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OPTIMUM CHECKING OF REPLACEABLE SYSTEMS UNDER RESTRICTIONS OF THE LOSS COST PER UNIT TIME

1. Introduction. In this paper we consider a model whose principles have been already described in [1], [2], [4], [5]: At time $t = 0$ a system starts working. The time to its failure (lifetime) is a random variable X with the cumulative distribution function (cdf) $F(t)$, $F(+0) = 0$. System failures are known only by inspection. Each inspection entails a fixed cost c_1, $0 < c_1 < \infty$, but takes only negligible time. On the other hand, a downtime t of the system (i.e., time between system failure and the starting point of the new system) gives rise to cost $c_2 t$, $0 \leqslant c_2 < \infty$. After detection of a failure the system is immediately replaced by a new one with the same cdf of lifetime as the former one. The average replacement cost is denoted by c_3 ($c_1 \leqslant c_3 < \infty$) and the average replacement time by d ($0 \leqslant d < \infty$). The time between two neighbouring replacements is called a *cycle*.

Let $S = \{t_k\}$ be an inspection strategy, i.e., an unbounded increasing sequence of numbers such that $0 = t_0 < t_1 < \ldots$ (at time t_k the k-th inspection takes place if no failure has been detected before) and let σ be the set of all inspection strategies. Using $S = \{t_k\}$, the expected length $L(S)$ of a cycle is given by

$$L(S) = \mu + \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} (t_{k+1} - t)\, dF(t) + d = \sum_{k=0}^{\infty} t_{k+1}\big(F(t_{k+1}) - F(t_k)\big) + d,$$

where $\mu = \mathrm{E}(X) < \infty$. Consequently, the long-run availability $A(S)$ of the system is

$$A(S) = \frac{\mu}{\sum\limits_{k=0}^{\infty} t_{k+1}\big(F(t_{k+1}) - F(t_k)\big) + d}.$$

An availability of the system arbitrarily close to its supremum $\mu/(\mu + d)$ can be secured by using strategies $S = \{t_k\}$ such that $\sup \delta_k$

($\delta_k = t_{k+1} - t_k$) is sufficiently small. But in this case the expected (total) loss cost $K(S)$ per unit time would tend to infinity. Then we have

$$K(S) = \frac{C(S)}{L(S)},$$

where

$$C(S) = \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} [(k+1)c_1 + c_2(t_{k+1} - t)]\,dF(t) + c_2 d + c_3$$

is the mean total loss cost per cycle. Hence in this paper we consider the problem of achieving the maximum availability under a restriction of the expected loss cost per unit time by application of a proper inspection strategy. We obtain results in the cases of full and partial information on the lifetime distribution of the system.

The problem of maximizing the system availability under restrictions of the expected total loss cost per cycle has been already considered in [6]. System availability without cost restrictions has been investigated in [8].

Throughout Sections 1-3 we assume

$$\frac{c_1 + c_3}{\mu} < c_2. \tag{1}$$

Otherwise, the expected loss per unit time by "ideal inspection and renewal" (i.e. failure of the system is detected immediately, and replacement occurs in negligible time) would be larger than or equal to the cost per unit downtime of the system. But then inspection and replaces ment of the system are uneconomical.

The following lemma (Lagrange multiplier method) will be used in Section 2.

LEMMA 1. *Let $M(S)$ and $N(S)$ be real-valued functions of S, $S \in \sigma$, and let $D(S, \lambda) = M(S) + \lambda N(S)$, where λ is a real number. For every $\lambda \geqslant 0$ there exists a strategy $S(\lambda) \in \sigma$ such that*

$$D\big(S(\lambda), \lambda\big) = \min_{S \in \sigma} D(S, \lambda).$$

If there exists a $\lambda_0 \in \Lambda = \{\lambda;\ S(\lambda) \in \sigma_0\}$, $\sigma_0 = \{S;\ N(S) = 0\}$, such that

$$D\big(S(\lambda_0),\ \lambda_0\big) = \min_{\lambda \in \Lambda} D\big(S(\lambda), \lambda\big),$$

then

$$M\big(S(\lambda_0)\big) = \min_{S \in \sigma_0} M(S).$$

Proof. If S_0 satisfies $M(S_0) = \min_{S\in\sigma_0} M(S)$, then for any $\lambda \in \Lambda$ we have

$$M(S_0) = \min_{S\in\sigma_0} D(S,\lambda) \geqslant \min_{\lambda\in\Lambda}\min_{S\in\sigma} D(S,\lambda) = \min_{\lambda\in\Lambda} D\big(S(\lambda),\lambda\big)$$
$$= D\big(S(\lambda_0),\lambda_0\big) = M\big(S(\lambda_0)\big).$$

Thus $S_0 = S(\lambda_0)$ and the lemma is proved.

2. Exponentially distributed lifetime. Let $F(t) = 1-e^{-t/\mu}$, $t \geqslant 0$. From [5] we know that under this assumption the optimum strategy S^* with respect to $K(S)$ is strictly periodic. (A strategy $S = \{t_k\}$ is called *strictly periodic with the inspection interval* δ if $\delta = t_{k+1}-t_k$ $(k = 0, 1, \ldots)$. In this case we write $S = S^{(\delta)}$.) In particular, we have

$$K(S^{(\delta)}) = \frac{c_1+c_2\delta-(c_2\mu-c_3)(1-e^{-\delta/\mu})}{\delta+d(1-e^{-\delta/\mu})}.$$

Hence the inspection interval δ^* of S^* satisfies

$$1-e^{-\delta/\mu}\left\{\frac{\delta}{\mu}+1-\frac{c_1 d}{\big(c_3-(\mu+d)c_2\big)\mu}\right\} = \frac{c_1}{(\mu+d)c_2-c_3}.$$

In view of (1) there always exists a solution.

Let K_0 be a fixed boundary for $K(S)$,

$$K(S^*) < K_0 < c_2. \tag{2}$$

Next our aim is to find a strategy $S = S_0$ such that

$$A(S_0) = \max_{S\in\sigma_0} A(S) \tag{3}$$

with $\sigma_0 = \{S;\ K(S) = K_0\}$. We write this problem in the form

$$L(S) \to \min, \quad C(S)-K_0L(S) = 0. \tag{4}$$

Note that with respect to maximization of $A(S)$ there arises no additional profit if σ_0 is substituted by $\{S;\ K(S) \leqslant K_0\}$.

According to Lemma 1 let us introduce

$$D(S,\lambda) = L(S)+\lambda\big(C(S)-K_0L(S)\big).$$

Since the functional structure of $D(S,\lambda)$ is the same as that of $C(S)$ we know from [5] (where the problem of minimizing $C(S)$ has been solved) that for every $\lambda \geqslant 0$ the strategy $S(\lambda)$ is strictly periodic. Hence the strategy $S_0 = S(\lambda_0)$, which is a solution of (3), is also strictly periodic. Moreover, the inspection interval δ_0 of S_0 must satisfy the equation

$C(S^{(\delta)}) - K_0 L(S^{(\delta)}) = 0$ or

$$e^{-\delta/\mu} + \frac{c_2 - K_0}{\mu c_2 - (c_2 - K_0)d - c_3} = 1 + \frac{c_1}{\mu c_2 - (c_2 - K_0)d - c_3}.$$

By (2) there exist exactly two positive solutions. Since

$$L(S^{(\delta)}) = \frac{\delta}{1 - e^{-\delta/\mu}} + d$$

is increasing in δ, δ_0 is the smallest one.

3. Known expectation of lifetime. We now consider the case where the probability distribution of the system lifetime X is unknown but its mathematical expectation is $\mu = \mathrm{E}(X)$, i.e. we know only that the cdf of X belongs to the set $\mathscr{F}_\mu$ of those cdf's F which have the properties

$$F(+0) = 0 \quad \text{and} \quad \mu = \int_0^\infty \bar{F}(t)\,dt, \quad 0 < \mu < \infty.$$

Our aim is to make use of this partial information on the lifetime distribution with respect to proper scheduling of inspections.

Let us now write more exactly $A(S, F)$, $C(S, F)$, $L(S, F)$, and $K(S, F)$ instead of $A(S)$, $C(S)$, $L(S)$, and $K(S)$, respectively, and let us put

$$A_\mu(S) = \inf_{F \in \mathscr{F}_\mu} A(S, F), \qquad C_\mu(S) = \sup_{F \in \mathscr{F}_\mu} C(S, F),$$
$$L_\mu(S) = \sup_{F \in \mathscr{F}_\mu} L(S, F), \qquad K_\mu(S) = \sup_{F \in \mathscr{F}_\mu} K(S, F).$$

Let σ_p be the set of all strictly periodic inspection strategies and let $\sigma_0 = \{S \in \sigma_p;\ K_\mu(S) = K_0\}$. The problem considered here consists in finding a strategy $S_\mu \in \sigma_0$ such that

(5) $$A_\mu(S_\mu) = \max_{S \in \sigma_0} A_\mu(S).$$

Analogously to (4) we write this problem in the equivalent form

$$L_\mu(S) \to \min, \quad K_\mu(S) - K_0 = 0.$$

K_0 is assumed to satisfy

(6) $$K_\mu(S_\mu^*) < K_0 < c_2,$$

where S_μ^* is the optimum strategy with respect to $K_\mu(S)$. According to [5], S_μ^* is strictly periodic with the inspection interval

$$\delta_\mu^* = \sqrt{c_1 \mu / (c_2 - K_\mu(S_\mu^*))},$$

where

$$(7)\quad K_\mu(S_\mu^*) = \frac{1}{\mu+d}\left[c_1+c_2d+c_3-\frac{2\mu c_1}{\mu+d}+\right.$$
$$\left.+2\sqrt{\frac{\mu c_1}{\mu+d}\left\{c_2\mu-c_1-c_3+\frac{\mu c_1}{\mu+d}\right\}}\,\right].$$

For any $F\in\mathscr{F}_\mu$ we have

$$C(S^{(\delta)},F) = \sum_{k=0}^{\infty}\int_{k\delta}^{(k+1)\delta}\big[(k+1)c_1+c_2\big((k+1)\delta-t\big)\big]dF(t)+c_2d+c_3$$
$$\leqslant \sum_{k=0}^{\infty}\int_{k\delta}^{(k+1)\delta}[(k+1)c_1+c_2\delta]dF(t)+c_2d+c_3$$
$$= \frac{c_1}{\delta}\sum_{k=0}^{\infty}(k\delta)\big[F\big((k+1)\delta\big)-F(k\delta)\big]+c_2(\delta+d)+c_3$$
$$\leqslant \frac{c_1}{\delta}\mu+c_2(\delta+d)+c_1+c_3.$$

This estimation is sharp. Therefore,

$$(8)\qquad C_\mu(S^{(\delta)}) = \frac{c_1}{\delta}\,\delta+c_2(\delta+d)+c_1+c_3,$$

$$(9)\qquad L_\mu(S^{(\delta)}) = \mu+\delta+d.$$

Evidently, the condition $K_\mu(S)-K_0=0$ is equivalent to

$$(10)\qquad N_\mu(S) = \sup_{F\in\mathscr{F}_\mu}\big(C(S,F)-K_0L(S,F)\big) = 0.$$

By (6) the functional structure of $C(S,F)-K_0L(S,F)$ is the same as that of $C(S,F)$. Thus, (8) and (10) imply

$$K_\mu(S^{(\delta)}) = \frac{\mu c_1/\delta+(\delta+d)c_2+c_1+c_3}{\mu+\delta+d}.$$

The condition $K_\mu(S^{(\delta)})=K_0$ leads to the quadratic equation

$$\delta^2-\frac{(\mu+d)K_0-c_1-c_2d-c_3}{c_2-K_0}\,\delta+\frac{\mu c_1}{c_2-K_0}=0.$$

By (6) and (7) there exist exactly two positive solutions. Taking into account (9) we infer that the inspection interval δ_μ of the strategy

$S_\mu = S^{(\delta_\mu)}$, being a solution of problem (5), is the smallest one:

$$\delta_\mu = \frac{1}{2(c_2 - K_0)} \Big[(\mu + d)K_0 - c_1 - c_2 d - c_3 - \\ - \sqrt{\big((\mu+d)K_0 - c_1 - c_2 d - c_3\big)^2 - 4\mu c_1 (c_2 - K_0)}\Big]$$

(using (7) we easily see from (6) that the radicand is positive).

Example 1. Let $\mu = 200$, $c_1 = 10$, $c_2 = 5$, $c_3 = 100$, and $d = 0$. Fig. 1 shows the availabilities $A(S_0)$, $A_\mu(S_\mu)$, and $A(S_\mu)$ (dependent on K_0) if $F(t) = 1 - e^{-t/\mu}$, $t \geqslant 0$. It should be noted that $K(S^*) = 1.154$ and $K_\mu(S_\mu^*) = 1.399$.

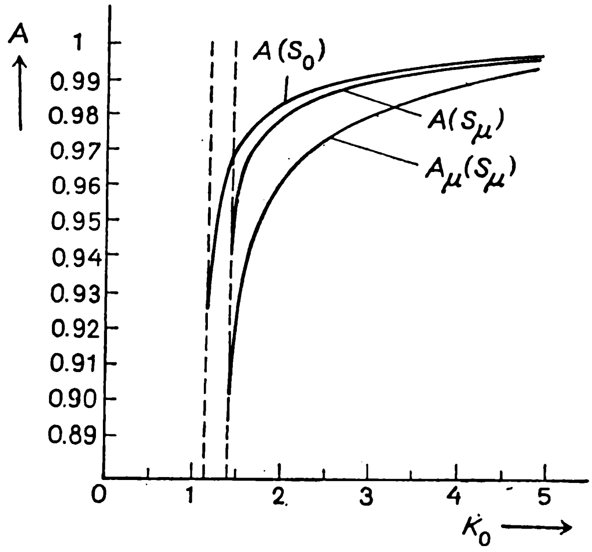

Fig. 1. Comparison of availability criterions in case of $c_3 > 0$ (Example 1)

If the underlying lifetime distribution of the system is of the exponential type and we do not know this fact, then by application of the inspection interval δ_μ we would actually get the system availability $A(S_\mu)$. Hence in Fig. 1 this function is also plotted.

4. Expected inspection and replacement cost per unit time. In many cases (e.g. military defensive weapons) the downtime cost per unit time is negligible small. However, the system has to be available for certain emergency situations (see [3], [4]). Then one can solve problems (3) and (5) in case of $c_2 = 0$ if conditions (2) and (6), respectively, are replaced by

(11) $$0 < K_0 < c_3/d.$$

As in Section 2 we conclude that S_0 is a strictly periodic strategy. However, λ is now, without loss of generality, restricted by

$$0 \leqslant \lambda < 1/K_0. \tag{12}$$

Otherwise, the case $S(\lambda)$ would be the trivial strategy "no inspection and replacement at all". But a solution S_0 of problem (3) cannot be of this type, since the only admissible value of K_0 were then $K_0 = 0$, which contradicts (11).

The inspection interval δ_0 of S_0 is the unique solution of the equation

$$e^{-\delta/\mu} + \frac{K_0}{c_3 - K_0 d}\,\delta = 1 + \frac{c_1}{c_3 - K_0 d}.$$

It is easy to see that the corresponding λ_0-value ($S_0 = S(\lambda_0)$) satisfies (12).

Let us now consider problem (5) for $c_2 = 0$. Unfortunately, to compute $N_\mu(S^{(\delta)})$ as defined by (10) we cannot directly use the previous results for $C(S, F)$ and $C(S, F) - K_0 L(S, F)$ which have now different functional structures. In what follows we assume for obvious reasons that

$$0 < c_1/\delta < K_0. \tag{13}$$

LEMMA 2. *We have*

$$N_\mu(S^{(\delta)}) = \begin{cases} \mu c_1/\delta + c_3 - K_0(\mu + d) & \text{if } 0 < \delta \leqslant \mu, \\ c_1 + c_3 - K_0(\delta + d) & \text{if } \delta > \mu. \end{cases}$$

Proof. Within a more general model, Hoeffding [7] has shown that

$$\sup_{F \in \mathscr{F}_\mu} N(S^{(\delta)}, F) = \sup_{F \in \mathscr{F}_\mu^{(2)}} N(S^{(\delta)}, F), \tag{14}$$

where $\mathscr{F}_\mu^{(2)}$ is the set of all those elements of $\mathscr{F}_\mu$ which have exactly two points of increase. If $F \in \mathscr{F}_\mu^{(2)}$, then there exist two numbers u_0 and u_1, $0 < u_0 < \mu < u_1 < \infty$, such that

$$P(X = u_0) = \frac{u_1 - \mu}{u_1 - u_0} \quad \text{and} \quad P(X = u_1) = \frac{\mu - u_0}{u_1 - u_0}.$$

For fixed $S^{(\delta)} = \{t_k\}$, $t_k = k\delta$, let $t_r < u_0 \leqslant t_{r+1}$ and $t_s < u_1 \leqslant t_{s+1}$. Using these F and $S^{(\delta)}$ we have

$$C(S^{(\delta)}, F) - K_0 L(S^{(\delta)}, F) = G(u_0, u_1; t_{r+1}, t_{s+1}) + c_3 - K_0(\mu + d), \tag{15}$$

where

$$G(u_0, u_1;\ t_{r+1}, t_{s+1}) = \frac{u_1-\mu}{u_1-u_0}\,[(r+1)c_1 - K_0(t_{r+1}-u_0)] + \\ + \frac{\mu-u_0}{u_1-u_0}\,[(s+1)c_1 - K_0(t_{s+1}-u_1)].$$

Let the integer m for $S = S^{(\delta)}$ be defined by $t_m < \mu \leqslant t_{m+1}$. To prove the lemma for the case $0 < \delta \leqslant \mu$, we have to consider two possibilities:

1. $0 \leqslant r < m$.

In view of (13), by straightforward estimations, we get

$$G(u_0, u_1;\ t_{r+1}, t_{s+1}) \leqslant G(t_{r+1}, t_{s+1};\ t_{r+1}, t_{s+1}) = \frac{\mu}{\delta}c_1. \tag{16}$$

2. $0 \leqslant r = m$.

In this case we have

$$G(u_0, u_1;\ t_{r+1}, t_{s+1}) \leqslant G(\mu, t_{s+1};\ t_{r+1}, t_{s+1}) \\ = (r+1)c_1 - K_0(t_{r+1}-\mu) \leqslant \frac{\mu}{\delta}c_1. \tag{17}$$

Inequality (16) is sharp. Combining (14)-(17) we prove the first part of the lemma. The second part ($\delta > \mu$) can be proved analogously.

Lemma 2 yields

$$K_\mu(S^{(\delta)}) = \begin{cases} \dfrac{\mu c_1/\delta + c_3}{\mu + d} & \text{if } 0 < \delta \leqslant \mu, \\ \dfrac{c_1 + c_3}{\delta + d} & \text{if } \mu < \delta. \end{cases}$$

Thus we have

$$\delta_\mu = \begin{cases} \dfrac{c_1 + c_3 - K_0 d}{K_0} & \text{if } 0 < K_0 < \dfrac{c_1+c_3}{\mu+d}, \\ \dfrac{\mu c_1}{(\mu+d)K_0 - c_3} & \text{if } \dfrac{c_1+c_3}{\mu+d} \leqslant K_0. \end{cases}$$

Example 2. As in Example 1 let $\mu = 200$, $c_1 = 10$, $c_3 = 100$, and $d = 0$. Fig. 2 shows the availabilities $A(S_0)$, $A_\mu(S_\mu)$, and $A(S_\mu)$ dependent on K_0 ($F(t) = 1 - e^{-t/\mu}$, $t \geqslant 0$). In this case there exist no nontrivial inspection strategies S^* and S_μ^*. Indeed, we have $\delta^* = \delta_\mu^* = \infty$ and $K(S^*) = K_\mu(S_\mu^*) = 0$.

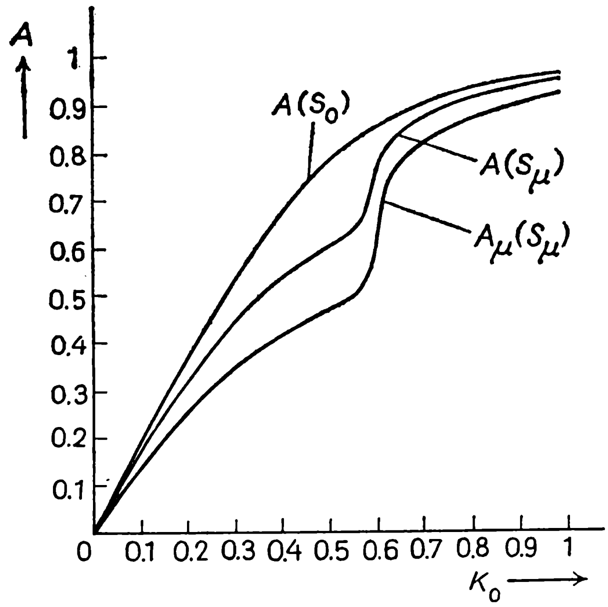

Fig. 2. Comparison of availability criterions in case of $c_2 = 0$ (Example 2)

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OPTYMALNA INSPEKCJA SYSTEMÓW Z ODNOWĄ PRZY OGRANICZENIACH NAŁOŻONYCH NA KOSZT JEDNOSTKOWY

STRESZCZENIE

W pracy rozpatruje się system o znanym rozkładzie czasu życia. Awarie systemu można stwierdzić tylko przez inspekcję i usunąć przez odnowę systemu. Inspekcja jest natychmiastowa, ale ma ustalony koszt, mniejszy od kosztu odnowy. Z drugiej strony, koszt awarii systemu jest proporcjonalny do czasu jej trwania. Podane są optymalne strategie inspekcji, pozwalające osiągnąć maksymalną niezawodność systemu. Wyniki otrzymano dla przypadku pełnej i niepełnej informacji o rozkładzie czasu życia systemu.