]. In what follows two random variables X and Y are said to be *equivalent*, in symbols $X \sim Y$, if they have the same probability distribution. We say that a symmetrically distributed random variable X has the *weakly stable probability distribution* whenever for any quadruple X_1, X_2, Y_1, Y_2 of independent random variables such that $X_1 \sim X_2 \sim X$ there exist independent random variables X_3 and Y_3 with the property

$$X_1 Y_1 + X_2 Y_2 \sim X_3 Y_3$$

and $X_3 \sim X$. This definition is equivalent to the following one: for every pair a, b of positive numbers there exists a non-negative random variable Y_{ab} independent upon X_3 such that

$$aX_1 + bX_2 \sim X_3 Y_{ab}. \tag{1}$$

Note that the case when Y_{ab} is constant with probability 1 corresponds to stable probability measures.

Throughout this paper we denote by $\mathscr{S}$ the set of all symmetric probability measures on the real line. $\mathscr{P}$ will denote the set of all probability measures concentrated on the half-line $[0, \infty)$. Further, δ_a will denote the probability measure concentrated at the point a. We denote the convolution of two measures μ and ν by $\mu * \nu$. Moreover, $\mu\nu$ will denote the probability distribution of the product XY of independent random variables X and Y with the probability distribution μ and ν, respectively. Consequently,

$$(\mu\nu)(E) = \int_{u \neq 0} \mu(u^{-1}E)\nu(du) + \nu(\{0\})\,\delta_0(E) \tag{2}$$

for every Borel subset E of the real line. Moreover, the characteristic function of $\mu\nu$ is given by the formula

$$\widehat{\mu\nu}(t) = \int_{-\infty}^{\infty} \mu(tu)\nu(du).$$

The set of all probability measures is equipped with the topology of the weak convergence. The following statements are evident:

(i) if $\mu_n \to \mu$, then $\mu_n\nu \to \mu\nu$ for every ν,

(ii) if $\nu \neq \delta_0$ and the sequence $\{\mu_n\nu\}$ is conditionally compact, then the sequence $\{\mu_n\}$ is conditionally compact, too.

We note that the definition of the weak stability can be formulated as follows. A probability measure $\mu \in \mathscr{S}$ is *weakly stable* if and only if for every pair ν_1, ν_2 of probability measures there exists a probability measure ν such that

$$(\nu_1\mu) * (\nu_2\mu) = \nu\mu. \tag{3}$$

Moreover, condition (1) can be written in terms of characteristic functions as follows:

$$\hat{\mu}(at)\hat{\mu}(bt) = \int_0^{\infty} \hat{\mu}(ut)\nu_{ab}(du), \tag{4}$$

where $\nu_{ab} \in \mathscr{P}$.

Now we shall quote some examples of symmetric weakly stable probability measures.

1. *Symmetric stable measures.* $\hat{\mu}(t) = e^{-|t|^p}$ $(0 < p \leqslant 2)$. In this case $\nu_{ab} = \delta_{(a^p+b^p)^{1/p}}$.

2. $\mu = \frac{1}{2}(\delta_c + \delta_{-c})$ $(-\infty < c < \infty)$, $\nu_{ab} = \frac{1}{2}(\delta_{a+b} + \delta_{|a-b|})$.

3. $$\mu(E) = \frac{\Gamma(q+3/2)}{\sqrt{\pi}\Gamma(q+1)} \int_{E\cap[-1,1]} (1-u^2)^q du \quad (q > -1),$$

$$\nu_{ab}(E) = \frac{\Gamma(q+3/2)}{\sqrt{\pi}\Gamma(q+1)4^q a^{2q+1} b^{2q+1}} \int_{E\cap[|a-b|,a+b]} [((a+b)^2-u^2)(u^2-(a-b)^2)]^q u\,du.$$

4. $$\mu(E) = \frac{1}{2c}\int_{E\cap[-c,c]} du \quad (c > 0), \qquad \nu_{ab}(E) = \frac{1}{2ab}\int_{E\cap[|a-b|,a+b]} u\,du.$$

5. $\hat{\mu}(t) = (1-|t|^p)^n$ if $|t| \leqslant 1$ and $\hat{\mu}(t) = 0$ otherwise $(0 < p \leqslant 1$, $n = 1, 2, \ldots)$. For $a \leqslant b$

$$\nu_{ab}(E) = (1-a^p/b^p)^n \delta_b(E) + \sum_{k=1}^{n} p(n+1)\binom{n}{k}\binom{n}{k-1} a^{p(n+1-k)} b^{pk} \times$$
$$\times \int_{E\cap[b,\infty)} (u^p-a^p)^{k-1}(u^p-b^p)^{n-k} u^{-2np-1} du.$$

The main aim of this paper is to characterize Gaussian probability measures among weakly stable ones. Namely we shall prove the following theorems.

THEOREM 1. *A measure μ from $\mathscr{S}$ is weakly stable, infinitely divisible and $\int_{-\infty}^{\infty} |u|^p \mu(du) < \infty$ for $0 < p < 2$ if and only if it is Gaussian.*

THEOREM 2. *A measure μ from $\mathscr{S}$ is weakly stable, $\int_{-\infty}^{\infty} |u|^p \mu(du) < \infty$ for $0 < p < 2$, $\int_{-\infty}^{\infty} e^{u^{2+\varepsilon}} \mu(du) = \infty$ for all $\varepsilon > 0$ if and only if it is non-degenerate Gaussian.*

THEOREM 3. *A measure μ from $\mathscr{S}$ is weakly stable and has at least one atom if and only if it is of the form $\mu = \frac{1}{2}(\delta_c + \delta_{-c})$ where $c \geqslant 0$.*

The necessity of the conditions of all three theorems is evident We ought to prove their sufficiency. Before proceeding to prove it we shal establish some lemmas.

For a given probability measure μ we introduce the notation

$$\tilde{\mu}(x) = \mu((-\infty, -x]\cup[x, \infty)) \quad (x \geqslant 0).$$

Put

$$\varkappa(\mu) = \varliminf_{x\to\infty} \frac{\log\log\tilde{\mu}(x)^{-1}}{\log x}.$$

Then from the inequality

$$e^{a|x|^p}\tilde{\mu}(x) \leqslant \int_{-\infty}^{\infty} e^{a|u|^p}\mu(du) \quad (p > 0)$$

and the formula

$$\int_{-\infty}^{\infty} e^{a|u|^p}\mu(du) = ap\int_0^{\infty} u^{p-1}e^{a|u|^p}\tilde{\mu}(u)\,du + \tilde{\mu}(0) \tag{5}$$

we get the following lemma.

LEMMA 1. *If for some positive a and p, $\int_{-\infty}^{\infty} e^{a|u|^p}\mu(du) < \infty$, then $\varkappa(\mu) \geqslant p$. Conversely, if $\varkappa(\mu) > p$, then $\int_{-\infty}^{\infty} e^{c|u|^p}\mu(du) < \infty$ for all positive numbers c.*

LEMMA 2. *Suppose that* $\mu \in \mathcal{S}$ *and* $\int_{-\infty}^{\infty} |u|^p \mu(du) < \infty$ *for* $0 < p < 2$. *If* μ *is weakly stable, then* $\int_{-\infty}^{\infty} e^{au^2} \mu(du) < \infty$ *for a certain positive number* a.

Proof. Since for $\mu = \delta_0$ our statement is obvious, we may assume that $\mu \neq \delta_0$. Given $0 < p < 2$, we put

$$\omega_p(E) = p \int_{E \cap [1,\infty)} u^{-p-1} du.$$

Then, by (2), we have the formula

$$\widetilde{\omega_p \mu}(x) = p x^{-p} \int_0^x \tilde{\mu}(u) u^{p-1} du.$$

Consequently, by (5) and the assumption $\mu \neq \delta_0$, the function $x^p \widetilde{\omega_p \mu}(x)$ has a finite positive limit when $x \to \infty$. Thus for all positive numbers y,

$$\lim_{x\to\infty} \frac{\widetilde{\omega_p \mu}(x)}{\widetilde{\omega_p \mu}(xy)} = y^p.$$

In other words the probability measure $\omega_p \mu$ belongs to the domain of attraction of the symmetric stable probability measure λ_p with the exponent p, i.e. $\hat{\lambda}_p(t) = e^{-|t|^p}$ ([1], Chapter XVII, § 5, Th. 1a). Consequently, there exists a sequence $\{a_n\}$ of positive numbers such that

$$(\delta_{a_n} \omega_p \mu)^{*n} \to \lambda_p \tag{6}$$

when $n \to \infty$. By the weak stability of μ (formula (3)) we infer that there exist probability measures $\nu_{n,p}$ satisfying the condition

$$(\delta_{a_n} \omega_p \mu)^{*n} = \nu_{n,p} \mu.$$

By (6) and (ii) the sequence $\nu_{n,p}$ $(n = 1, 2, \ldots)$ is conditionally compact. Let ν_p be its cluster point. By (i) we have the formula $\nu_p \mu = \lambda_p$ $(0 < p < 2)$. But λ_p tends to the Gaussian probability measure λ when $p \to 2$, because $\hat{\lambda}_p(t) \to e^{-t^2}$. Thus, by (ii), the family ν_p $(1 \leqslant p < 2)$ is conditionally compact. Denoting by ν its cluster point we have, by (i), the equation $\nu\mu = \lambda$, where $\hat{\lambda}(t) = e^{-t^2}$. Consequently, for $c < 1/4$ the integral $\int_{-\infty}^{\infty} e^{cu^2} \lambda(du)$ is finite, which, by (2), yields the finiteness of the integral $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{cu^2v^2} \mu(du)\nu(dv)$. Since $\nu \neq \delta_0$, our assertion is the consequence of the Fubini Theorem.

We are now in a position to prove the sufficiency of the conditions of our theorem.

Proof of Theorem 1. Suppose that μ fulfils the conditions of the theorem. Then, by Lemma 2, $\int_{-\infty}^{\infty} e^{au^2} \mu(du) < \infty$ for a certain positive number a.

Thus its characteristic function $\hat{\mu}$ can be extended to an entire function without zeros on the complex plane ([4], p. 187). Hence it follows that μ is Gaussian ([1], Chapter XV, § 8) which completes the proof.

Proof of Theorem 2. Suppose that μ fulfils the conditions of the theorem. Then, by Lemma 2, $\int_{-\infty}^{\infty} e^{au^2} \mu(du) < \infty$ for a certain positive number a. Hence it follows that its characteristic function $\hat{\mu}$ can be extended to an entire function on the complex plane. Moreover, by Lemma 1, we have the inequality $\varkappa(\mu) \geqslant 2$. Further, the assumption $\int_{-\infty}^{\infty} e^{u^{2+\varepsilon}} \mu(du) = \infty$ $(\varepsilon > 0)$ yields, by Lemma 1, the inequality $\varkappa(\mu) \leqslant 2 + \varepsilon$. Thus $\varkappa(\mu) = 2$. Let $\varrho(\mu)$ denote the order of $\hat{\mu}$. Then

$$\frac{1}{\varrho(\mu)} + \frac{1}{\varkappa(\mu)} = 1$$

([3], p. 54). Consequently, $\varrho(\mu) = 2$. It is well known that the maximum modulus of $\hat{\mu}$ in the circle $|z| \leqslant r$ is equal to $\max\{\hat{\mu}(ir), \hat{\mu}(-ir)\}$ ([4], p. 134). Since $\mu \in \mathcal{S}$, we have $\hat{\mu}(ir) = \hat{\mu}(-ir)$. Then there exists a Valiron function f defined and differentiable on the positive half-line with the properties

$$\lim_{t\to\infty} f(t) = 2, \tag{7}$$

$$\lim_{t\to\infty} f'(t)\, t \log t = 0, \tag{8}$$

$$\overline{\lim_{t\to\infty}} \frac{\log \hat{\mu}(it)}{t^{f(t)}} = 1, \tag{9}$$

$$\log \hat{\mu}(it_n) = t_n^{f(t_n)} \quad (n = 1, 2, \ldots) \tag{10}$$

for a certain sequence $t_1 < t_2 < \ldots$ tending to ∞ ([2], p. 52). By the Lagrange mean value theorem we have, in view of (8), the formula

$$\lim_{t\to\infty} \big(f(t) - f(c^{-1}t)\big) \log t = 0$$

for every positive number c. Hence, by virtue of (7) and (10) we get the formula

$$\lim_{n\to\infty} \frac{\log \hat{\mu}(icr_n)}{r_n^{f(r_n)}} = c^2, \tag{11}$$

where $r_n = c^{-1} t_n$ and the sequence $\{t_n\}$ is defined by (10).

By (3) we have a measure $\nu \in P$ with the property

$$\mu * \mu = \mu\nu \tag{12}$$

or in terms of the characteristic functions

$$\hat{\mu}(t)^2 = \int_0^\infty \hat{\mu}(tu)\nu(du).$$

To prove that this equation remains true in the complex plane, i.e.

(13) $$\hat{\mu}(z)^2 = \int_0^\infty \hat{\mu}(zu)\nu(du)$$

for all complex numbers z it suffices to prove that the right-hand side of (13) is an entire function or, equivalently, that for all $t > 0$

(14) $$\int_0^\infty \hat{\mu}(itu)\nu(du) < \infty.$$

This inequality is obvious when ν has a compact support. Therefore we consider the case when ν is not concentrated on a compact set. We already know that $\int_{-\infty}^{\infty} e^{au^2}\mu(du) < \infty$ for a certain positive number a. Since

$$\int_{-\infty}^{\infty} e^{(a/2)u^2}(\mu*\mu)(du) \leqslant \Big(\int_{-\infty}^{\infty} e^{au^2}\mu(du)\Big)^2,$$

we have, by virtue of (12),

$$\int_0^\infty \int_{-\infty}^{\infty} e^{(a/2)u^2v^2}\mu(du)\nu(dv) < \infty.$$

Taking into account that ν has an unbounded support and $\mu \neq \delta_0$ because of $\int_{-\infty}^{\infty} e^{u^{2+\varepsilon}}\mu(du) = \infty$ $(\varepsilon > 0)$ and applying the Fubini theorem, we have the inequalities

$$\int_{-\infty}^{\infty} e^{au^2}\mu(du) < \infty, \qquad \int_0^\infty e^{bv^2}\nu(dv) < \infty$$

for all $a > 0$ and a certain $b > 0$. Since $|tuv| \leqslant b^{-1}t^2u^2 + bv^2$, we get the inequality

$$\int_0^\infty \hat{\mu}(itu)\nu(du) \leqslant \int_0^\infty \int_{-\infty}^{\infty} e^{|tuv|}\mu(du)\nu(dv)$$
$$\leqslant \int_{-\infty}^{\infty} e^{b^{-1}t^2u^2}\mu(du)\int_0^\infty e^{bv^2}\nu(du),$$

which yields (14) and, consequently, completes the proof of (13).

Let c be an arbitrary positive number with the property $\nu([c,\infty)) > 0$. Let $\{r_n\}$ be the sequence appearing in (11). By (13) we have the inequality

$$\hat{\mu}(ir_n)^2 \geqslant \int_c^\infty \hat{\mu}(iur_n)\nu(du) \geqslant \hat{\mu}(icr_n)\nu([c,\infty)).$$

Consequently, by (9) and (11),

$$2 \geqslant \varlimsup_{n\to\infty} \frac{\log\hat{\mu}(ir_n)^2}{r_n^{f(r_n)}} \geqslant c^2,$$

which shows that $c \leqslant \sqrt{2}$ or, in other words, ν is concentrated on the interval $[0, \sqrt{2}]$. Differentiating (13) we get the equation

$$2\int_{-\infty}^{\infty} v^2\mu(dv) = \int_{-\infty}^{\infty} v^2\mu(dv)\int_0^{\sqrt{2}} u^2\nu(du)$$

which yields

$$\int_0^{\sqrt{2}} u^2\nu(du) = 2$$

because $\mu \neq \delta_0$. The last equation shows that $\nu = \delta_{\sqrt{2}}$ which implies, in view of (13), $\hat{\mu}(t)^2 = \hat{\mu}(\sqrt{2}\,t)$. Hence by induction we get the formula $\hat{\mu}(t) = \hat{\mu}(t/\sqrt{2^n})^{2^n}$ $(n = 1, 2, \ldots)$ which, by the Central Limit Theorem shows that the measure μ is Gaussian. The theorem is thus proved.

Proof of Theorem 3. Let A be the set of all atoms of the measure μ. Of course, A is at most denumerable and, by the symmetry of μ, $\mu(\{c\}) = \mu(\{-c\})$ for $c \in A$. For every pair a, b of positive numbers, ν_{ab} will denote a measure from P satisfying condition (4) or, equivalently, the equation

(15) $$(\delta_a\mu)*(\delta_b\mu) = \mu\nu_{ab}.$$

First consider the case $A = \{0\}$. Then the last equation yields

$$\mu(\{0\})^2 = (\mu*\mu)(\{0\}) = \mu(\{0\})\big(1-\nu_{11}(\{0\})\big)+\nu_{11}(\{0\}).$$

which implies the inequality $\mu(\{0\})^2 \geqslant \mu(\{0\}) > 0$. Consequently, $\mu(\{0\}) = 1$, i.e. $\mu = \delta_0$ which completes the proof.

Now let us assume that $0 \in A$ and $A \smallsetminus \{0\} \neq \emptyset$. Let a, b be a pair of linearly independent positive numbers over the denumerable field generated by the set A. Then from (15) we get the equation

$$\mu(\{0\})^2 = (\delta_a\mu)*(\delta_b\mu)(\{0\}) = \mu(\{0\})\big(1-\nu_{ab}(\{0\})\big)+\nu_{ab}(\{0\})$$

whence the inequality $\mu(\{0\})^2 \geqslant \mu(\{0\}) > 0$ follows. Thus $\mu(\{0\}) = 1$, i.e. $\mu = \delta_0$ which completes the proof.

Finally consider the case $0 \notin A$. Passing to $\delta_c \mu$ if necessary, we may assume without loss of generality that

(16) $$1 \in A.$$

Let a, b be a pair of linearly independent positive numbers over the field generated by the set A. Let A_{ab} and B_{ab} denote the set of all atoms of the measures $(\delta_a \mu) * (\delta_b \mu)$ and ν_{ab}, respectively. From (15) we get the equations

(17) $$A_{ab} = \{ac_1 + bc_2 : c_1, c_2 \in A\},$$

(18) $$A_{ab} = \{cd : c \in A,\ d \in B_{ab}\}.$$

Thus

(19) $$0 \notin B_{ab}$$

and, by (16),

(20) $$B_{ab} \subset A_{ab}.$$

The last inclusion shows that every number d from B_{ab} has the unique representation $d = ag + bh$, where $g, h \in A$. Let C_{ab} be the set of all such coefficients g and h. By (18) for every $c \in A$ we have the relation $cd = acg + bch \in A_{ab}$. Hence, by the linear independence of a, b we infer that cg, $ch \in A$. Consequently, denoting by $\mathrm{sem}(C_{ab})$ the multiplicative semi-group generated by C_{ab} we have the inclusion

(21) $$\mathrm{sem}(C_{ab}) \subset A.$$

Further, by the linear independence of a and b, we have the equation

$$(\delta_a \mu) * (\delta_b \mu)(\{ac_1 + bc_2\}) = \mu(\{c_1\}) \mu(\{c_2\}) \quad (c_1, c_2 \in A),$$

which, by (15), yields

(22) $$\mu(\{c_1\}) \mu(\{c_2\}) = \sum_{d \in B_{ab}} \mu\big(\{d^{-1}(ac_1 + bc_2)\}\big) \nu_{ab}(\{d\})$$

for all $c_1, c_2 \in A$. In particular, for every $c \in A$ we have the formula

(23) $$\mu(\{c\})^2 = \sum_{d \in B_{ab}} \mu\big(\{d^{-1} c(a+b)\}\big) \nu_{ab}(\{d\}).$$

Suppose that $c \in A$, $d \in B_{ab}$ and $g = d^{-1} c(a+b) \in A$. Then $g^{-1} c(a+b) \in B_{ab}$ and, consequently, by (20), $g^{-1} c(a+b) \in A_{ab}$. By the linear independence of a, b and by (17) we infer that $h = g^{-1} c \in A$. Moreover, $d = h(a+b)$. Thus setting

$$H_{ab} = \{h : h \in A,\ h(a+b) \in B_{ab}\}$$

we have, by virtue of (23),

(24) $$\mu(\{c\})^2 = \sum_{h \in H_{ab}} \mu(\{ch^{-1}\}) \nu_{ab}\big(\{h(a+b)\}\big)$$

for every $c \in A$. Moreover,

(25) $$H_{ab} \subset C_{ab}.$$

Now we shall prove that

(26) $$H_{ab} = \{1\}.$$

Contrary to this let us assume that H_{ab} contains a positive number d different from 1. Then, by (21) and (25), $d^k \in A$ $(k = 1, 2, \ldots)$ and, by (24),

$$\mu(\{d^{k+1}\})^2 \geqslant \mu(\{d^k\}) \nu_{ab}\big(\{d(a+b)\}\big) \quad (k = 1, 2, \ldots).$$

Hence we get the inequality

$$\sum_{k=n+1}^{\infty} \mu(\{d^k\})^2 \geqslant \nu_{ab}\big(\{d(a+b)\}\big) \sum_{k=n}^{\infty} \mu(\{d^k\}) \quad (n = 1, 2, \ldots)$$

which yields

$$\nu_{ab}\big(\{d(a+b)\}\big) \leqslant \max_{k \geqslant n} \mu(\{d^k\}) \quad (n = 1, 2, \ldots).$$

Consequently, $\nu_{ab}\big(\{d(a+b)\}\big) = 0$ which contradicts the assumption $d \in H_{ab}$. Formula (26) is thus proved. Now equation (24) can be written in the form

$$\mu(\{c\})^2 = \mu(\{c\}) \nu_{ab}(\{a+b\}) \quad (c \in A).$$

Thus

$$\mu(\{c\}) = \nu_{ab}(\{a+b\}) \quad (c \in A).$$

Hence it follows that the set A is finite. Consequently, by (21), the set C_{ab} is contained in the two-point set $\{-1, 1\}$ which implies the inclusion

$$B_{ab} \subset \{|a-b|, a+b\}$$

because B_{ab} consists of positive numbers. Further, taking into account (18), we have the inclusion

(27) $$A_{ab} \subset \{c|a-b| : c \in A\} \cup \{c(a+b) : c \in A\}.$$

By (16) and the symmetry of μ we have the inclusion $\{-1, 1\} \subset A$. Hence by the linear independence of a, b it follows that inclusion (27) holds in the case $A = \{-1, 1\}$ only. Of course, we have then $B_{ab} = \{|a-b|, a+b\}$ which together with (22) yields

$$\begin{aligned} \mu(\{1\})^2 = \mu(\{1\}) \mu(\{-1\}) &= \mu\big(\{\mathrm{sgn}(a-b)\}\big) \nu_{ab}\big(\{|a-b|\}\big) \\ &= \mu(\{1\}) \nu_{ab}(\{|a-b|\}). \end{aligned}$$

Thus

(28) $$\nu_{ab}(\{|a-b|\}) = \mu(\{1\})$$

for every pair a, b of positive numbers linearly independent over the field generated by A, i.e. the field of rational numbers. Suppose now that $b \to a$, a, b being linearly independent. Then, by (i) and (15), the family $\{\nu_{ab}\}$ is conditionally compact. Let ν be its cluster point. Then by (15)

$$(\delta_a \mu) * (\delta_a \mu) = \mu\nu \tag{29}$$

and, by (28),

$$\nu(\{0\}) \geqslant \mu(\{1\}).$$

Since $A = \{-1, 1\}$, we get, by virtue of (29), the relation

$$2\mu(\{1\})^2 = (\delta_a \mu) * (\delta_a \mu)(\{0\}) = (\mu\nu)(\{0\}) = \nu(\{0\}) \geqslant \mu(\{1\}).$$

Thus $\mu(\{1\}) \geqslant 1/2$. Since $\mu(\{-1\}) = \mu(\{1\})$ and $\mu(\{-1\}) + \mu(\{1\}) \leqslant 1$, we have $\mu(\{1\}) = \mu(\{-1\}) = 1/2$ and, consequently, $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$ which completes the proof.

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Received March 5, 1982 (1741)

STUDIA MATHEMATICA, T. LXXVII. (1983)

Non-Leibniz algebras

by

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Dedicated to Professor J. Mikusiński on the 70th birthday

Abstract. We consider algebras with right invertible operators in the case when the Leibniz condition

$$D(xy) = xDy + yDx \quad \text{for} \quad x, y \in \operatorname{dom} D \tag{L}$$

(provided that $xy \in \operatorname{dom} D$) is not satisfied. In particular, it is shown that in a large class of non-Leibniz algebras all initial operators are averaging.

We shall consider algebras with right invertible operators in the non-Leibniz case, i.e., in the case where the condition

$$D(xy) = xDy + yDx \quad \text{for} \quad x, y \in \operatorname{dom} D \tag{L}$$

is not satisfied (provided that $xy \in \operatorname{dom} D$).

Some particular cases have been studied by Dudek [1] and by the author and von Trotha (cf. [3], [4], [9]).

In [5], [6], [7] we have shown that the Green formula, the Euler–Lagrange equation and the P'cone identity hold in the general non-Leibniz case. In [8] there was given a classification of non-Leibniz algebras. A large class of algebras which are in a sense "close" to Leibniz case has been distinguished. Properties of right invertible operators and their inverses in these algebras, in particular, Wroński theorems have been studied.

1. Preliminaries. Let X be a linear space over a field $\mathscr{F}$ of scalars. Let $L(X)$ be the set of all linear operators A such that the domain of A (denoted by $\operatorname{dom} A$) is a linear subset of X and $AX \subset X$. In particular, we write: $L_0(X) = \{A \in L(X)\colon \operatorname{dom} A = X\}$. Let $R(X)$ be the set of all right invertible operators belonging to $L(X)$. For a given $D \in R(X)$ we denote by $\mathscr{R}_D = \{R_\gamma\}_{\gamma \in \Gamma}$ the set of all right inverses of D. We shall assume that $R_\gamma \in L_0(X)$ for $\gamma \in \Gamma$. Here and in the sequel we shall assume also that $\dim\ker D > 0$, i.e., D is right invertible but not left invertible. Any element of $\ker D$ is a *constant* for D. By definition, F is an initial operator for D if is a projection *onto* $\ker D$ such that $FR = 0$ for a right inverse R.