On the coefficients of some classes of starlike functions

by RYSZARD MAZUR (Kielce)

Abstract. Let Ω be the class of functions ω , $\omega(0) = 0$, $|\omega(z)| < 1$, holomorphic in the unit disc K ; let a and b be arbitrary fixed numbers, $-1 \leq b < a \leq 1$, $\wp(a, b)$ — the class of functions P , $P(0) = 1$, holomorphic in K , such that $P \in \wp(a, b)$ iff $P(z) = (1 + a\omega(z))(1 + b\omega(z))^{-1}$ for some function $\omega \in \Omega$ and every z in K . Let $S^*(a, b)$ denote the class of functions f , $f(0) = 0$, $f'(0) = 1$, holomorphic in K , such that $f \in S^*(a, b)$ iff $zf'(z)(f(z))^{-1} = P(z)$ for some $P \in \wp(a, b)$; $\Sigma^*(a, b)$ — the class of meromorphic functions of the form $F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ and satisfying the condition: $zF'(z)(F(z))^{-1} = P(z)$, $P \in \wp(a, b)$.

The author obtains sharp estimates of the coefficients of functions of the families $S^*(a, b)$ and $\Sigma^*(a, b)$.

1. Introduction. Let Ω be the family of all functions of the form

$$(1.1) \quad \omega(z) = c_1 z + c_2 z^2 + \dots$$

which are holomorphic in the unit disc $K = \{z: |z| < 1\}$ and satisfy the condition $|\omega(z)| < 1$ for $z \in K$.

Janowski introduced (cf. [2]) a class $\wp(a, b)$ of holomorphic functions

$$(1.2) \quad P(z) = 1 + p_1 z + p_2 z^2 + \dots$$

in the unit disc; by definition, p is in $\wp(a, b)$ if and only if

$$(1.3) \quad P(z) = \frac{1 + a\omega(z)}{1 + b\omega(z)}, \quad z \in K,$$

for some function $\omega \in \Omega$; a, b are arbitrarily fixed numbers that satisfy the condition $-1 \leq b < a \leq 1$. It is easy to notice that $\wp(a, b) \subset \wp$ and $\wp(1, -1) \equiv \wp$, where $\wp$ is the well-known family of functions of Carathéodory type.

Let us denote by $S^*(a, b)$ (cf. [2]) the class of functions

$$(1.4) \quad f(z) = z + a_2 z^2 + \dots$$

holomorphic in K , defined by the condition: f is in $S^*(a, b)$ if and only if

there exists a function $P \in \wp(a, b)$ such that

$$(1.5) \quad \frac{zf'(z)}{f(z)} = P(z), \quad z \in K.$$

Since $\wp(a, b) \subset \wp$ and $\wp(1, -1) \equiv \wp$, we have $S^*(a, b) \subset S^*$ and $S^*(1, -1) \equiv S^*$, where S^* is the well-known class of starlike functions.

Moreover, let $\Sigma^*(a, b)$ denote the family of meromorphic functions

$$(1.6) \quad F(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \dots$$

in the unit disc such that

$$(1.7) \quad \frac{-zF'(z)}{F(z)} = P(z), \quad z \in K,$$

for some function P in $\wp(a, b)$.

It is not difficult to see that $\Sigma^*(a, b) \subset \Sigma^*$ and $\Sigma^*(1, -1) \equiv \Sigma^*$, where Σ^* is the class of starlike meromorphic functions.

In this paper we give estimates of the coefficients of the families $S^*(a, b)$ and $\Sigma^*(a, b)$.

Our results contain those of Clunie [1], Janowski [3], Kaczmariski [4], Plaskota [5], Pomerence [6], Robertson [7], and Wiczorek [8], as particular cases.

2. Estimates of the coefficients of functions of class $S^*(a, b)$. We shall prove the following result.

THEOREM 1. *If $f \in S^*(a, b)$ and $-1 \leq b < a \leq 1$, then*

$$(2.1) \quad |a_n| \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} |a - kb| \quad \text{for } n = 2, 3, \dots, p$$

and

$$(2.2) \quad |a_n| \leq \frac{1}{(n-1)(p-2)!} \prod_{k=1}^{p-1} |a - kb| \quad \text{for } n = p+1, \dots,$$

where $p \in \left(\frac{1+a}{1+b}, \frac{2+a+b}{1+b} \right)$ is a natural number. If $b = -1$, then

$$(2.3) \quad |a_n| \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (a+k) \quad \text{for } n = 2, 3, \dots$$

The bounds (2.1) and (2.3) are attained by the function

$$(2.4) \quad f(z) = \begin{cases} z(1 + \varepsilon bz)^{(a-b)/b} & \text{for } b \neq 0, \\ z_0 \exp(a\varepsilon z) & \text{for } b = 0, \end{cases} \quad |\varepsilon| = 1.$$

Proof. If $f \in S^*(a, b)$, then

$$(2.5) \quad \frac{zf'(z)}{f(z)} = \frac{1 + a\omega(z)}{1 + b\omega(z)}, \quad z \in K,$$

for some function $\omega \in \Omega$.

From (2.5) it follows that

$$zf'(z) - f(z) = (af(z) - bzf'(z))\omega(z);$$

hence

$$(2.6) \quad \sum_{k=1}^{\infty} (k-1) a_k z^k = \omega(z) \cdot \sum_{k=1}^{\infty} (a - kb) a_k z^k$$

and it is easy to see that $|a_2| \leq a - b$.

Thus estimate (2.1) holds true for $n = 2$. Now suppose that $n \geq 2$. Let us write equality (2.6) as follows:

$$\sum_{k=1}^n (k-1) a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k = \omega(z) \sum_{k=1}^{n-1} (a - kb) a_k z^k,$$

where the series $\sum_{k=n+1}^{\infty} d_k z^k$ is convergent in the unit disc. By using the method of Clunie's (cf. [1]) we get:

$$(2.7) \quad \left| \sum_{k=1}^n (k-1) a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k \right| < \left| \sum_{k=1}^{n-1} (a - kb) a_k z^k \right|.$$

Hence, putting $z = re^{it}$, $0 < r < 1$, $0 \leq t < 2\pi$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^n (k-1) a_k r^k e^{ik t} + \sum_{k=n+1}^{\infty} d_k r^k e^{ik t} \right|^2 dt \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{n-1} (a - kb) a_k r^k e^{ik t} \right|^2 dt.$$

Integrating we get

$$(2.8) \quad \sum_{k=1}^n (k-1)^2 |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} \leq \sum_{k=1}^{n-1} (a - kb)^2 |a_k|^2 r^{2k}.$$

In particular, from (2.8) follows

$$(2.9) \quad \sum_{k=1}^n (k-1)^2 |a_k|^2 r^{2k} \leq \sum_{k=1}^{n-1} (a - kb)^2 |a_k|^2 r^{2k}.$$

Passing in (2.9) to the limit as $r \rightarrow 1$ we obtain

$$\sum_{k=1}^n (k-1)^2 |a_k|^2 \leq \sum_{k=1}^{n-1} (a - kb)^2 |a_k|^2.$$

Thus

$$(n-1)^2 |a_n|^2 \leq \sum_{k=1}^{n-1} ((a-kb)^2 - (k-1)^2) |a_k|^2.$$

We easily observe that $(a-(n-1)b)^2 - (n-2)^2 \geq 0$ iff $n \leq p$ and $-1 < b < a \leq 1$; moreover, if $b = -1$, then $(a+n-1)^2 - (n-2)^2 \geq 0$ for $n = 2, 3, \dots$ and every $a \in (-1, 1)$. Estimates (2.1), (2.2) and (2.3) are obtained by induction. The function f defined by formula (2.4) belongs to the family $S^*(a, b)$. Let a_n^* denote the coefficient at z^n in the power series representing the function (2.4). Then we have

$$a_n^* = \frac{e^{n-1}}{(n-1)!} \prod_{k=1}^{n-1} (a-kb);$$

thus the estimates (2.1) and (2.3) are sharp. Q.E.D.

Theorem 1 implies the following corollaries:

COROLLARY 1. *The values of the function $f \in S^*(a, b)$ include the disc*

$$(2.10) \quad |w| < \frac{1}{2+a-b}.$$

In fact, let $f \in S^*(a, b)$ and $f(z) \neq w_0$ for every $z \in K$. Then the function

$$g(z) = \frac{f(z)}{1-f(z)/w_0} = z + (a_2 + 1/w_0)z^2 + \dots$$

belongs to the family S . Thus $|a_2 + 1/w_0| \leq 2$. Since $|a_2| \leq a-b$, then (2.10) follows.

By specifying the values of the parameters appearing in Theorem 1 we obtain some interesting particular cases.

COROLLARY 2. *By putting $a = 1-2\alpha$, $b = -1$ in (2.3) we obtain the result of [7] for starlike functions of order α .*

COROLLARY 3. *For $a = 1$ and $b = \frac{1}{M} - 1$, $M \geq 1$, we obtain the result of Janowski (cf. [3]).*

COROLLARY 4. *By substituting $b = -a$, $0 < a < 1$, in (2.1), (2.2) and (2.10) we get corresponding results obtained in [8].*

COROLLARY 5. *If $a = 1 - \lambda - \lambda m$, $b = -m$, then we obtain the result of [5].*

COROLLARY 6. *By applying Theorem 1 to the case where $a = 1$ and $b = -1$, we obtain $|a_n| \leq n$ for starlike functions of order 0.*

3. Estimates for coefficients of functions of class $\Sigma^*(a, b)$. Now we shall find estimates for coefficients of functions belonging to the family $\Sigma^*(a, b)$.

THEOREM 2. If $F \in \Sigma^*(a, b)$, then

$$(3.1) \quad |a_n| \leq \frac{a-b}{n+1} \quad \text{for } n = 0, 1, 2, \dots$$

The bound is attained by the function

$$(3.2) \quad F^*(z) = \begin{cases} \frac{1}{z} (1 + bz^{n+1})^{(b-a)/b(n+1)} & \text{for } b \neq 0, \\ \frac{1}{z} \exp\left(\frac{-a}{n+1} z^{n+1}\right) & \text{for } b = 0. \end{cases}$$

Proof. It follows from the definitions of the families $\Sigma^*(a, b)$ and $\wp(a, b)$ that $F \in \Sigma^*(a, b)$ if and only if

$$(3.3) \quad -\frac{zF'(z)}{F(z)} = \frac{1+a\omega(z)}{1+b\omega(z)}, \quad z \in K,$$

for some function $\omega \in \Omega$.

From (3.3) and (1.6) we have

$$(3.4) \quad \sum_{k=0}^{\infty} (k+1) a_k z^{k+1} = -(a-b + \sum_{k=0}^{\infty} (a+kb) a_k z^{k+1}) \omega(z).$$

Hence and from (1.1) we have

$$(3.5) \quad |a_0| \leq a-b, \quad |a_1| \leq (a-b)/2.$$

Let $n \geq 2$. By applying the method of Clunie [1] we finally obtain the inequality

$$(3.6) \quad (n+1)^2 |a_n|^2 \leq (a-b)^2 + \sum_{k=0}^{n-1} ((a+kb)^2 - (k+1)^2) |a_k|^2.$$

Since $(a+lb)^2 - (l+1)^2 \leq 0$ for $l = 0, 1, 2, \dots$ and $-1 \leq b < a \leq 1$, we have

$$(3.7) \quad (n+1)^2 |a_n|^2 \leq (a-b)^2.$$

From inequalities (3.5) and (3.7) we conclude that the estimate (3.1) holds true for $n = 0, 1, 2, \dots$

The function F defined by formula (3.2) belongs to the family $\Sigma^*(a, b)$, and if $F^*(z) = (1/z) + a_n^* z^n + \dots$, then $|a_n^*| = (a-b)/(n+1)$ for $n = 0, 1, 2, \dots$, then estimate (3.1) is sharp.

COROLLARY 1. If $F \in \Sigma^*(a, b)$, then

$$(3.8) \quad \sum_{k=0}^{\infty} ((1-b^2)k^2 + 2(1-ab)k + (1-a^2)) |a_k|^2 \leq (a-b)^2.$$

In fact, by equality (3.4) and the condition $|\omega(z)| < 1$, we have

$$\sum_{k=0}^{\infty} |(k+1) a_k z^{k+1}|^2 < |a-b + \sum_{k=0}^{\infty} (a+kb) a_k z^k|^2;$$

thus

$$\int_0^{2\pi} \left| \sum_{k=0}^{\infty} (k+1) a_k r^{k+1} e^{i(k+1)t} \right|^2 dt \leq \int_0^{2\pi} \left| a-b + \sum_{k=0}^{\infty} (a+kb) a_k r^k e^{i(k+1)t} \right|^2 dt.$$

Hence, by integrating and passing to the limit as r tends to 1, we obtain (3.8).

COROLLARY 2. *By putting $a = 1 - \alpha - \alpha m$, $b = -m$ in (3.1) and (3.8) we get the corresponding result obtained in [4], and by letting $b = -a$, $0 < a \leq 1$, we get the result of [8]; finally, we remark that if $b = -1$ and $a = 1 - 2\alpha$, then we get the result obtained in [6].*

References

- [1] J. Clunie, *On meromorphic schlicht*, J. London Math. Soc. 34 (1959), p. 215–216.
- [2] W. Janowski, *Some extremal problems for certain families of analytic functions I*, Ann. Polon. Math. 28 (1973), p. 297–326.
- [3] —, *Extremal problems for a family of functions with positive real part and for some related families*, ibidem 23 (1970), p. 159–177.
- [4] J. Kaczmariski, *On the coefficients of some classes of starlike functions*, Bull. Acad. Polon. Sci. 17 (1969), p. 495–501.
- [5] W. Plaskota, *Limitation des coefficients dans une famille de fonctions holomorphes dans le cercle $|z| < 1$* , Ann. Polon. Math. 24 (1970), p. 65–70.
- [6] Ch. Pomeranke, *On meromorphic starlike functions*, Pacific J. Math. 13 (1963), p. 221–235.
- [7] M. S. Robertson, *On the theory of univalent functions*, Ann. of Math. 37 (1936), p. 169–185.
- [8] Z. Wieczorek, *On the coefficients of starlike functions of some classes*, Comment. Math. 18 (1974), p. 113–119.

Reçu par la Rédaction le 3. 1. 1978