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Pseudo-Markov transformations

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Abstract. Pseudo-Markov transformations on intervals are defined, extending the notion of pseudo-r-adic transformations introduced by Lasota [7]. Ergodic Markov invariant measures for such a transformation are shown to exist. A sufficient condition for the transformation to be mixing and hence exact under the Markov invariant measure is given. Continuous maps of intervals with orbits of odd period $\geqslant 3$ are known to be pseudo-Markov and to satisfy the above condition.

1. Introduction. Recently many authors are interested in continuous maps with orbits of period three. For instance the map

$$f(x) = rx(1-x),$$

which is originated in a biological problem of the population growth of a single species (cf. [6]), has an orbit of period three if $3.83 < r \leqslant 4$. Li and Yorke [9] proved that period three implies chaos. This suggests to us the existence of a continuous invariant measure. Indeed, Lasota [7] showed that any continuous map with an orbit of period three possesses a continuous ergodic invariant measure.

In this note we are concerned with a class of piecewise continuous maps, namely the pseudo-Markov transformations, of intervals including the maps mentioned above. We will show the existence of an ergodic Markov invariant measure for a pseudo-Markov transformation. For a continuous map τ with an orbit of period three, this Markov measure m turns out to be mixing and so (τ, m) is exact in the sense of Rohlin [11].

On the other hand, expanding piecewise C^2 maps of intervals were studied by several authors. Lasota and Yorke [8] showed that such a map g possesses an absolutely continuous (with respect to Lebesgue measure) ergodic invariant measure m. Bowen [2] proved the weak Bernoulliness of (g, m) under suitable conditions. In this case the natural extension of (g, m) is Bernoulli. In our case it seems that one can not expect in general the existence of an invariant measure with such strong properties as absolute continuity and Bernoulliness, because of the generality of our setting.

The main part of this note is Section 3, where we prove the existence of

Markov invariant measures for pseudo-Markov transformations. The method is an imitation of Lasota's one ([7]). We construct a Borel injection from a Markov subshift to the considered transformation on an interval. Then the image measure of a Markov invariant measure of the subshift is invariant under the transformation. In Section 2 we give the definition of pseudo-Markov transformations and we explain some known things about Markov subshifts. In Section 4 we consider a simple example, in which the Markov invariant measure equivalent to Lebesgue measure does not attain the maximum of the metrical entropies. Thus this invariant measure is not the most random one in a sense among invariant measures.

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2. Pseudo-Markov transformations. Let I be an interval (bounded or not) of the real line. A Borel map $\tau\colon I \to I$ is called a *pseudo-Markov transformation* if there are disjoint bounded non-empty open intervals $I_1, I_2, \ldots, I_d \subset I$, $d \geqslant 2$, with the following two properties:

(i) Define

$$m_{ij} = \begin{cases} 1, & \text{if } \tau(I_i) \supset I_j, \\ 0, & \text{otherwise,} \end{cases}$$

then the matrix $M = (m_{ij})_{1 \leqslant i,j \leqslant d}$ is irreducible, i.e. for any i and j there are $i_0 = i, i_1, \ldots, i_n = j$ such that $m_{i_{k-1}i_k} = 1$ for all $1 \leqslant k \leqslant n$, and (ii) if $m_{ij} = 1$, then there is a non-empty open interval $I_{ij} \subset I_i$ such that $\tau(I_{ij}) = I_j$ and $\tau|_{I_{ij}}$ extends to a continuous function τ_{ij} on the closure $\bar{I}_{ij}$ of I_{ij}. The matrix M is called a *structure matrix* of τ.

The pseudo-d-adic transformations in the sense of Lasota [7] is pseudo-Markov, in which $m_{ij} = 1$ for all $1 \leqslant i, j \leqslant d$. A continuous map of an interval with an orbit of period three is also pseudo-Markov. Indeed, if $a = \tau^3(a) < \tau(a) < \tau^2(a)$ is an orbit, then putting $I_1 = (a, \tau(a))$ and $I_2 = (\tau(a), \tau^2(a))$ we obtain the pseudo-Markov property with

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{1}$$

If $a = \tau^3(a) > \tau(a) > \tau^2(a)$, then putting $I_1 = (\tau(a), a)$ and $I_2 = (\tau^2(a), \tau(a))$ we have the same M.

Now let M be a $d \times d$ structure matrix and N the set of all natural numbers. Define $\Omega = \{\omega = (\omega_1, \omega_2, \ldots);\ m_{\omega_n \omega_{n+1}} = 1 \text{ for all } n \geqslant 1\} \subset \{1, 2, \ldots, d\}^N$. Endowed with the product of discrete topology, $\{1, 2, \ldots, d\}^N$ is a compact metrizable space and Ω is its closed subset. Let T be the shift of Ω defined by $(T\omega)_n = \omega_{n+1}$, $n \geqslant 1$. The pair (Ω, T) is called a *Markov subshift*. Let $P = (p_{ij})$ be a $d \times d$ non-negative matrix such that

(i) $\sum_j p_{ij} = 1$ for all i and (ii) P is consistent with M, i.e. $p_{ij} > 0$ iff $m_{ij} = 1$. Because of the irreducibility of P and the above property (i), Perron–Frobenius Theorem (cf. [3]) implies the existence of a positive row vector $\pi = (\pi_1, \pi_2, \ldots, \pi_d)$ (i.e. $\pi_i > 0$ for all i) such that $\sum_i \pi_i = 1$ and $\sum_i \pi_i p_{ij} = \pi_j$ for all j. Then there exists a T-invariant ergodic probability measure μ such that

$$\mu(\{\omega \in \Omega;\ \omega_k = i_0, \ldots, \omega_{k+n} = i_n\}) = \pi_{i_0} p_{i_0 i_1} \ldots p_{i_{n-1} i_n}$$

for all $i_0, \ldots, i_n \in \{1, \ldots, d\}$, $n \geqslant 0$ and $k \geqslant 1$. This μ is called a *Markov measure*, P a *transition matrix* and π the *stationary measure* for P.

There is a special T-invariant Markov measure as follows. By Perron–Frobenius Theorem, there are positive right (column) eigenvector $(\xi_1, \ldots, \xi_d)$ and positive left (row) eigenvector $(\eta_1, \ldots, \eta_d)$ associated with the largest eigenvalue $\lambda \geqslant 1$ of M. Then a Markov measure μ_0 is defined by

$$p_{ij} = \frac{m_{ij}\xi_j}{\lambda\xi_i}, \quad 1 \leqslant i, j \leqslant d,$$

and

$$\pi_i = \frac{\xi_i \eta_i}{\sum_k \xi_k \eta_k}, \quad 1 \leqslant i \leqslant d.$$

It is proved by Parry [10] that

$$h(T, \mu_0) = \log\lambda > h(T, \mu)$$

for any T-invariant probability measure $\mu \neq \mu_0$, where $h(T, \mu)$ denotes the metrical entropy of T with respect to μ.

Throughout the sequel we mean by measure a Borel probability measure.

3. Invariant measures. Let τ be a pseudo-Markov transformation on an interval I with sub-intervals $I_1, \ldots, I_d$ and the structure matrix $M = (m_{ij})$. In order to prove the existence of an invariant measure of τ, we will construct a Borel injection from the Markov subshift defined by M to I.

First we will define inductively open intervals $I_{i_1 \ldots i_n} = (a_{i_1 \ldots i_n}, b_{i_1 \ldots i_n})$ or $\emptyset$, $1 \leqslant i_k \leqslant d$, $1 \leqslant k \leqslant n$, $n \geqslant 1$, with the following properties:

$$\tau(I_{i_1 \ldots i_n}) = I_{i_2 \ldots i_n}, \tag{2}$$

$$I_{i_1 \ldots i_n} \subset I_{i_1 \ldots i_{n-1}}, \tag{3}$$

$$I_{i_1 \ldots i_n} = \emptyset \quad \text{iff} \quad m_{i_k i_{k+1}} = 0 \quad \text{for some } 1 \leqslant k \leqslant n-1, \tag{4}$$

and hence

$$(5) \qquad I_{i_1\dots i_n} \cap I_{i'_1\dots i'_n} = \emptyset \quad \text{if} \quad (i_1, \dots, i_n) \neq (i'_1, \dots, i'_n),$$

$$(6) \qquad I_{i_1\dots i_n} \subset I_{i_1} \cap \tau^{-1}(I_{i_2\dots i_n}).$$

Putting $I_{ij} = \emptyset$ if $m_{ij} = 0$, we have intervals $\{I_i\}$, $\{I_{ij}\}$ satisfying (2)–(6). Suppose that intervals $\{I_{i_1\dots i_n}\}$ are already defined up to n. Put $I_{i_1\dots i_{n+1}} = \emptyset$ if $m_{i_k i_{k+1}} = 0$ for some $1 \leqslant k \leqslant n$. If $m_{i_k i_{k+1}} = 1$ for all $1 \leqslant k \leqslant n$, we define $E^a_{i_1\dots i_{n+1}} = \{x \in \bar{I}_{i_1\dots i_n};\ \tau_{i_1 i_2}(x) = a_{i_2\dots i_{n+1}}\}$ and $E^b_{i_1\dots i_{n+1}} = \{x \in \bar{I}_{i_1\dots i_n};\ \tau_{i_1 i_2}(x) = b_{i_2\dots i_{n+1}}\}$. Since $I_{i_1\dots i_n} \neq \emptyset$, $I_{i_2\dots i_{n+1}} \neq \emptyset$ and $\tau(I_{i_1\dots i_n}) \supset I_{i_2\dots i_{n+1}}$, we have $E^a_{i_1\dots i_{n+1}} \neq \emptyset$ and $E^b_{i_1\dots i_{n+1}} \neq \emptyset$. Hence there exists an open interval $I_{i_1\dots i_{n+1}} = (a_{i_1\dots i_{n+1}}, b_{i_1\dots i_{n+1}}) \neq \emptyset$ satisfying properties (2), (3) and so (5), (6) (cf. [7]).

Now let (Ω, T) be a Markov subshift defined by M (cf. Section 2). Put $\varphi(\omega) = \bigcap_{n=1}^{\infty} I_{\omega_1\dots\omega_n} \subset I$ for $\omega \in \Omega$ and $\Omega_0 = \{\omega \in \Omega;\ \varphi(\omega) \text{ is a singleton}\}$. Then it is not hard to see that $\Omega \backslash \Omega_0$ is countable (cf. [7]) and $\varphi(\omega) \cap \varphi(\omega') = \emptyset$ if $\omega \neq \omega'$. We have

$$(7) \qquad T\Omega_0 \subset \Omega_0,$$

$$(8) \qquad \varphi(T(\omega)) = \tau(\varphi(\omega)) \quad \text{for } \omega \in \Omega_0.$$

Indeed for $\omega \in \Omega_0$, $\varphi(T(\omega)) = \bigcap_n I_{\omega_2\dots\omega_n} = \bigcap_n \tau(I_{\omega_1\dots\omega_n}) \supset \tau(\bigcap_n I_{\omega_1\dots\omega_n}) = \tau(\varphi(\omega))$. If $x \in \varphi(T(\omega))$ there is $x_n \in I_{\omega_1\dots\omega_n}$ such that $\tau(x_n) = x$ for all $n \geqslant 1$. By the definition x_n converges to $\varphi(\omega)$. Thus we have $x = \tau(\varphi(\omega))$ hence $\varphi(T(\omega)) = \tau(\varphi(\omega))$ and $T(\omega) \in \Omega_0$.

Since $\varphi\colon \Omega_0 \to I$ is a limit of Borel maps $\varphi_n(\omega) = \inf I_{\omega_1\dots\omega_n}$, φ is itself a Borel injection which satisfies (8). Therefore for any T-invariant continuous measure μ on Ω, the image measure $m = \mu\circ\varphi^{-1}$ is a τ-invariant continuous measure on I. If μ is an ergodic T-invariant Markov measure, then m is an ergodic τ-invariant Markov measure, under which $\{I_1 \cap \varphi(\Omega_0), \dots, I_d \cap \varphi(\Omega_0)\}$ is a Markov generator for τ.

Thus appealing the argument in Section 2, we have

THEOREM. *For any pseudo-Markov transformation τ with a structure matrix M, there exists an ergodic Markov invariant measure m such that $h(\tau, m) = \log\lambda$, where $\lambda \geqslant 1$ is the largest eigenvalue of M.*

COROLLARY 1. *If there exists $n \geqslant 1$ such that $M^n = (m^{(n)}_{ij}) > 0$ (i.e. $m^{(n)}_{ij} > 0$ for all i and j) for the structure matrix M of a pseudo-Markov transformation τ, then for any ergodic τ-invariant Markov measure m, the dynamical system (τ, m) is exact in the sense of Rohlin* [11].

Indeed, letting $m = \mu\circ\varphi^{-1}$ we are enough to show that (T, μ) is exact.

In this case (T, μ) is mixing (cf. [1]) and hence exact (in fact its natural extension is Bernoulli, cf. [5]).

Since the structure matrix (1) satisfies the condition of Corollary 1, we have

COROLLARY 2. *For any continuous map τ of an interval with an orbit of period three, there exists an ergodic Markov invariant measure m such that (τ, m) is exact and $h(\tau, m) = \log\{(1+\sqrt{5})/2\}$, where $(1+\sqrt{5})/2$ is the largest eigenvalue of the matrix* (1) *in Section* 2.

Since the topological entropy of a continuous map of a compact metric space bounds its metrical entropy for any invariant measure (cf. [4]), we have

COROLLARY 3. *For any continuous map of a compact interval with an orbit of period three, its topological entropy is not less than* $\log\{(1+\sqrt{5})/2\}$.

4. Examples. In this section we continue to use the same notation as in the preceding sections. The simplest example of a continuous map with an orbit of period three is as follows.

Let $I = [0, 1]$ and

$$\tau(x) = \begin{cases} \dfrac{1-q}{q}x+q, & 0 \leqslant x \leqslant q, \\ \dfrac{1}{1-q}(1-x), & q \leqslant x \leqslant 1, \end{cases}$$

where $0 < q < 1$. For any $0 < p < 1$, the matrix

$$P_p = \begin{pmatrix} 0 & 1 \\ p & 1-p \end{pmatrix}$$

is a transition matrix consistent with the structure matrix of τ which is given by (1) in Section 2. Its stationary measure is $\pi_p = (p/(1+p), 1/(1+p))$. In this case $I \setminus \varphi(\Omega_0)$ is countable where the Borel injection φ is defined in Section 3.

Let μ_p be the Markov measure defined by P_p and π_p on the space Ω of the Markov subshift, and $m_p = \mu_p \circ \varphi^{-1}$ on I. Then Birkoff ergodic theorem implies that m_p and $m_{p'}$ are mutually singular if $p \neq p'$. Thus we have

$$h(\tau, m_p) < h(\tau, m_{p_0}) = \log\{(1+\sqrt{5})/2\}$$

for any $p \neq p_0$, where $p_0 = ((1+\sqrt{5})/2)^{-2} = (3-\sqrt{5})/2$. It is easy to see that m_q is the unique ergodic τ-invariant measure equivalent to Lebesgue measure. Therefore except the case $q = p_0$, the ergodic τ-invariant measure equivalent to Lebesgue measure does not attain the maximum of the metrical entropies.

Let us now consider a continuous map τ of an interval with an orbit of period $d > 3$. In general τ is not pseudo-Markov because of the lack of irreducibility of the structure matrix. But if τ has an orbit of odd period, then it is shown to be pseudo-Markov as follows.

Assume that the continuous map τ has an orbit ω of odd period $d = 2k+1 \geqslant 5$ and it has no orbit of odd period d', $3 \leqslant d' < d$. Štefan [12] investigated such a map. It is proved that numbering $\omega = \{a_1 < a_2 < \ldots < a_d\}$ (or $\omega = \{a_1 > a_2 > \ldots > a_d\}$) we have

$$\tau(a_1) = a_{k+1}, \quad \tau(a_i) = \begin{cases} a_{2k+3-i} & 2 \leqslant i \leqslant k+1, \\ a_{2k+2-i}, & k+2 \leqslant i \leqslant 2k+1. \end{cases}$$

Define $2k$ intervals $I_{2i-1} = (a_{k+1}, a_{k+i+1})$, $I_{2i} = (a_{k+1-i,k+2-i})$, $1 \leqslant i \leqslant k$. Then we obtain the $2k \times 2k$ structure matrix

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 1 & 0 & & 1 & 0 \end{bmatrix}$$

with the characteristic polynomial $f(\lambda) = (\lambda^d - 2\lambda^{d-2} - 1)(\lambda+1)^{-1}$. It is seen that the largest eigenvalue λ of M is bigger than $\sqrt{2}$ (cf. [12]). It is also easily seen that $M^{4k} > 0$. Thus we obtain an ergodic τ-invariant Markov measure m such that (τ, m) is exact and $h(\tau, m) = \log\lambda > (\log 2)/2$.

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