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Received on 2. 10. 1981
and in revised form on 14. 10. 1982 (1270)

ACTA ARITHMETICA
XLIII (1984)

Oscillatory properties of $M(x) = \sum_{n\le x}\mu(n)$, III

by

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1. In part I [6] of this series we proved that if $\zeta(\varrho_0) = \zeta(\beta_0+i\gamma_0) = 0$, then for $Y > e^{|\gamma_0|+4}$

$$(1.1)\qquad \max_{Y/(100\log Y)\le x\le Y} |M(x)| \ge \frac{1}{Y}\int_{Y/(100\log Y)}^{Y} |M(x)|\,dx \ge \frac{Y^{\beta_0}}{6\,|\varrho_0|^3}.$$

This implies by easy calculation that for $Y > 2$

$$(1.2)\qquad \max_{Y/(100\log Y)\le x\le Y} |M(x)| \ge \frac{1}{Y}\int_{Y/(100\log Y)}^{Y} |M(x)|\,dx > \frac{\sqrt{Y}}{17000}.$$

In part II [7] we showed that $M(x)$ changes sign in every interval of the form

$$(1.3)\qquad [Y\exp(-3\log_2^{3/2} Y),\ Y]$$

for $Y > c_1$, where $\log_\nu Y$ denotes the ν times iterated logarithmic function, and $c_1, c_2, \ldots$ denote explicitly calculable positive absolute constants. Concerning these problems, it is natural to ask how large are the oscillations of $M(x)$ in positive and negative directions and what kind of estimates can be proved for $\max_{x\le Y} M(x)$ and $\min_{x\le Y} M(x)$.

The first results in this field are due to S. Knapowski. By the application of Turán's method he proved in [4] that the Riemann hypothesis implies for $Y > c_2$ the inequality

$$(1.4)\qquad \max_{x\le Y} M(x) \ge \max_{A(Y)\le x\le Y} M(x) \ge \sqrt{Y}\exp\left(-15\frac{\log Y}{\log_2 Y}\log_3 Y\right)$$

and the corresponding inequality for $\min_{x\le Y} M(x)$, where

$$(1.5)\qquad A(Y) = Y\exp\left(-c_3\frac{\log Y}{\log_2 Y}\log_3 Y\right).$$

In 1965–68 I. Kátai [1], [2], [3] proved somewhat weaker results but without making use of any unproved conjecture. He showed [3] that for $Y > c_4$

$$\max_{Y^{0.36} \leqslant x \leqslant Y} \frac{M(x)}{\sqrt{x}} > c_5, \qquad \min_{Y^{0.36} \leqslant x \leqslant Y} \frac{M(x)}{\sqrt{x}} < -c_5. \tag{1.6}$$

This implies for $\max\limits_{x \leqslant Y} M(x)$ the effective lower bound $c_5 Y^{0.18}$ but he also proved the ineffective estimate $Y^{\theta-\varepsilon}$ [2], where θ denotes the least upper bound of the real parts of the zeta-zeros. Under assumption of the Riemann hypothesis he improved [1] the estimate (1.4) of Knapowski to $\sqrt{Y}\exp(-c_6 \log_2^2 Y)$ for $Y > c_7$.

2. A refinement of the method of proving (1.3) (cf. [7]) enables us to show the following effective theorem.

THEOREM. *If*

$$\zeta(\varrho_1) = \zeta(\beta_1 + i\gamma_1) = 0 \ (\beta_1 \geqslant 1/2) \quad \textit{and} \quad Y > \max\big(\exp(2|\gamma_1|), c_8\big),$$

then there exist

$$x', x'' \in [Y\exp(-5\log_2^{3/2} Y), Y] \tag{2.1}$$

such that

$$M(x') > \frac{(x')^{\beta_1}}{48|\varrho_1|^3} \tag{2.2}$$

and

$$M(x'') < -\frac{(x'')^{\beta_1}}{48|\varrho_1|^3}. \tag{2.3}$$

Choosing $\varrho_1 = 1/2 + i\cdot 14.13\ldots$, the first zero above the real axis, we get

COROLLARY 1. *For $Y > c_9$ there exist*

$$x', x'' \in [Y\exp(-5\log_2^{3/2} Y), Y] \tag{2.4}$$

such that

$$M(x') > \frac{\sqrt{x'}}{136000}, \qquad M(x'') < -\frac{\sqrt{x''}}{136000}. \tag{2.5}$$

These results naturally yield effective estimates for $\max\limits_{x \leqslant Y} M(x)$ and $\min\limits_{x \leqslant Y} M(x)$. Namely, we have

COROLLARY 2. *If* $\zeta(\varrho_1) = \zeta(\beta_1 + i\gamma_1) = 0$ $(\beta_1 \geqslant 1/2)$ *and* $Y > \max\big(\exp(2|\gamma_1|), c_{10}\big)$, *then*

$$\max_{A(Y) \leqslant x \leqslant Y} M(x) > Y^{\beta_1}\exp(-6\log_2^{3/2} Y) \tag{2.6}$$

and the corresponding inequality holds for $\min\limits_{A(Y) \leqslant x \leqslant Y} M(x)$ *too, where*

$$A(Y) = Y\exp(-5\log_2^{3/2} Y). \tag{2.7}$$

COROLLARY 3. *For $Y > c_{11}$ one has*

$$\max_{A(Y) \leqslant x \leqslant Y} M(x) > \sqrt{Y}\exp(-3\log_2^{3/2} Y) \tag{2.8}$$

and the corresponding inequality holds for $\min\limits_{A(Y) \leqslant x \leqslant Y} M(x)$ *too, where $A(Y)$ is given in* (2.7).

3. To prove our theorem we shall need two lemmas.

LEMMA 1. *Let $L \geqslant 1000$ be an integer, $\lambda \geqslant 10$, $R \geqslant 1$ and u real numbers; further, let us define*

$$\mathscr{F}_{L,\lambda,R}(u) \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{(2)} \left(\frac{e^{Rs} - e^{-Rs}}{2Rs}\right)^L e^{us + (Rs)^2/\lambda}\, ds. \tag{3.1}$$

Then for every u

$$\mathscr{F}_{L,\lambda,R}(u) \geqslant 0; \tag{3.2}$$

for $|u| \leqslant R\sqrt{L/6}$ we have

$$\mathscr{F}_{L,\lambda,R}(u) > 1/4R\sqrt{L}, \tag{3.3}$$

further, for $|u| \geqslant R(L+2)$,

$$\mathscr{F}_{L,\lambda,R}(u) \leqslant \exp\left(-\lambda\left(\frac{|u|}{R} - L - 1\right)\right). \tag{3.4}$$

This is a slightly sharpened form of Lemma 1 in [7]. Since (3.2) and (3.4) were proved in Lemma 2.1 of [5], it is sufficient to show (3.3). This will be proved in the Appendix.

LEMMA 2. *If $\zeta(s)$ has no zero in the domain*

$$\sigma > \sigma_0, \quad |t| \leqslant \lambda + 1, \tag{3.5}$$

where $\lambda > c_{12}$, then for

$$\sigma \geqslant \sigma_0 + 2/\lambda, \quad 10 \leqslant |t| \leqslant \lambda \tag{3.6}$$

the estimate

$$\left|\frac{1}{\zeta(s)}\right| \leqslant \lambda^{c_{13}} |t|^{\log\lambda} \tag{3.7}$$

holds.

This lemma is proved in [7]. Let

$$L = [\log(100\log Y)]+1, \tag{3.8}$$

$$R = \sqrt{6L}, \tag{3.9}$$

and

$$\lambda = \log Y - R(L+4). \tag{3.10}$$

If there exists a ζ-zero in the domain

$$\sigma > \beta_1 + 4L/\lambda, \quad |t| \leqslant \lambda+1, \tag{3.11}$$

then let $\varrho_0 = \beta_0 + i\gamma_0$ be any zero with maximal real part among those satisfying (3.11).

If the domain (3.11) is free of zeros of $\zeta(s)$, then let

$$\varrho_0 \overset{\text{def}}{=} \varrho_1. \tag{3.12}$$

If $\varrho_0 \neq \varrho_1$, i.e. ϱ_0 is contained in (3.11), then for x in (2.1)

$$x^{\beta_0-\beta_1} > (e^{(3/4)\lambda})^{4L/\lambda} = e^{3L} > \lambda^3 > (|\varrho_0|/|\varrho_1|)^3, \tag{3.13}$$

that is,

$$\frac{x^{\beta_0}}{|\varrho_0|^3} > \frac{x^{\beta_1}}{|\varrho_1|^3}. \tag{3.14}$$

Thus in view of (3.8)–(3.10) and (3.12)–(3.14) it is sufficient to show the existence of

$$x', x'' \in [e^{\lambda-R(L+4)}, e^{\lambda+R(L+4)}] \tag{3.15}$$

such that the inequalities

$$M(x') > \frac{(x')^{\beta_0}}{48|\varrho_0|^3}, \tag{3.16}$$

$$M(x'') < -\frac{(x'')^{\beta_0}}{48|\varrho_0|^3} \tag{3.17}$$

hold.

4. Since the proofs for (3.16) and (3.17) are nearly the same, we shall only prove (3.17). Let us suppose contrary to (3.17) that

$$f(x) \overset{\text{def}}{=} M(x) + \frac{x^{\beta_0}}{48|\varrho_0|^3} \geqslant 0 \tag{4.1}$$

in the whole interval (3.15). In this case Lemma 1 gives for

$$U \overset{\text{def}}{=} \int\limits_{\exp(\lambda-L)}^{\exp(\lambda+L)} \frac{f(x)}{x^{1+\beta_0}}\,dx \tag{4.2}$$

the upper estimate

$$\begin{aligned} U &\leqslant 4R\sqrt{L} \int\limits_{\exp(\lambda-L)}^{\exp(\lambda+L)} \frac{f(x)}{x^{1+\beta_0}} \mathscr{F}_{L,\lambda,R}(\lambda-\log x)\,dx \\ &\leqslant 4\sqrt{6}L \int\limits_{\exp(\lambda-R(L+4))}^{\exp(\lambda+R(L+4))} \frac{f(x)}{x^{1+\beta_0}} \mathscr{F}_{L,\lambda,R}(\lambda-\log x)\,dx \\ &= 4\sqrt{6}L \int\limits_1^\infty \frac{f(x)}{x^{1+\beta_0}} \mathscr{F}_{L,\lambda,R}(\lambda-\log x)\,dx + \\ &\quad + O\left(L \int\limits_1^{\exp(\lambda-R(L+4))} \exp\left(-\lambda\left(\frac{\lambda-\log x}{R}-L-1\right)\right)dx\right) + \\ &\quad + O\left(L \int\limits_{\exp(\lambda+R(L+4))}^{\infty} \exp\left(-\lambda\left(\frac{\log x-\lambda}{R}-L-1\right)\right)dx\right).^{(1)} \end{aligned} \tag{4.3}$$

Introducing the definition

$$V \overset{\text{def}}{=} \int\limits_1^\infty \frac{f(x)}{x^{1+\beta_0}} \cdot \frac{1}{2\pi i} \int\limits_{(2)} \left(\frac{e^{Rs}-e^{-Rs}}{2Rs}\right)^L e^{(Rs)^2/\lambda} \cdot \frac{e^{\lambda s}}{x^s}\,ds\,dx, \tag{4.4}$$

(4.3) gives by easy calculation

$$U \leqslant 4\sqrt{6}LV + O(e^{-\lambda}). \tag{4.5}$$

Considering

$$\begin{aligned} \int\limits_1^\infty \frac{f(x)}{x^{s+\beta_0+1}}\,dx &= \int\limits_1^\infty \frac{M(x)}{x^{s+\beta_0+1}}\,dx + \frac{1}{48|\varrho_0|^3} \int\limits_1^\infty \frac{1}{x^{s+1}}\,dx \\ &= \frac{1}{(s+\beta_0)\zeta(s+\beta_0)} + \frac{1}{48|\varrho_0|^3 s} \quad (\sigma > 1), \end{aligned} \tag{4.6}$$

and interchanging the order of integrations we get

$$\begin{aligned} V &= \frac{1}{2\pi i} \int\limits_{(2)} \left(\frac{e^{Rs}-e^{-Rs}}{2Rs}\right)^L e^{(Rs)^2/\lambda+\lambda s} \int\limits_1^\infty \frac{f(x)}{x^{s+\beta_0+1}}\,dx\,ds \\ &= V' + \frac{1}{48|\varrho_0|^3} V'', \end{aligned} \tag{4.7}$$

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(1) Writing the last equality sign we used $|f(x)| \leqslant 2x$ and (3.4) which imply the convergence of the infinite integral.

where

$$V' = \frac{1}{2\pi i}\int\limits_{(2)}\left(\frac{e^{Rs}-e^{-Rs}}{2Rs}\right)^L e^{(Rs)^2/\lambda+\lambda s}\frac{1}{(s+\beta_0)\zeta(s+\beta_0)}\,ds, \tag{4.8}$$

$$V'' = \frac{1}{2\pi i}\int\limits_{(2)}\left(\frac{e^{Rs}-e^{-Rs}}{2Rs}\right)^L e^{(Rs)^2/\lambda+\lambda s}\frac{1}{s}\,ds. \tag{4.9}$$

It is easy to evaluate V''; by shifting the path of integration onto $\sigma = -2$ we have by (3.8)–(3.10)

$$V'' = 1+\frac{1}{2\pi i}\int\limits_{(-2)}\left(\frac{e^{Rs}-e^{-Rs}}{2Rs}\right)^L e^{(Rs)^2/\lambda+\lambda s}\frac{1}{s}\,ds = 1+O(e^{-\lambda}). \tag{4.10}$$

To give an upper bound for $|V'|$ we transform the path of integration onto the broken line Γ defined for $t\geqslant 0$ by

$$\begin{array}{lll} I_1:\ \sigma = 2 & \text{for} & t\geqslant\lambda,\\ I_2:\ 5L/\lambda\leqslant\sigma\leqslant 2 & \text{for} & t=\lambda,\\ I_3:\ \sigma = 5L/\lambda & \text{for} & 10\leqslant t\leqslant\lambda,\\ I_4:\ -1/4\leqslant\sigma\leqslant 5L/\lambda & \text{for} & t=10,\\ I_5:\ \sigma=-1/4 & \text{for} & 0\leqslant t\leqslant 10, \end{array} \tag{4.11}$$

and for $t\leqslant 0$ by reflection on the real axis. Taking into account the definition of ϱ_0 in (3.11) and (3.12), there are no zeros of $(s+\beta_0)\zeta(s+\beta_0)$ right of Γ and on Γ, so we have

$$V' = \frac{1}{2\pi i}\int\limits_{\Gamma}\left(\frac{e^{Rs}-e^{-Rs}}{2Rs}\right)^L e^{(Rs)^2/\lambda+\lambda s}\frac{1}{(s+\beta_0)\zeta(s+\beta_0)}\,ds. \tag{4.12}$$

Further in view of (3.11) and (3.12) the conditions of Lemma 2 are satisfied for

$$\sigma_0 = \beta_0+4L/\lambda, \tag{4.13}$$

so considering (3.8)–(3.10), Lemma 2 gives

$$\left|\frac{1}{(s+\beta_0)\zeta(s+\beta_0)}\right| \leqslant \lambda^{c_{13}}t^{\log\lambda-1}\leqslant\lambda^{c_{13}}t^{L-2}\quad\text{for}\quad s\in I_2\cup I_3. \tag{4.14}$$

Since $((s+\beta_0)\zeta(s+\beta_0))^{-1}$ is bounded by an absolute constant for $s\in I_1\cup\cup I_4\cup I_5$ we have the following estimates for the integrals V'_ν on I_ν $(1\leqslant\nu\leqslant 5)$:

$$|V'_1| \ll \exp\left(2RL+\frac{R^2(4-\lambda^2)}{\lambda}+2\lambda\right)\int\limits_{\lambda}^{\infty}\frac{1}{t^L}\,dt \ll \exp\left(-\frac{R^2\lambda}{2}\right),$$

$$\begin{gathered} |V'_2| \ll \exp\left(2RL+\frac{R^2(4-\lambda^2)}{\lambda}+2\lambda\right)\frac{\lambda^{c_{13}}\lambda^{L-2}}{\lambda^L} \ll \exp\left(-\frac{R^2\lambda}{2}\right),\\ |V'_3| \ll \int\limits_{10}^{\lambda}\left(\frac{2e^{\frac{5L}{\lambda}R}}{2Rt}\right)^L e^{5L}\lambda^{c_{13}}t^{L-2}\,dt \ll \lambda^{c_{13}}\left(\frac{e^5}{R}\right)^L,\\ |V'_4| \ll \max_{-1/4\leqslant\sigma\leqslant 5L/\lambda}\left(\frac{(2e^{R|\sigma|})^L e^{\lambda\sigma}}{(20R)^L}\right) \ll \left(\frac{e^5}{10R}\right)^L,\\ |V'_5| \ll \left(\frac{2e^{R/4}}{2R\cdot(1/4)}\right)^L e^{-\lambda/4} \ll e^{-\lambda/5}, \end{gathered} \tag{4.15}$$

that is,

$$V' \ll \lambda^{c_{13}}\left(\frac{e^5}{R}\right)^L \ll \left(\frac{e^{5+c_{13}}}{\sqrt{6L}}\right)^L \ll L^{-L/3}, \tag{4.16}$$

and so by (4.5), (4.7), (4.10) and $|\varrho_0|<\lambda+2<e^L$

$$U < \frac{10L}{48|\varrho_0|^3}. \tag{4.17}$$

5. Now, we shall give a lower estimate for the integral U (defined in (4.2)) with the aid of (1.1). Let

$$H(y) = \int\limits_{e^{\lambda-L}}^{y}|M(x)|\,dx. \tag{5.1}$$

For $y\geqslant e^\lambda$ (1.1) implies

$$H(y) > \frac{y^{1+\beta_0}}{6|\varrho_0|^3}. \tag{5.2}$$

Thus we get

$$\begin{aligned} U' &\overset{\text{def}}{=} \int\limits_{e^{\lambda-L}}^{e^{\lambda+L}}\frac{|M(x)|}{x^{1+\beta_0}}\,dx\\ &= \frac{H(e^{\lambda+L})}{(e^{\lambda+L})^{1+\beta_0}}+(1+\beta_0)\int\limits_{e^{\lambda-L}}^{e^{\lambda+L}}\frac{H(x)}{x^{2+\beta_0}}\,dx\\ &\geqslant (1+\beta_0)\int\limits_{e^{\lambda}}^{e^{\lambda+L}}\frac{1}{6|\varrho_0|^3}\cdot\frac{x^{1+\beta_0}}{x^{2+\beta_0}}\,dx = \frac{1+\beta_0}{6|\varrho_0|^3}L. \end{aligned} \tag{5.3}$$

So in view of (4.1) and (4.2)

$$U \geqslant U' - \frac{1}{48|\varrho_0|^3}\int_{e^{\lambda-L}}^{e^{\lambda+L}} \frac{x^{\beta_0}}{x^{1+\beta_0}}dx \tag{5.4}$$

$$\geqslant \frac{3/2}{6|\varrho_0|^3}L - \frac{1}{48|\varrho_0|^3}\cdot 2L = \frac{5}{24|\varrho_0|^3}L,$$

which contradicts (4.17) and thus proves the theorem.

Appendix

In order to prove (3.3) first we remark that since the function $\mathscr{F}_{L,\lambda,R}(u)$ is decreasing for $u \geqslant 0$ (this is proved in Lemma 2.1 of [5]), it is sufficient to prove (3.3) for $u = R\sqrt{L/6}$. Transforming the path of integration for $\sigma = 0$, we get

$$\mathscr{F}_{L,\lambda,R}(R\sqrt{L/6}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin Rt}{Rt}\right)^L \cos(R\sqrt{L/6}\,t)e^{-R^2t^2/\lambda}dt$$

$$= \frac{\sqrt{6}}{R\sqrt{L}}\cdot\frac{1}{\pi}\int_0^{\infty}\left(\frac{\sin(y/\sqrt{L/6})}{y/\sqrt{L/6}}\right)^L \cos y\cdot e^{-6y^2/(\lambda L)}dy$$

$$= \frac{\sqrt{6}}{\pi R\sqrt{L}}\left\{\int_0^1 + \int_1^{\pi/2} + \int_{\pi/2}^{\pi-1} + \int_{\pi-1}^{\sqrt{L/6}} + \int_{\sqrt{L/6}}^{\infty}\right\}$$

$$\stackrel{\text{def}}{=} \frac{\sqrt{6}}{\pi R\sqrt{L}}\{\mathscr{F}_1+\mathscr{F}_2+\mathscr{F}_3+\mathscr{F}_4+\mathscr{F}_5\}.$$

In view of the inequality

$$0.99e^{-x^2/6} \leqslant \frac{\sin x}{x} \leqslant e^{-x^2/6} \quad \text{for} \quad |x| \leqslant 1,$$

we have

$$\mathscr{F}_1 \geqslant 0.99\int_0^1 e^{-y^2}\cdot\frac{1}{2}dy > \frac{0.99}{2}\left(\frac{\sqrt{\pi}}{2} - \frac{1}{2e}\right),$$

$$|\mathscr{F}_4| \leqslant \int_{\pi-1}^{\sqrt{L/6}} e^{-y^2}dy < \frac{1}{2(\pi-1)}e^{-(\pi-1)^2}.$$

Since for $0 < x < \pi/2$

$$\left(\frac{\sin((\pi/2-x)/\sqrt{L/6})}{(\pi/2-x)/\sqrt{L/6}}\right)^L e^{-\frac{6(\pi/2-x)^2}{\lambda L}} > \left(\frac{\sin((\pi/2+x)/\sqrt{L/6})}{(\pi/2+x)/\sqrt{L/6}}\right)^L e^{\frac{-6(\pi/2+x)^2}{\lambda L}},$$

we have

$$\mathscr{F}_2 > -\mathscr{F}_3 > 0.$$

Further estimating $\mathscr{F}_5$ trivially, we get

$$|\mathscr{F}_5| < \int_{\sqrt{L/6}}^{\infty}\left(\frac{y}{\sqrt{L/6}}\right)^{-L}dy = \frac{\sqrt{L/6}}{L-1}. \ \blacksquare$$

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Received on 26. 10. 1981
and in revised form on 9. 8. 1982 (1276)