Families of convex sets closed under intersections, homotheties and uniting increasing sequences of sets

by

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Abstract. Modifying a condition in a characterization of the family of all convex sets of Euclidean space, we study some variants of generalized convexity.

The considerations of this paper concern the n-dimensional Euclidean space R^n. Let e denote the metric of R^n. Our terminology follows [11], [13] and [10]. Particularly, we use the definitions, symbols and properties of convex half-spaces given in [10].

Analogically to the proof of Theorem 12 in [8], we can prove the following characterization of classic convexity in R^n (the characterization is also a special case of the first part of Theorem 26).

The family of all convex sets in R^n is identical with the smallest family $\mathscr{C}$ of sets which contains the Euclidean unit ball $B_e = \{x;\ e(x, 0) \leqslant 1\}$ and fulfils the following conditions:

(M) $\mathscr{C}$ is multiplicative (i.e. closed under arbitrary intersections),

(U) $\mathscr{C}$ is closed under uniting increasing sequences of sets,

(H) $\mathscr{C}$ is closed under homotheties J_a^λ with arbitrary centres a and positive coefficients λ.

Modifying the condition "$\mathscr{C}$ is the smallest family which contains B_e", we study some variants of generalized convexity.

In the first part we consider properties of an arbitrary family of convex sets fulfilling conditions (M), (U) and (H). In the second part $\mathscr{C}$ is the smallest family which contains a given family of convex half-spaces and fulfils conditions (M), (U) and (H). In the last part we discuss the following reinforcement of condition (U):

(U*) $\mathscr{C}$ is closed under uniting families of sets linearly ordered by inclusion.

Here are examples of families of convex sets fulfilling conditions (M), (U) and (H).

EXAMPLE 1. The family of d-convex sets. A set $D \subset R^n$ is said to be d-convex (where d denotes the metric induced by a fixed norm in R^n) if for any $a \in D$, $b \in R^n$, $c \in D$, the equality $d(a, b)+d(b, c) = d(a, c)$ implies $b \in D$. The bibliography and properties of d-convexity are presented in [1].

EXAMPLE 2. The family of B-convex sets. A set is called B-convex if, containing any finite number of points, it also contains the intersection of all closed balls (in the sense of a fixed norm in R^n) containing those points [8, 9]. At the end of this paper we shall show that the family of B-convex sets is identical with the smallest family of sets which contains the unit ball B and fulfils conditions (M), (U) and (H).

1. Arbitrary families of convex sets satisfying conditions (M), (U) and (H). Let $\mathscr{C}$ denote in this section an arbitrary family of convex sets of R^n which fulfils conditions (M), (U) and (H). The sets from the family $\mathscr{C}$ will be called $\mathscr{C}$-convex.

The following theorem shows that the family $\mathscr{C}$ is closed under some operations (see also part 1 of Theorem 4).

THEOREM 1. *If C is $\mathscr{C}$-convex, then the following sets are $\mathscr{C}$-convex: any translate of C, the closure $\bar{C}$, the affine hull* aff C, *the relative interior* ri C, *the interior* int C, $C-L=\{c-l;\ c\in C,\ l\in L\}$, *where L is an arbitrary direction of recession*(1) *of C. If the sets C_i, $i=1, 2, \ldots$, are $\mathscr{C}$-convex, then*(2) $\operatorname*{Lim\,inf}_{i\to\infty} C_i$ *and* $\operatorname*{Lim}_{i\to\infty} C_i$, *if it exists, are $\mathscr{C}$-convex.*

Proof. We first notice that the translation by a vector w is the composition $J_0^{1/2}\circ J_x^2$ of two homotheties, where $v_{0x}=-2w$. Therefore $\mathscr{C}$-convexity of the translate $C+w$ follows from (H).

We shall show the remaining properties for $C\neq\emptyset$ only, because for $\emptyset$ they are obvious.

Since the relative interior of any nonempty convex set is nonempty, we can assume that there exists a point $a\in\operatorname{ri} C$. Obviously, $J_a^\lambda(C)\in\mathscr{C}$ for $\lambda>0$. The equality $\bar{C}=\bigcap_{\lambda>1} J_a^\lambda(C)$ implies that $\bar{C}\in\mathscr{C}$. From $\operatorname{aff} C=\bigcup_{m=1}^{\infty} J_a^m(C)$ and $J_a^1(C)\subset J_a^2(C)\subset\ldots$, we get $\operatorname{aff} C\in\mathscr{C}$. Since $\operatorname{ri} C=\bigcup_{m=2}^{\infty} J_a^{1-1/m}(C)$ and $J_a^{1-1/2}(C)\subset J_a^{1-1/3}(C)\subset\ldots$, we have $\operatorname{ri} C\in\mathscr{C}$. Hence $\operatorname{int} C$ belongs to $\mathscr{C}$.

Let $L=\{\lambda b;\ \lambda\geqslant 0\}$, $b\neq 0$, be a direction of recession of C. Obviously, $C-L=\bigcup_{\lambda\geqslant 0}(C-\lambda b)$. Since the inclusion $C-\lambda_1 b\subset C-\lambda_2 b$ holds if and only if $\lambda_1\leqslant\lambda_2$, we have $C-L=\bigcup_{m=0}^{\infty}(C-mb)$. Hence $C-L$ is $\mathscr{C}$-convex as the union of the increasing sequence of sets $C-mb\in\mathscr{C}$, $m=0, 1, \ldots$

(1) I.e. L is such a half-line with the vertex 0 that $L+a\subset C$ for arbitrary $a\in C$ (see [11], p. 61).

(2) $\operatorname*{Lim\,inf}_{i\to\infty} C_i=\bigcup_{i=1}^{\infty}\bigcap_{j=i}^{\infty} C_j$, $\operatorname*{Lim\,sup}_{i\to\infty} C_i=\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} C_j$, if $\operatorname*{Lim\,inf}_{i\to\infty} C_i=C=\operatorname*{Lim\,sup}_{i\to\infty} C_i$, then $\operatorname*{Lim}_{i\to\infty} C_i=C$.

Since $C_i\in\mathscr{C}$, $i=1, 2, \ldots$, we have $\bigcap_{j=i}^{\infty} C_j\in\mathscr{C}$. From the inclusions $\bigcap_{j=i}^{\infty} C_j\subset\bigcap_{j=i+1}^{\infty} C_j$, $i=1, 2, \ldots$, we get $\bigcup_{i=1}^{\infty}\bigcap_{j=i}^{\infty} C_i\in\mathscr{C}$, i.e. that $\operatorname*{Lim\,inf}_{i\to\infty} C_i$ is $\mathscr{C}$-convex. If the limit $\operatorname*{Lim}_{i\to\infty} C_i$ exists, then $\operatorname*{Lim}_{i\to\infty} C_i=\operatorname*{Lim\,inf}_{i\to\infty} C_i\in\mathscr{C}$.

THEOREM 2. *The set R^n is $\mathscr{C}$-convex. The empty set $\emptyset$ is $\mathscr{C}$-convex (with only one trivial exception when $\mathscr{C}=\{R^n\}$). If there exists a bounded nonempty $\mathscr{C}$-convex set, then any one-point set is $\mathscr{C}$-convex.*

Proof. The set R^n is $\mathscr{C}$-convex as the intersection of the empty family of $\mathscr{C}$-convex subsets of R^n.

Let $\mathscr{C}\neq\{R^n\}$. Therefore a $\mathscr{C}$-convex set $A\neq R^n$ exists. Let $b\notin A$. For arbitrary $x\in R^n$ the translate $A+v_{bx}$ is $\mathscr{C}$-convex and $x\notin(A+v_{bx})$. Hence the set $\bigcap_{x\in R^n}(A+v_{bx})=\emptyset$ is $\mathscr{C}$-convex.

Let there exist a bounded nonempty $\mathscr{C}$-convex set C. Let $x\in\operatorname{ri} C$. Since

$$\{x\}=\bigcap_{0<\lambda<1} J_x^\lambda(C),$$

the one-point set $\{x\}$ is $\mathscr{C}$-convex. Consequently, any one-point set is $\mathscr{C}$-convex as a translate of $\{x\}$.

DEFINITION 1. We denote by $\mathscr{C}$-conv A the intersection of all sets from the family $\mathscr{C}$ which contain a given set A and we call it the *$\mathscr{C}$-hull of A*. Obviously, $\mathscr{C}$-conv A is the smallest $\mathscr{C}$-convex set containing A.

THEOREM 3. *The $\mathscr{C}$-hull of an arbitrary set $A\subset R^n$ has the following properties (where $x\in R^n$ and $\lambda>0$):*

$$J_x^\lambda(\mathscr{C}\text{-conv } A)=\mathscr{C}\text{-conv } J_x^\lambda(A),\qquad \overline{\mathscr{C}\text{-conv } A}\supset\mathscr{C}\text{-conv } \bar{A},$$

$$\operatorname{aff}\mathscr{C}\text{-conv } A\supset\mathscr{C}\text{-conv aff } A,\qquad \operatorname{int}\mathscr{C}\text{-conv } A\supset\mathscr{C}\text{-conv int } A,$$

if A is an open set, then the set $\mathscr{C}$-conv A is also open.

Proof. The inclusion $\mathscr{C}\text{-conv} A\supset A$ implies the ones $J_x^\lambda(\mathscr{C}\text{-conv} A)\supset J_x^\lambda(A)$, $\overline{\mathscr{C}\text{-conv } A}\supset\bar{A}$, $\operatorname{aff}\mathscr{C}\text{-conv } A\supset\operatorname{aff} A$, $\operatorname{int}\mathscr{C}\text{-conv } A\supset\operatorname{int} A$. By (H) and Theorem 1 the left sides of the inclusions are $\mathscr{C}$-convex. So $J_x^\lambda(\mathscr{C}\text{-conv} A)\supset\mathscr{C}\text{-conv} J_x^\lambda(A)$, $\overline{\mathscr{C}\text{-conv} A}\supset\mathscr{C}\text{-conv}\bar{A}$, $\operatorname{aff}\mathscr{C}\text{-conv} A\supset\mathscr{C}\text{-conv} A$, and $\operatorname{int}\mathscr{C}\text{-conv } A\supset\mathscr{C}\text{-conv int } A$.

Now, we shall show the inclusion $J_x^\lambda(\mathscr{C}\text{-conv } A)\subset\mathscr{C}\text{-conv } J_x^\lambda(A)$. Obviously, $J_x^\lambda(A)\subset\mathscr{C}\text{-conv } J_x^\lambda(A)$. So

$$A=J_x^{1/\lambda}(J_x^\lambda(A))\subset J_x^{1/\lambda}(\mathscr{C}\text{-conv } J_x^\lambda(A)).$$

The right side of the inclusion is $\mathscr{C}$-convex by condition (H). Hence $\mathscr{C}\text{-conv } A\subset J_x^{1/\lambda}(\mathscr{C}\text{-conv } J_x^\lambda(A))$. Therefore

$$J_x^\lambda(\mathscr{C}\text{-conv } A)\subset J_x^\lambda\big(J_x^{1/\lambda}(\cancel{\mathscr{C}\text{-conv}}\ J_x^\lambda(A))\big)=\mathscr{C}\text{-conv } J_x^\lambda(A).$$

If A is open, then from $\operatorname{int}\mathscr{C}\text{-conv}\, A \supset \mathscr{C}\text{-conv}\, \operatorname{int} A$ and from $A = \operatorname{int} A$ we get $\operatorname{int}\mathscr{C}\text{-conv}\, A \supset \mathscr{C}\text{-conv}\, A$. Hence $\mathscr{C}\text{-conv}\, A$ is open.

The proof is complete.

The intersection of all cones with the (included or excluded) vertex x which contain a given set A is called the *induced cone of A with the vertex x* and is denoted by $\operatorname{cone}_x A$. Obviously, $\operatorname{cone}_x A$ is the smallest cone with the vertex x which contains A and

$$\operatorname{cone}_x A = \bigcup_{y\in A} \{x+\lambda(y-x) \text{ for all } \lambda > 0\}.$$

Obviously, $x\in\operatorname{cone}_x A$ if and only if $x\in A$.

DEFINITION 2. If a cone is $\mathscr{C}$-convex, then we call it a *$\mathscr{C}$-cone*. We denote by $\mathscr{C}\text{-cone}_x A$ the intersection of all $\mathscr{C}$-cones with the vertex x which contain a given set $A\subset R^n$ and we call it the induced $\mathscr{C}$-cone of A with the vertex x.

Obviously, $\mathscr{C}\text{-cone}_x A$ is the smallest $\mathscr{C}$-cone with the vertex x which contains A.

THEOREM 4. *Here are some properties of $\mathscr{C}$-cones:*

1. *If C is $\mathscr{C}$-convex and if $C\cup\{x\}$ is convex (particularly: if $x\in C$), then the induced cone $\operatorname{cone}_x C$ is $\mathscr{C}$-convex.*

2. *If $\{x\}\cup\mathscr{C}\text{-conv}\, A$ is convex (particularly: if $x\in\mathscr{C}\text{-conv}\, A$), then $\operatorname{cone}_x \mathscr{C}\text{-conv}\, A \supset \mathscr{C}\text{-conv}\, \operatorname{cone}_x A$.*

3. *If K is a cone with a vertex x, then $\mathscr{C}\text{-conv}\, K$ is a $\mathscr{C}$-cone with the vertex x.*

4. *The equality $\mathscr{C}\text{-cone}_x A = \mathscr{C}\text{-conv}\, \operatorname{cone}_x A$ holds for arbitrary $A\subset R^n$ and $x\in R^n$.*

Proof. 1. From $x\in C\cup\{x\}$ and from the convexity of $C\cup\{x\}$ we get

$$J_x^1(C\cup\{x\}) \subset J_x^2(C\cup\{x\}) \subset \ldots$$

Since $J_x^m(\{x\}) = \{x\}$, we have

$$J_x^m(C\cup\{x\}) = J_x^m(C)\cup\{x\}, \quad m = 1, 2, \ldots$$

Hence $J_x^1(C)\subset J_x^2(C)\subset\ldots$ because $x\in J_x^m(C)$ if and only if $x\in C$, $m = 1, 2, \ldots$ Since $C\in\mathscr{C}$, we have $J_x^m(C)\in\mathscr{C}$, $m = 1, 2, \ldots$ Thus by the previous inclusions we have that $\bigcup_{m=1}^{\infty} J_x^m(C)$ is $\mathscr{C}$-convex.

Now, we shall show that $\operatorname{cone}_x C \subset \bigcup_{m=1}^{\infty} J_x^m(C)$. Let $u\in\operatorname{cone}_x C$. If $u = x$, then $u\in\operatorname{cone}_u C$ implies $u\in C\subset \bigcup_{m=1}^{\infty} J_x^m(C)$. Let $u\neq x$. There are $y\in C$ and $\lambda > 0$ such that $u = x+\lambda(y-x)$. Let $k\geqslant\lambda$ be a natural number. Obviously, $x+k(y-x)\in J_x^k(C)$. Since $J_x^k(C)\cup\{x\} = J_x^k(C\cup\{x\})$ is convex,

$$\{x+\gamma(y-x);\ 0\leqslant\gamma\leqslant k\} \subset J_x^k(C)\cup\{x\}.$$

Since $u = x+\lambda(y-x)$ and $0<\lambda\leqslant k$, we have $u\in J_x^k(C)$. Hence $u\in \bigcup_{m=1}^{\infty} J_x^m(C)$.

Consequently, $\operatorname{cone}_x C\subset \bigcup_{m=1}^{\infty} J_x^m(C)$.

The inverse inclusion is obvious. Therefore

$$\operatorname{cone}_x C = \bigcup_{m=1}^{\infty} J_x^m(C)$$

and consequently $\operatorname{cone}_x C$ is $\mathscr{C}$-convex.

2. By the first part of the proof the set $\operatorname{cone}_x \mathscr{C}\text{-conv}\, A$ is $\mathscr{C}$-convex. Hence the inclusion $\operatorname{cone}_x \mathscr{C}\text{-conv}\, A \supset \operatorname{cone}_x A$ implies that $\operatorname{cone}_x \mathscr{C}\text{-conv}\, A \supset \mathscr{C}\text{-conv}\, \operatorname{cone}_x A$.

3. If K is a cone with a vertex x, then $J_x^\lambda(K) = K$ for any $\lambda > 0$. Hence by Theorem 3 we get

$$J_x^\lambda(\mathscr{C}\text{-conv}\, K) = \mathscr{C}\text{-conv}\, J_x^\lambda(K) = \mathscr{C}\text{-conv}\, K$$

for any $\lambda > 0$. Therefore $\mathscr{C}\text{-conv}\, K$ is a $\mathscr{C}$-cone with the vertex x.

4. By the previous part of the proof $\mathscr{C}\text{-conv}\, \operatorname{cone}_x A$ is a $\mathscr{C}$-cone with the vertex x. Hence $\mathscr{C}\text{-cone}_x \operatorname{cone}_x A \subset \mathscr{C}\text{-conv}\, \operatorname{cone}_x A$ and consequently

$$\begin{aligned}\mathscr{C}\text{-cone}_x A &\subset \mathscr{C}\text{-cone}_x \operatorname{cone}_x A \subset \mathscr{C}\text{-conv}\, \operatorname{cone}_x A \\ &\subset \mathscr{C}\text{-conv}\, \mathscr{C}\text{-cone}_x A = \mathscr{C}\text{-cone}_x A.\end{aligned}$$

Therefore $\mathscr{C}\text{-cone}_x A = \mathscr{C}\text{-conv}\, \operatorname{cone}_x A$.

LEMMA 1. *For any convex set $A\subset R^n$ we have*

$$A = \bigcap \{\operatorname{cone}_x A;\ x\in R^n \text{ and the set } A\cup\{x\} \text{ is convex}\}.$$

Proof. In the proof we denote by $[a, b]$ the closed segment joining the points a, b and by (a, b) the open segment joining a and b.

We can assume $A\neq\emptyset$ and $A\neq R^n$ because for the sets $\emptyset$ and R^n the lemma is obvious. It is sufficient to show that for arbitrary $y\notin A$ there is an $x\in R^n$ such that $A\cup\{x\}$ is convex and that $y\notin\operatorname{cone}_x A$.

Case 1. The set $A\cup\{y\}$ is convex. We put $y = x$. Since $y = x\notin A$, we have $y\notin\operatorname{cone}_x A$. Obviously, $A\cup\{x\}$ is convex.

Case 2. There exists $x\in A$ such that $(y, x)\cap A = \emptyset$. Such a point $x\in A$ will be called visible from $y\notin A$. Since x is visible from y and since A is convex, we have $y\notin\operatorname{cone}_x A$. Moreover, $A\cup\{x\}$ is convex because $x\in A$.

Case 3. Cases 1 and 2 do not hold. In other words: $A\cup\{y\}$ is not convex and no point of the set A is visible from y. Since $A\cup\{y\}$ is not convex, and since A is convex, there exists $z\in A$ such that $(z, y)\setminus A\neq\emptyset$. Since A is convex and no point of A is visible from y, for a point $x\in(z, y)$ we have $(x, z)\subset A$ and $\{x+\lambda(y-x);\ \lambda\geqslant 0\}\cap A = \emptyset$. Hence $y\notin\operatorname{cone}_x A$.

At the end we shall show that $A\cup\{x\}$ is convex. By the convexity of A it is sufficient to show the inclusion $(v, x)\subset A$ for any $v\in A$.

If v lies on the line L through x and z, then $(v, x) \subset [v, z] \cup (z, x) \subset A$.

Now, let $v \notin L$. Since v is not visible from y, thus $(y, v) \cap A \neq \emptyset$. Let $t \in (y, v) \cap A$. Obviously, $[t, v] \subset A$. The construction implies that there exists a point $w \in (z, t) \cap (x, v)$. From the inclusions $[v, w] \subset A$ and $(w, x) \subset \operatorname{conv}((x, z) \cup \{t\}) \subset A$ we get $(v, x) \subset A$.

Definition 3. A family $\mathscr{B} \subset \mathscr{C}$ will be called an *intersection basis of the family* $\mathscr{C}$ (or shortly: an *intersection $\mathscr{C}$-basis*) if any set from $\mathscr{C}$ is an intersection of sets from $\mathscr{B}$.

By Lemma 1 and by the first part of Theorem 4 we get

Theorem 5. *The family of all $\mathscr{C}$-cones is an intersection $\mathscr{C}$-basis.*

Let $W \subset R^n$ be a closed convex body. By a regular point of W we understand any point x of the boundary bd W such that there exists exactly one hyperplane supporting W at x. The closed half-space bounded by the hyperplane and containing W will be denoted by $W(x)$.

Theorem 6. *If x is a regular point of a $\mathscr{C}$-convex closed body W, then the closed half-space $W(x)$ is $\mathscr{C}$-convex.*

Proof. Obviously, $W(x) = \overline{\operatorname{cone}_x W}$. From Theorem 1 and the first part of Theorem 4 we infer that $W(x)$ is $\mathscr{C}$-convex.

Theorem 7. *Any $\mathscr{C}$-convex closed body W is an intersection of $\mathscr{C}$-convex closed half-spaces. More exactly: $W = \bigcap \{W(x);\ x$ is a regular point of $W\}$.*

Proof. If $W = R^n$, then W is the intersection of the empty family of closed half-spaces.

Let $W \neq R^n$. Let $z \notin W$. From $\operatorname{int} W \neq \emptyset$ we get $(\operatorname{bd} W) \cap J_z^\lambda(\operatorname{int} W) \neq \emptyset$ for a number λ, $0 < \lambda < 1$. Since $J_z^\lambda(\operatorname{int} W)$ is open, the set $(\operatorname{bd} W) \cap J_z^\lambda(\operatorname{int} W)$ is open in bd W, i.e. it is open in the sense of the induced topology in bd W. Since the set of regular points of W is dense in bd W ([2], p. 88), there exists a regular point $x \in (\operatorname{bd} W) \cap J_z(\operatorname{int} W)$. From $J_z^{1/\lambda}(x) \in \operatorname{int} W \subset \operatorname{int} W(x)$ and $1/\lambda > 1$ it follows that $z \notin W(x)$. Since $z \notin W$ was an arbitrary point, $W \supset \bigcap \{W(x);\ x$ is regular point of $W\}$. The inverse inclusion is obvious.

Theorem 8. *Any $\mathscr{C}$-convex closed set is the intersection of a family of $\mathscr{C}$-convex planes and $\mathscr{C}$-convex closed half-planes.*

Proof. Let A be a $\mathscr{C}$-convex closed set. If $A = \operatorname{aff} A$, then the theorem is obvious. Let $A \neq \operatorname{aff} A$. By Theorem 1 the set aff A is $\mathscr{C}$-convex. The family $\mathscr{C}_1$ of $\mathscr{C}$-convex subsets of the space aff A is a family of convex sets which fulfils the conditions (M), (U), and (H). Therefore we can apply Theorem 7 to the family $\mathscr{C}_1$. We find that A is an intersection of $\mathscr{C}_1$-convex (thus $\mathscr{C}$-convex) closed half-planes of the plane aff A.

In the next theorem we give an analogue of the classical separation theorem: two opposite closed half-spaces bounded by a separating hyperplane (in the classical theorem) are replaced by two closed $\mathscr{C}$-cones.

Theorem 9. *If for $\mathscr{C}$-convex sets C and D the equality $\operatorname{ri} C \cap \operatorname{ri} D = \emptyset$ holds, then there exist closed $\mathscr{C}$-cones $S \supset C$ and $T \supset D$ such that $\operatorname{ri} S \cap \operatorname{ri} T = \emptyset$. More exactly: there exist two $\mathscr{C}$-cones as above which have a common vertex (if $\operatorname{aff} C \cap \operatorname{aff} D \neq \emptyset$) or which are planes (if $\operatorname{aff} C \cap \operatorname{aff} D = \emptyset$).*

Proof. Since the closure of any $\mathscr{C}$-convex set is $\mathscr{C}$-convex (and $\operatorname{aff} \bar{G} = \operatorname{aff} G$), it is sufficient to consider only the case where C and D are closed.

If $\operatorname{aff} C \cap \operatorname{aff} D = \emptyset$, then we put $S = \operatorname{aff} C$ and $T = \operatorname{aff} D$. By Theorem 1 the planes S and T are $\mathscr{C}$-convex. So they are $\mathscr{C}$-cones.

If $\operatorname{aff} C \cap \operatorname{aff} D \neq \emptyset$, then we consider two cases. Put

$$e(C, D) = \inf \{e(c, d);\ c \in C \text{ and } d \in D\}.$$

Case 1; $e(C, D) = 0$. If a point $x \in C \cap D$ exists, then we take the cones $S = \overline{\operatorname{cone}_x C}$ and $T = \overline{\operatorname{cone}_x D}$. They are $\mathscr{C}$-convex, closed, have x as a common vertex, $S \supset C$ and $T \supset D$. Suppose $\operatorname{ri} S \cap \operatorname{ri} T \neq \emptyset$. Then there exists a ray

$$L = \{x + \lambda a;\ \lambda > 0\} \subset \operatorname{ri} S \cap \operatorname{ri} T.$$

Since $L \subset \operatorname{ri} S = \operatorname{ri} \operatorname{cone}_x C = \operatorname{cone}_x \operatorname{ri} C$, there exists a $\lambda_1 > 0$ such that

$$\{x + \lambda a;\ 0 < \lambda < \lambda_1\} \subset \operatorname{ri} C.$$

Analogously, there exists a $\lambda_2 > 0$ such that

$$\{x + \lambda a;\ 0 < \lambda < \lambda_2\} \subset \operatorname{ri} D.$$

Hence $\operatorname{ri} C \cap \operatorname{ri} D \neq \emptyset$. The contradiction with the assumptions implies that $\operatorname{ri} S \cap \operatorname{ri} T = \emptyset$.

If $C \cap D = \emptyset$, then we recurrently define the following sets. Let $C_0 = C$, $D_0 = D$. There exists (see Lemma 4 in [9]) a common direction L_1 of recession of C_0 and D_0. If $C_{k-1} \cap D_{k-1} = \emptyset$, then we put $C_k = \overline{C_{k-1} - L_k}$, $D_k = \overline{D_{k-1} - L_k}$, where L_k is a common direction of recession of C_{k-1} and D_{k-1} which is perpendicular to directions $L_1, \ldots, L_{k-1}$. We will show that such a direction exists. Let Π_{k-1} be the orthogonal complement of the smallest subspace which contains $L_1, \ldots, L_{k-1}$. Hence the sets $C_{k-1} \cap \Pi_{k-1}$ and $D_{k-1} \cap \Pi_{k-1}$ are closed, nonempty, disjoint and

$$e(C_{k-1} \cap \Pi_{k-1}, D_{k-1} \cap \Pi_{k-1}) = 0.$$

So they have a common direction L_k of recession (see Lemma 4 in [9]). Obviously, L_k is perpendicular to $L_1, \ldots, L_{k-1}$ and it is a common direction of recession of C_{k-1} and D_{k-1}.

For some number $m < n$ there exists a point $x \in C_m \cap D_m$. By Theorem 1 the sets C_m and D_m are $\mathscr{C}$-convex. Since L_m is a direction of recession of the two sets C_{m-1}, D_{m-1} and since $C_{m-1} \cap D_{m-1} = \emptyset$, we have

$$(C_{m-1} - L_m) \cap (D_{m-1} - L_m) = \emptyset.$$

Hence $\operatorname{ri} C_m \cap \operatorname{ri} D_m = \emptyset$. Moreover, C_m and D_m are closed. We have shown that the assumptions of the theorem and of the first part of Case 1 are fulfilled for the sets $C_m \supset C$ and $D_m \supset D$. Hence $S = \operatorname{cone}_x C_m$ and $T = \operatorname{cone}_x D_m$ are the required $\mathscr{C}$-cones.

Case 2; $e(C, D) > 0$. Let $c \in \operatorname{ri} C$. If a minimum $\lambda \geqslant 1$ exists such that $e(J_c^\lambda(C), D) = 0$, then the assumptions of the theorem and Case 1 hold for $J_c^\lambda(C)$ and D. Hence there exist closed $\mathscr{C}$-cones S and T with a common vertex such that $S \supset J_c^\lambda(C) \supset C$, $T \supset D$, $\operatorname{ri} S \cap \operatorname{ri} T \neq \emptyset$.

If $e(J_c^\lambda(C), D) > 0$ for any $\lambda \geqslant 1$, then $e(\operatorname{aff} C, D) > 0$. Let $d \in \operatorname{ri} D$. Since $\operatorname{aff} C \cap \operatorname{aff} D = \emptyset$, there exists a minimum $\gamma \geqslant 1$ such that $e(\operatorname{aff} C, J_d^\gamma(D)) = 0$. The assumptions of the theorem and Case 1 are fulfilled for $\operatorname{aff} C$ and $J_d^\gamma(D)$. So there exist closed $\mathscr{C}$-cones S and T with a common vertex such that $S \supset \operatorname{aff} C \supset C$, $T \supset J_d^\gamma(D) \supset D$, $\operatorname{ri} S \cap \operatorname{ri} T = \emptyset$.

THEOREM 10. *$\mathscr{C}$-convex convex half-spaces have the following properties:*

1. *If $B_x(v_1, \ldots, v_k) \in \mathscr{C}$, then $B_x(v_1, \ldots, v_i) \in \mathscr{C}$ and $P_x(v_1, \ldots, v_i) \in \mathscr{C}$, $i = 1, \ldots, k$.*

2. *If $P_x(v_1, \ldots, v_k) \in \mathscr{C}$, then $B_x(v_1, \ldots, v_i) \in \mathscr{C}$ and $P_x(v_1, \ldots, v_i) \in \mathscr{C}$, $i = 1, \ldots, k$.*

3. This is a special case of the preceding properties.

4. *The family of all $\mathscr{C}$-convex convex half-spaces is compact in the sense of the limit* Lim.

5. *The family of $\mathscr{C}$-convex blunt convex half-spaces with a fixed vertex x is compact in the sense of the limit* Lim. *This is also true in the cases where we replace the word "blunt" by "pointed" or if we omit the word.*

Proof. 1. Any translate of $\mathscr{C}$-convex set $B_x(v_1, \ldots, v_k)$ has the form $B_y(v_1, \ldots, v_k)$ and it is $\mathscr{C}$-convex. Consequently, $B_x(v_1, \ldots, v_i)$, where $1 \leqslant i \leqslant k$, is $\mathscr{C}$-convex as the union of the increasing sequence of $\mathscr{C}$-convex sets $B_{x_m}(v_1, \ldots, v_k)$, where $v_{xx_m} = (1/m)v_i$, $m = 1, 2, \ldots$ The set $P_x(v_1, \ldots, v_i)$, where $1 \leqslant i \leqslant k$, is $\mathscr{C}$-convex as the intersection of $\mathscr{C}$-convex sets $B_{y_m}(v_1, \ldots, v_k)$, where $v_{xy_m} = -(1/m)v_i$, $m = 1, 2, \ldots$

2. The argumentation is similar.

3. This is a special case of the preceding properties.

4. The conclusion holds for classic convexity (see Theorem 2 in [10]). By Theorem 1 the family of $\mathscr{C}$-convex sets is closed with respect to the limit Lim. These two facts imply that the family of $\mathscr{C}$-convex convex half-spaces is a closed subfamily of the family of all convex half-spaces. Hence the family of $\mathscr{C}$-convex convex half-spaces is compact as a closed subfamily of a compact family.

5. This property results from the preceding property and from Corollary 1 in [10].

Let us recall that a convex subset F of a convex set A is called a *face*

of A if, for any $a \in A$ and $b \in A$, from

$$\{(1-\alpha)a + \alpha b;\ 0 < \alpha < 1\} \cap F \neq \emptyset$$

it follows $a \in F$ and $b \in F$ (see [11], p. 162).

THEOREM 11. *The family $\mathscr{C}$ is closed under symmetries with centre 0 if and only if $\mathscr{C}$ is closed under homotheties J_a^λ with arbitrary coefficients $\lambda \neq 0$ and arbitrary centres a. For such family $\mathscr{C}$ some additional properties hold:*

1. *Any face of an arbitrary $\mathscr{C}$-convex set is $\mathscr{C}$-convex (with the exception of the trivial case $\mathscr{C} = \{R^n\}$).*

2. *The bounding $(k-1)$-dimensional plane of any $\mathscr{C}$-convex k-dimensional half-plane is $\mathscr{C}$-convex, $1 \leqslant k \leqslant n$.*

3. *The complement of an arbitrary $\mathscr{C}$-convex convex half-space is also $\mathscr{C}$-convex (with the exception of the trivial case $\mathscr{C} = \{R^n\}$).*

4. *The hyperplane supporting a $\mathscr{C}$-convex closed body at a regular point is $\mathscr{C}$-convex.*

Proof. Let $\mathscr{C}$ be closed under symmetries with centre 0. Moreover, by Theorem 1, the family $\mathscr{C}$ is closed under translations. Hence $\mathscr{C}$ is closed under symmetries with arbitrary centres $a \in R^n$. The homothety J_a^λ, where $\lambda < 0$, is the composition of the homothety $J_a^{-\lambda}$, where $-\lambda > 0$, and of the symmetry with the centre a. Hence by (H) we infer that $\mathscr{C}$ is closed under homotheties J_a^λ, where $\lambda \neq 0$.

The inverse implication is obvious.

1. Let F be a face of a $\mathscr{C}$-convex set K. If $F = \emptyset$, then by Theorem 2 the face F is $\mathscr{C}$-convex. Let $F \neq \emptyset$. Consequently, $\operatorname{ri} F \neq \emptyset$. Let $x \in \operatorname{ri} F$. First we shall show three equalities: $F \cap J_x^{-1}(F) = K \cap J_x^{-1}(K)$, $\operatorname{aff}(F \cap J_x^{-1}(F)) = \operatorname{aff} F$, $F = K \cap \operatorname{aff} F$.

Let $p \in K \cap J_x^{-1}(K)$. If $p = x$, then $p \in F \cap J_x^{-1}(F)$. Let $p \neq x$ and let r be the point symmetric to p with respect to x. Obviously, $r \in K$. Since F is a face of K and $p \in K$, $r \in K$, $x \in F$, we have $p \in F$ and $r \in F$. Hence $p \in F$ and $p \in J_x^{-1}(F)$. Therefore $F \cap J_x^{-1}(F) \supset K \cap J_x^{-1}(K)$. The inverse inclusion is obvious.

The sets $\operatorname{ri} F$ and $\operatorname{ri} J_x^{-1}(F)$ are open in $\operatorname{aff} F$ and contain the point x. Therefore the set $\operatorname{ri} F \cap \operatorname{ri} J_x^{-1}(F)$ is nonempty and open in $\operatorname{aff} F$. So

$$\operatorname{aff} F = \operatorname{aff}(\operatorname{ri} F \cap \operatorname{ri} J_x^{-1}(F)) \subset \operatorname{aff}(F \cap J_x^{-1}(F)) \subset \operatorname{aff} F.$$

Hence the second equality holds.

Let $y \in K \cap \operatorname{aff} F$. If $y = x$, then $y \in F$. Let $y \neq x$. Since $x \in \operatorname{ri} F$ and $y \in \operatorname{aff} F$, there exists a $z \in \operatorname{ri} F$ such that

$$x \in \{(1-\alpha)y + \alpha z;\ 0 < \alpha < 1\}.$$

Thus from $y \in K$, $z \in K$ and $x \in F$ we get $y \in F$. Hence $K \cap \operatorname{aff} F \subset F$. The inverse inclusion is obvious.

The three equalities imply that $F = K \cap \operatorname{aff}(K \cap J_x^{-1}(K))$. From $K \in \mathscr{C}$ we get $J_x^{-1}(K) \in \mathscr{C}$. Thus by (M) and Theorem 1 the face F is $\mathscr{C}$-convex.

2. It is a special case of property 1.

3. $\mathscr{C}$-convex convex half-spaces can have only the forms $\emptyset$, R^n, $B_x(v_1, \ldots, v_k)$ or $P_x(v_1, \ldots, v_k)$ (see [10], Theorem 1).

Since $\mathscr{C} \neq \{R^n\}$, by Theorem 2 the complementary sets $\emptyset$ and R^n are $\mathscr{C}$-convex.

Let $B_x(v_1, \ldots, v_k) \in \mathscr{C}$. By Theorem 10 the set $P_x(v_1, \ldots, v_k)$ is $\mathscr{C}$-convex. Thus the set $J_x^{-1}(P_x(v_1, \ldots, v_k)) = P_x(-v_1, \ldots, -v_k) = R^n \setminus B_x(v_1, \ldots, v_k)$ is also $\mathscr{C}$-convex.

Analogically, $P_x(v_1, \ldots, v_k) \in \mathscr{C}$ implies $R^n \setminus P_x(v_1, \ldots, v_k) \in \mathscr{C}$.

4. This follows immediately from property 2 and from Theorem 6.

2. Families fulfilling conditions (M), (U), (H) and generated by convex half-spaces. In this section we shall use definitions, properties and symbols of convex half-spaces from [10]. We denote by $\mathscr{G}$ a family of convex half-spaces of R^n and by $\mathscr{C}$ the smallest family of sets which contains $\mathscr{G}$ and fulfils conditions (M), (U) and (H).

The above definition of $\mathscr{C}$ is correct. Indeed, the intersection of an arbitrary number of families fulfilling conditions (M), (U) and (H) is a family for which the conditions also hold. Moreover, there exists in R^n at least one family containing $\mathscr{G}$ and fulfilling conditions (M), (U), (H), namely the family of all convex sets in R^n. Therefore $\mathscr{C}$ exists as the intersection of all families of subsets of R^n containing $\mathscr{C}$ and fulfilling conditions (M), (U), (H). We also infer that any set from $\mathscr{C}$ is convex.

More important are the cases where $\mathscr{G}$ is a family of closed half-spaces and where $\mathscr{G}$ is a family of open half-spaces. In the first case $\mathscr{C}$ is identical with the smallest family which fulfils condition (U) and contains the family of H-convex sets (see [1]) generated by $\mathscr{G}$ in R^n.

The following notation will be useful. If $\mathscr{V}$ is an arbitrary family of subsets of R^n, then $\mathscr{V}_T$ denotes the family of all translates of sets from $\mathscr{V}$. Analogically, $\mathscr{V}_M$ denotes the family of all intersections of sets from $\mathscr{V}$, $\mathscr{V}_U$ denotes the family of all unions of increasing sequences of sets from $\mathscr{V}$, and $\mathscr{V}_L$ denotes the family of sets $\operatorname*{Lim}_{i\to\infty} A_i$, where $A_i \in \mathscr{V}$, $i = 1, 2, \ldots$

THEOREM 12. *If, starting from the family $\mathscr{G}$, we create in turn all translates, all intersections, all unions of increasing sequences of sets and again all intersections, then we get the family $\mathscr{C}$. In other words: $\mathscr{G}_{TMUM} = \mathscr{C}$. The equality $\mathscr{G}_{TLM} = \mathscr{C}$ also holds.*

Proof. If $\operatorname*{Lim}_{i\to\infty} A_i$ exists, then $\operatorname*{Lim}_{i\to\infty} A_i = \operatorname*{Lim\,inf}_{i\to\infty} A_i = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_j$. Hence $\mathscr{V}_L \subset \mathscr{V}_{MU}$ for an arbitrary family $\mathscr{V}$. Consequently, $\mathscr{G}_{TLM} \subset \mathscr{G}_{TMUM}$. From the definition of $\mathscr{C}$ and Theorem 1 we get the inclusion $\mathscr{G}_{TMUM} \subset \mathscr{C}$. Therefore to prove the theorem it is sufficient to show the inclusion $\mathscr{C} \subset \mathscr{G}_{TLM}$, i.e. it is sufficient to show that $\mathscr{G}_{TLM}$ is a family of convex sets containing $\mathscr{G}$ and fulfilling conditions (M), (U) and (H).

Obviously, $\mathscr{G}_{TLM} \supset \mathscr{G}$ and $\mathscr{G}_{TLM}$ is multiplicative. Since the sets from $\mathscr{G}$ are convex and since the family of all convex sets is closed under translations, intersections and the limit Lim, the sets from $\mathscr{G}_{TLM}$ are also convex.

Note that any homothetical image (with an arbitrary centre and a positive coefficient) of an arbitrary convex half-space is equal to a translate of the convex half-space. Hence condition (H) holds for the family $\mathscr{G}_T$. Consequently, (H) holds for $\mathscr{G}_{TL}$ and also for $\mathscr{G}_{TLM}$.

Finally we shall prove (U) for the family $\mathscr{G}_{TLM}$. Let $V = \bigcup_{i=1}^{\infty} V_i$, where $V_i \in \mathscr{G}_{TLM}$, $V_i \subset V_{i+1}$, $i = 1, 2, \ldots$ We can assume $V \neq R^n$ since the case $V = R^n$ is trivial. Let $x \notin V$. Since $V_i \in \mathscr{G}_{TLM}$ and $x \notin V_i$, there exists a $Z_i \in \mathscr{G}_{TL}$ such that $x \notin Z_i \supset V_i$, $i = 1, 2, \ldots$ All members of the family $\mathscr{G}_{TL}$ are convex half-spaces as limits Lim of sequences of convex half-spaces ([10], condition (1) in Theorem 2). Therefore the sets Z_i, $i = 1, 2, \ldots$, are convex half-spaces. From the sequence Z_i, $i = 1, 2, \ldots$, a subsequence Z_{i_j}, $j = 1, 2, \ldots$, can be selected such that the limit $\operatorname*{Lim}_{j\to\infty} Z_{i_j} = Z$ exists ([10], condition (6) in Theorem 2). Moreover, $Z \in \mathscr{G}_{TL}$ ([10], condition (5) in Theorem 2). Obviously, $x \notin Z \supset V$. Thus V is the intersection of sets from the family $\mathscr{G}_{TL}$. So $V \in \mathscr{G}_{TLM}$.

Remark 1. Each stage in Theorem 12 is necessary. For any of the three first stages an example can easily be found. Example 3 will show that the last stage cannot be omitted even in the case where $\mathscr{G}$ is the family of all open half-spaces of the space R^2. The family $\mathscr{G}_M = \mathscr{G}_{TM}$ is identical with the family of sets called evenly convex sets ([4], see also [7]). $\mathscr{C} = \mathscr{G}_{TMUM}$ is the family of all convex sets because $\mathscr{G}_{TMUM}$ contains the Euclidean ball B_e (compare the beginning of this paper). Example 3 will show that $\mathscr{G}_{TMU} \neq \mathscr{C}$.

EXAMPLE 3. We shall construct a convex set $T \subset R^2$ which is not the union of an increasing sequence of evenly convex sets (evenly convex sets in R^2 are the intersections of open half-spaces of R^2). Let C denote the Cantor set and let E denote the set of the end-points of the removed segments in the construction of C. We put

$$p_\lambda = (\cos 2\pi\lambda, \sin 2\pi\lambda) \subset R^2 \quad \text{for} \quad 0 \leqslant \lambda \leqslant 1.$$

Let $D = \{p_\lambda;\ \lambda \in C\}$, $F = \{p_\lambda;\ \lambda \in E\}$, $G = D \setminus F$. If $a \in E$ and $b \in E$ are the end-points of a removed segment, then let P_{ab} denote the union of the one-point set $\{\frac{1}{2}(p_a + p_b)\}$ and of the open half-space containing the point $(0, 0)$ and bounded by the line through p_a and p_b. Obviously, P_{ab} is convex. Therefore the intersection T of all such P_{ab} is also convex.

Suppose $T = \bigcup_{i=1}^{\infty} T_i$, where $T_i \subset T_{i+1}$ and T_i are evenly convex,

$i=1,2,\ldots$ We shall show that $\bar{T}_i \cap F = \emptyset$, $i=1,2,\ldots$ Suppose the contrary, i.e. suppose $\bar{T}_k \cap F \neq \emptyset$ for an index k. Let $p_c \in \bar{T}_k \cap F$. Obviously, c is an end-point of a removed segment in the construction of C. Let d be the other end-point of this segment. Hence $\frac{1}{2}(p_c+p_d) \in T$. So $\frac{1}{2}(p_c+p_d) \in T_j$ for an index $j \geqslant k$. From $p_c \in \bar{T}_k$ we get $p_c \in \bar{T}_j$. Since $p_c \in \bar{T}_j$, $\frac{1}{2}(p_c+p_d) \in T_j$ and T_j is the intersection of a family of open half-spaces, we have

$$\tfrac{1}{2}\cdot\tfrac{1}{2}(p_c+p_d)+\tfrac{1}{2}p_c \in T_j \subset T.$$

On the other hand, $\frac{1}{2}\cdot\frac{1}{2}(p_c+p_d)+\frac{1}{2}p_c \notin P_{cd} \supset T$. The contradiction implies $\bar{T}_i \cap F = \emptyset$, $i=1,2,\ldots$

From $\bar{T}_i \cap F = \emptyset$ and $F \cup G = D$ we get $\bar{T}_i \cap D = \bar{T}_i \cap G_i$, $i=1,2,\ldots$ Since F is dense in D and since $\bar{T}_i \cap D = \bar{T}_i \cap G$ is a closed subset of $G = D\backslash F$, the set $\bar{T}_i \cap G$ is nowhere dense in D, $i=1,2,\ldots$ Moreover, the equality $G = \bigcup_{i=1}^{\infty} (\bar{T}_i \cap G)$ holds because $G \subset T = \bigcup_{i=1}^{\infty} T_i \subset \bigcup_{i=1}^{\infty} \bar{T}_i$. Also any one-point subset of F is nowhere dense in D and F is countable. Thus

$$D = F \cup G = \bigcup_{f\in F} \{f\} \cup \bigcup_{i=1}^{\infty} (\bar{T}_i \cap G)$$

is the union of countably many sets which are nowhere dense in D. This contradicts the Baire theorem because D is a complete space as a closed subset of R^2.

Therefore the convex set T cannot be the union of an increasing sequence of evenly convex sets.

The following theorem is of practical value for the description of $\mathscr{C}$-convex half-spaces and $\mathscr{C}$-convex sets. For instance compare Example 4.

THEOREM 13. *If all convex half-spaces from $\mathscr{G}$ are blunt, then $\mathscr{C} = \mathscr{G}_{TL'M}$, where L' denotes the creating of the limits* Lim *of sequences of convex half-spaces with a common vertex.*

Proof. By Theorem 12 we have $\mathscr{C} = \mathscr{G}_{TLM}$. So $\mathscr{G} \subset \mathscr{G}_{TL'M} \subset \mathscr{G}_{TLM} = \mathscr{C}$. Therefore it is sufficient to show that conditions (M), (U) and (H) hold for the family $\mathscr{G}_{TL'M}$. Condition (M) is obvious. Condition (H) can be shown in the same way as in Theorem 12.

We shall show that (U) holds. Let $V = \bigcup_{i=1}^{\infty} V_i$, where $V_i \in \mathscr{G}_{TL'M}$ and $V_i \subset V_{i+1}$ for $i=1,2,\ldots$ Let $x \notin V$. Since $V_i \in \mathscr{G}_{TL'M}$, there exists a set $Z_i^* \in \mathscr{G}_{TL'}$ such that $x \notin Z_i^* \supset V_i$, $i=1,2,\ldots$ Since Z_i^* belongs to $\mathscr{G}_{TL'}$, it is a blunt convex half-space, $i=1,2,\ldots$ (compare [10], Corollary 1). If Z_i is such a translate of the blunt convex half-space Z_i^* that x is a vertex of Z_i, then the inclusion $Z_i^* \subset Z_i$ holds ([10], part 3 of Theorem 1). Consequently, $x \notin Z_i \supset V_i$ for $i=1,2,\ldots$ From the sequence Z_i, $i=1,2,\ldots$, we choose a subsequence Z_{i_j}, $j=1,2,\ldots$, convergent to a convex half-space Z with the vertex x (see [10], Corollary 1). Since $Z_i \in \mathscr{G}_{TL'}$, we have $Z \in \mathscr{G}_{TL'}$ (see [10], condition (5) in Theorem 2). Obviously, $x \notin Z \supset V$. Hence $V \in \mathscr{G}_{TL'M}$.

THEOREM 14. *The family $\mathscr{G}_{TL}$ and the family of all $\mathscr{C}$-convex convex half-spaces are intersection $\mathscr{C}$-bases. If at least one nonempty convex half-space belongs to $\mathscr{G}$, then the above families are identical.*

Proof. Obviously, $\mathscr{G}_{TL}$ is an intersection basis of $\mathscr{G}_{TLM}$. By Theorem 12 we have $\mathscr{G}_{TLM} = \mathscr{C}$. So $\mathscr{G}_{TL}$ is an intersection $\mathscr{C}$-basis.

Sets from $\mathscr{G}_T$ are $\mathscr{C}$-convex convex half-spaces. The family of $\mathscr{C}$-convex sets (by Theorem 1) and the family of convex half-spaces (by Theorem 2 in [10]) are closed with respect to the limit Lim. Therefore sets from $\mathscr{G}_{TL}$ are $\mathscr{C}$-convex convex half-spaces. Thus the family of all $\mathscr{C}$-convex convex half-spaces is also an intersection $\mathscr{C}$-basis.

To end the proof it is sufficient to show that if at least one nonempty convex half-space belongs to $\mathscr{G}$, then any $\mathscr{C}$-convex convex half-space G belongs to $\mathscr{G}_{TL}$. From Theorem 1 in [10] we infer that G can have only the forms $B_x(v_1,\ldots,v_k)$, $P_x(v_1,\ldots,v_k)$, R^n or $\emptyset$.

Let $G = B_x(v_1,\ldots,v_k) \in \mathscr{C}$. Since $B_x(v_1,\ldots,v_k) \in \mathscr{C} = \mathscr{G}_{TLM}$ and $x \notin B_x(v_1,\ldots,v_k)$, there exists a set $S \in \mathscr{G}_{TL}$ such that $x \notin S \supset B_x(v_1,\ldots,v_k)$. Therefore S as a convex half-space can have only the forms $S = B_z(v_1,\ldots,v_l)$ or $S = P_z(v_1,\ldots,v_l)$, where $k \leqslant l \leqslant n$, and where $x \notin S \supset B_x(v_1,\ldots,v_k)$. Consequently, $S_1 \subset S_2 \subset \ldots$, where $S_m = S+(1/m)v_k$, $m=1,2,\ldots$ (compare [10], part 3 of Theorem 1). Moreover, $S_m \in \mathscr{G}_{TL}$, $m=1,2,\ldots$ and $B_x(v_1,\ldots,v_k) = \bigcup_{m=1}^{\infty} S_m$. Since $\mathscr{G}_{TL}$ is closed under the limit Lim, (U) holds for $\mathscr{G}_{TL}$. Thus $B_x(v_1,\ldots,v_k) \in \mathscr{G}_{TL}$.

Let $G = P_x(v_1,\ldots,v_k) \in \mathscr{C}$. By Theorem 10 the set $B_x(v_1,\ldots,v_k)$ is $\mathscr{C}$-convex. By the preceding part of the proof, $B_x(v_1,\ldots,v_k) \in \mathscr{G}_{TL}$. Let $T_m = B_{y_m}(v_1,\ldots,v_k)$, where $v_{xy_m} = -(1/m)v_k$, $m=1,2,\ldots$ Obviously, $P_x(v_1,\ldots,v_k) = \bigcap_{m=1}^{\infty} T_m$. Since $T_1 \supset T_2 \supset \ldots$, we have $\bigcap_{m=1}^{\infty} T_m = \operatorname{Lim}_{m\to\infty} T_m$. Now, from $T_m \in \mathscr{G}_{TL}$ we get $P_x(v_1,\ldots,v_k) \in \mathscr{G}_{TLL}$. Consequently, $G \in \mathscr{G}_{TL}$ (compare [10], condition (5) in Theorem 2).

If $G = R^n \in \mathscr{C}$, then $G \in \mathscr{G}_{TL}$. Indeed, $\mathscr{G}$ contains at least one nonempty convex half-space H; thus $G = R^n$ can be obtained as the limit Lim of a sequence of translates of H.

If $G = \emptyset \in \mathscr{C}$, then $\mathscr{G} \neq \{R^n\}$. Let $A \in \mathscr{G}$ and $A \neq R^n$. The set $G = \emptyset$ can be obtained as the limit Lim of a sequence of translates of A.

THEOREM 15. *The family of all $\mathscr{C}$-convex blunt convex half-spaces is an intersection $\mathscr{C}$-basis. More exactly: if $C \in \mathscr{C}$ and $x \notin C$, then a $\mathscr{C}$-convex blunt convex half-space with the vertex x contains C.*

Proof. Let $P_x(v_1,\ldots,v_k)$ be a $\mathscr{C}$-convex pointed convex half-space. By Theorems 10 and 1 the half-spaces $B_x(v_1,\ldots,v_k)$ and $B_{x_m}(v_1,\ldots,v_k)$ are

$\mathscr{C}$-convex, where $v_{xx_m} = -(1/m)v_k$, $m = 1, 2, \ldots$ Obviously, $P_x(v_1, \ldots, v_k) = \bigcap_{m=1} B_{x_m}(v_1, \ldots, v_k)$. Moreover, the set R^n is the intersection of empty family of $\mathscr{C}$-convex blunt convex half-spaces. Therefore any $\mathscr{C}$-convex pointed convex half-space is an intersection of $\mathscr{C}$-convex blunt convex half-spaces. From Theorem 14 we infer that the family of all $\mathscr{C}$-convex blunt convex half-spaces is an intersection $\mathscr{C}$-basis.

Let $C \in \mathscr{C}$ and $x \notin C$. If $C \neq \emptyset$, then by the previous part there exists $B_y(v_1, \ldots, v_k) \in \mathscr{C}$ such that $x \notin B_y(v_1, \ldots, v_k) \supset C$. Since $x \notin B_y(v_1, \ldots, v_k)$, we have $B_x(v_1, \ldots, v_k) \supset B_y(v_1, \ldots, v_k)$ (see [10], part 3 of Theorem 1). Hence $B_x(v_1, \ldots, v_k)$ is the required $\mathscr{C}$-convex blunt convex half-space. If $C = \emptyset$, then $\emptyset$ is the required $\mathscr{C}$-convex blunt convex half-space.

It is known ([6], Theorem 1.5) that the smallest intersection basis of the family of convex sets in R^n consists of all semi-spaces, i.e. sets of the form $B_x(v_1, \ldots, v_n)$ (the set R^n is also added in [6], we omit the set R^n because we understand it as the intersection of the empty family of semi-spaces of R^n). We generalize the result to the family of $\mathscr{C}$-convex sets:

THEOREM 16. *The family $\mathscr{A}$ of all maximal $\mathscr{C}$-convex blunt convex half-spaces from all those with a fixed vertex x (for all $x \in R^n$) is the smallest intersection $\mathscr{C}$-basis.*

Proof. First, we shall show that $\mathscr{A}$ is an intersection $\mathscr{C}$-basis. In connection with Theorem 15 it is sufficient to prove that any $\mathscr{C}$-convex blunt convex half-space A is an intersection of sets from $\mathscr{A}$. If $A \in \mathscr{A}$, then this is obvious. Let $A \in \mathscr{C} \setminus \mathscr{A}$. If A has the form $B_x(v_1, \ldots, v_k)$, then there exist a number k, $i < k \leqslant n$, and unit vectors $v_{i+1}, \ldots, v_k$ such that $v_1, \ldots, v_k$ are perpendicular and that $B_x(v_1, \ldots, v_k) \in \mathscr{A}$. We notice that

$$B_x(v_1, \ldots, v_i) = \bigcap_{m=1}^{\infty} B_{x_m}(v_1, \ldots, v_k),$$

where $v_{xx_m} = mv_{i+1}$, $m = 1, 2, \ldots$ Since $B_{x_m}(v_i, \ldots, v_k) \in \mathscr{A}$, $m = 1, 2, \ldots$, $B_x(v_1, \ldots, v_i)$ is an intersection of sets from $\mathscr{A}$. If $A = \emptyset$, then a maximal blunt convex half-space with a vertex x has the form $\emptyset$ or $B_x(u_1, \ldots, u_l)$. The first case is obvious. In the second case $A = \emptyset$ is the intersection of $\mathscr{C}$-convex blunt convex half-spaces $B_{y_m}(u_1, \ldots, u_l) \in \mathscr{A}$, where $v_{xy_m} = -mu_1$, $m = 1, 2, \ldots$

We shall now show that $\mathscr{A}$ is the smallest $\mathscr{C}$-basis. It is sufficient to show that $\mathscr{A} \subset \mathscr{B}$ for an arbitrary $\mathscr{C}$-basis $\mathscr{B}$. Let $Z \in \mathscr{A}$ be a blunt convex half-space with a vertex x. Since $x \notin Z$, there exists a $C \in \mathscr{B}$ such that $x \notin C \supset Z$. By Theorem 15 there exists a $\mathscr{C}$-convex blunt convex half-space T with the vertex x such that $T \supset C \supset Z$. Since Z is a maximal blunt convex half-space with the vertex x, we have $T = C = Z$. Hence $Z \in \mathscr{B}$.

Remark 2. In connection with Theorem 16 let us notice that $B_x(v_1, \ldots, v_k) \in \mathscr{A}$ if and only if $B_x(v_1, \ldots, v_k) \in \mathscr{C}$ and if (when $k < n$) $B_x(v_1, \ldots, v_k, v_{k+1}) \notin \mathscr{C}$ for any unit vector v_{k+1} perpendicular to $v_1, \ldots, v_k$. This results from Theorem 10. Moreover, $\emptyset \in \mathscr{A}$ if and only if $\mathscr{C} = \{\emptyset, R^n\}$.

THEOREM 17. *For arbitrary $A \subset R^n$ and $x \in R^n$ we have: $x \in \mathscr{C}$-$\mathrm{cone}_x A$ if and only if $x \in \mathscr{C}$-$\mathrm{conv}\, A$. If $x \in \mathscr{C}$-$\mathrm{conv}\, A$ and $\mathrm{cone}_x A = \mathrm{cone}_x C$ then $x \in \mathscr{C}$-$\mathrm{conv}\, C$.*

Proof. Let $x \in \mathscr{C}$-$\mathrm{conv}\, A$. Since $\mathscr{C}$-$\mathrm{cone}_x A$ is $\mathscr{C}$-convex and contains A, we have $\mathscr{C}$-$\mathrm{cone}_x A \supset \mathscr{C}$-$\mathrm{conv}\, A$. Hence $x \in \mathscr{C}$-$\mathrm{cone}_x A$.

Let $x \notin \mathscr{C}$-$\mathrm{conv}\, A$. By Theorem 15 there exists a $\mathscr{C}$-convex blunt convex half-space T with the vertex x such that $T \supset \mathscr{C}$-$\mathrm{conv}\, A$. Since T is a $\mathscr{C}$-cone with the vertex x and $x \notin T$, we have $x \notin \mathscr{C}$-$\mathrm{cone}_x A$.

Let $x \in \mathscr{C}$-$\mathrm{conv}\, A$ and $\mathrm{cone}_x A = \mathrm{cone}_x C$. The equality implies $\mathscr{C}$-$\mathrm{cone}_x A = \mathscr{C}$-$\mathrm{cone}_x C$. From the first part of the theorem and from $x \in \mathscr{C}$-$\mathrm{conv}\, A$ we get $x \in \mathscr{C}$-$\mathrm{cone}_x A = \mathscr{C}$-$\mathrm{cone}_x C$. By the first part of the theorem $x \in \mathscr{C}$-$\mathrm{conv}\, C$.

THEOREM 18. *A hyperplane is $\mathscr{C}$-convex if and only if both closed half-spaces bounded by it are $\mathscr{C}$-convex.*

Proof. Let H be a hyperplane and let G and F be the opposite closed half-spaces bounded by H.

If $G \in \mathscr{C}$ and $F \in \mathscr{C}$, then $H = F \cap G \in \mathscr{C}$.

Let $H \in \mathscr{C}$. Let $x \notin G$. Consequently, $x \notin H$. By Theorem 15 there exists $B_x(v_1, \ldots, v_k) \in \mathscr{C}$ such that $B_x(v_1, \ldots, v_k) \supset H$. From Theorem 10 we infer that $P_x(v_1) \in \mathscr{C}$. Obviously, v_1 is perpendicular to H. Hence G is a translate of $P_x(v_1)$. So G is $\mathscr{C}$-convex. Analogically, $F \in \mathscr{C}$.

It is known ([5], pp. 225–226) that the family of all hypersubspaces of R^n is compact in the sense of the topological limit of sets Lt. An analogous property holds for the family of $\mathscr{C}$-convex hypersubspaces.

THEOREM 19. *The family $\mathscr{H}$ of all $\mathscr{C}$-convex hypersubspaces of R^n is compact in the sense of the topological limit of sets* Lt.

Proof. Since $\mathscr{H}$ is a subfamily of the compact family of all hypersubspaces, it is sufficient to show that $\mathscr{H}$ is closed with respect to the limit Lt. Let $H_i \in \mathscr{H}$, $i = 1, 2, \ldots$, and let $H = \mathop{\mathrm{Lt}}_{i \to \infty} H_i$. Let G_i and F_i be two opposite closed half-spaces bounded by H_i, $i = 1, 2, \ldots$ Since $H_i \in \mathscr{C}$, by Theorem 18 we get $G_i \in \mathscr{C}$ and $F_i \in \mathscr{C}$, $i = 1, 2, \ldots$ It is possible to select from the sequence G_i a subsequence G_{i_j}, $j = 1, 2, \ldots$ convergent, in the sense of the limit Lim, (see [10], Corollary 1). Let $G = \mathop{\mathrm{Lim}}_{j \to \infty} G_{i_j}$ and

$$F = R^n \setminus G = R^n \setminus \mathop{\mathrm{Lim}}_{j \to \infty} G_{i_j} = \mathop{\mathrm{Lim}}_{j \to \infty} (R^n \setminus G_{i_j}) = \mathop{\mathrm{Lim}}_{j \to \infty} F_{i_j}.$$

From $H = \mathop{\mathrm{Lt}}_{j \to \infty} H_{i_j}$ we infer that the sequence of the normal vectors of H_{i_j} (directed to the sides of G_{i_j}) converges to one of the normal vectors of H. Consequently, H is the bounding hyperplane of $\bar{G}$ and of $\bar{F}$ (compare

Theorem 3 in [10]). Moreover, by Theorem 1 the sets G and F, and consequently the sets $\bar{G}$ and $\bar{F}$, are $\mathscr{C}$-convex. Thus from Theorem 18 we get $H \in \mathscr{C}$. Hence $H \in \mathscr{H}$.

Remark 3. Some theorems, for instance 15, 18 and 19, are also true for any $\mathscr{C}$-convex subspace $R^m \subset R^n$. This results from the fact that the family of all $\mathscr{C}$-convex subsets of R^m is the smallest family of sets which fulfils conditions (M), (U), (H) and contains a family $\mathscr{G}_1$ of convex half-spaces of R^m. As $\mathscr{G}_1$ we can take the family of all $\mathscr{C}$-convex convex half-spaces of R^m. Indeed, intersections of $\mathscr{C}$-convex convex half-spaces of R^n with R^m are $\mathscr{C}$-convex convex half-spaces of R^m. Thus by Theorem 14 all $\mathscr{C}$-convex subsets of R^m are intersections of $\mathscr{C}$-convex half-spaces of R^m.

The vector v_1 of the convex half-space $B_x(v_1, \ldots, v_k)$ or $P_x(v_1, \ldots, v_k)$ will be called the first vector of the convex half-space.

THEOREM 20. *If $\mathscr{G}$ is the family of all open [analogically: all closed] half-spaces of R^n, then $\mathscr{C}$ is the family of all convex sets in R^n. Generally: $\mathscr{C}$ is the family of all convex sets in R^n if and only if the set of the first vectors of convex half-spaces from $\mathscr{G}$ is dense in the set of all unit vectors of R^n.*

Proof. Let $\mathscr{G}$ be the family of all open half-spaces of R^n. Then $\mathscr{G}_L$ contains all sets of the form $B_x(v_1, \ldots, v_n)$ (see Theorem 3 in [10]). Before presenting Theorem 16 we said that the sets form an intersection basis of the family of all convex sets. Hence $\mathscr{C}$ is the family of all convex sets.

Now, let $\mathscr{G}$ be an arbitrary family of convex half-spaces of R^n and let $W_{\mathscr{G}}$ be the set of all first vectors of the half-spaces from $\mathscr{G}$. Assume that $W_{\mathscr{G}}$ is dense in the set W of all unit vectors of R^n. Let $B_x(v_1)$ be an arbitrary open half-space. There exists a sequence $v_1^m \in W_{\mathscr{G}}$, $m = 1, 2, \ldots$, convergent to v_1. Since there exists a $\mathscr{C}$-convex convex half-space with the first vector v_1^m and with a vertex x_m, $m = 1, 2, \ldots$, by Theorems 10 and 1 the open half-space $B_x(v_1^m)$ is $\mathscr{C}$-convex, $m = 1, 2, \ldots$ From the sequence $B_x(v_1^m)$, $m = 1, 2, \ldots$, it is possible to select a convergent subsequence whose limit Lim has the form $B_x(u_1, \ldots, u_k)$ (see Corollary 1 in [10]). By Theorem 1 we have $B_x(u_1, \ldots, u_k) \in \mathscr{C}$. By Theorem 10 the open half-space $B_x(u_1)$ is $\mathscr{C}$-convex. Moreover, $u_1 = v_1$ (see Theorem 3 in [10]). Hence $B_x(v_1) \in \mathscr{C}$. Therefore any open half-space of R^n is $\mathscr{C}$-convex. From the first part of the theorem we infer that $\mathscr{C}$ is identical with the family of all convex sets.

Particularly, $\mathscr{G}$ can be the family of all closed half-spaces in the above considerations.

If $W_{\mathscr{G}}$ is not dense in W, then there exists a vector $v_1 \in W$ which is not the limit of a sequence from $W_{\mathscr{G}}$. Therefore the family $\mathscr{G}$, and so the families $\mathscr{G}_T$, $\mathscr{G}_{TL}$ and $\mathscr{G}_{TLM}$ do not contain convex half-spaces with the first vector v_1. Consequently, $\mathscr{G}_{TLM} = \mathscr{C}$ (compare Theorem 12) is not the family of all convex sets of R^n.

A family $\mathscr{C}$ will be called *generated by a family $\mathscr{Z}$ of subsets of R^n* [analogically: by a set Z] if $\mathscr{C}$ is the smallest family which fulfils conditions (M), (U), (H) and contains $\mathscr{Z}$ [analogically: Z].

THEOREM 21. *The following conditions are equivalent: $\mathscr{C}$ is generated by a family of open half-spaces, $\mathscr{C}$ is generated by a family of closed half-spaces, $\mathscr{C}$ is generated by an open convex body, $\mathscr{C}$ is generated by a closed convex body.*

Proof. It is known ([13], Theorems 1.16 and 1.17, p. 13) that $W = \overline{\operatorname{int} W}$ for any closed convex body W and that $A = \operatorname{int} A = \operatorname{int} \bar{A}$ for any open convex body A. Obviously, int W is an open body and $\bar{A}$ is a closed body. Moreover, from Theorem 1 we infer that $W \in \mathscr{C}$ implies int $W \in \mathscr{C}$ and $A \in \mathscr{C}$ implies $\bar{A} \in \mathscr{C}$. Hence the last two cases of the theorem are equivalent. Analogously, the first two cases are equivalent. Therefore it is sufficient to show the equivalence of the second and the last cases.

If $\mathscr{C}$ is generated by a closed convex body K, then it is also generated by the family of such closed half-spaces as in Theorem 7.

Now, let $\mathscr{C}$ be generated by a family of closed half-spaces G_λ, $\lambda \in \Lambda$. Let B_e be the Euclidean unit ball. By G_λ^* we denote the minimum translate of G_λ which contains B_e, $\lambda \in \Lambda$. Put $K = \bigcap_{\lambda \in \Lambda} G_\lambda^*$. Let $\mathscr{C}_K$ denote the family generated by K. Since $K \in \mathscr{C}$, so $\mathscr{C}_K \subset \mathscr{C}$. It follows from the smoothness of B_e and from the construction of K that the bounding hyperplane of G_λ^* supports K at a regular point. In virtue of Theorem 6 the half-space G_λ^* is $\mathscr{C}_K$-convex, $\lambda \in \Lambda$. Thus all half-spaces G_λ, $\lambda \in \Lambda$, are $\mathscr{C}_K$-convex. Consequently, $\mathscr{C} \subset \mathscr{C}_K$. Hence we get the equality $\mathscr{C} = \mathscr{C}_K$. Therefore the family $\mathscr{C}$ is generated by the closed convex body K.

EXAMPLE 4. Let the family $\mathscr{C}$ be generated by the open convex cone

$$S = \{(x_1, x_2, x_3);\ x_1^2 + x_2^2 < x_3^2 \text{ and } x_3 > 0\} \subset R^3.$$

Consequently, $\mathscr{C}$ is also generated by the family

$$\mathscr{G} = \{x_1 \cos\alpha + x_2 \sin\alpha + x_3 > 0;\ 0 \leqslant \alpha < 2\pi\}$$

of open half-spaces. From Theorems 13 and 16 of this paper and from Theorem 3 of [10] it results that the smallest intersection $\mathscr{C}$-basis $\mathscr{A}$ consists of all blunt convex half-spaces $B_x(v_1, v_2, v_3)$ such that

$$v_1 = ((\cos\alpha)/\sqrt{2},\ (\sin\alpha)/\sqrt{2},\ 1/\sqrt{2}),$$

$$v_2 = (-\sin\alpha,\ \cos\alpha,\ 0) \quad \text{or} \quad v_2 = (\sin\alpha,\ -\cos\alpha,\ 0),$$

$$v_3 = (-(\cos\alpha)/\sqrt{2},\ -(\sin\alpha)/\sqrt{2},\ 1/\sqrt{2}),$$

where $0 \leqslant \alpha < 2\pi$.

THEOREM 22. *If $\mathscr{C}$ is closed under symmetries with centre 0, then*

1. *If $R^k \subset R^m$ are $\mathscr{C}$-convex subspaces, then for any integer l such that $k \leqslant l \leqslant m$, there exists a $\mathscr{C}$-convex subspace R^l such that $R^k \subset R^l \subset R^m$.*

2. *Any $\mathscr{C}$-convex plane of dimension $k < n$ can be supplemented to a $\mathscr{C}$-convex plane of dimension $k+1$.*

3. *If for $\mathscr{C}$-convex sets C and D the condition $\operatorname{ri} C \cap \operatorname{ri} D = \emptyset$ holds, then there exist closed $\mathscr{C}$-cones $S \supset C$ and $T \supset D$ with a common vertex such that $\operatorname{ri} S \cap \operatorname{ri} T = \emptyset$.*

Proof. 1. By Theorem 15 and Remark 3 the set R^k is the intersection of a family of $\mathscr{C}$-convex blunt convex half-spaces of R^m. Let A be a member of the family. Obviously, $\bar{A}$ is a $\mathscr{C}$-convex closed half-space of R^m. By Theorem 11 the $(m-1)$-dimensional bounding plane H of $\bar{A}$ is $\mathscr{C}$-convex. Obviously, the subspace R^{m-1}, being the translate of H is $\mathscr{C}$-convex. From $\bar{A} \supset R^k$ we get $R^{m-1} \supset R^k$. If $l = m-1$, then R^{m-1} is the subspace we are looking for. If $l < m-1$, then we repeat this procedure a proper number of times. After $m-l$ times we get the required subspace R^l.

2. This follows from part 1 for $m = n$, $l = k+1$ and from the fact that $\mathscr{C}$ is closed under translations.

3. The case where $\operatorname{aff} C \cap \operatorname{aff} D \neq \emptyset$ follows from Theorem 9. Let $\operatorname{aff} C \cap \operatorname{aff} D = \emptyset$. We put $T = \operatorname{aff} D$ and $P_k = \operatorname{aff} C$, where k is the dimension of $\operatorname{aff} C$. By the preceding property there exist $\mathscr{C}$-convex planes $P_k \subset P_{k+1} \subset \ldots \subset P_n = R^n$. Let m be the smallest number for which $P_m \cap T \neq \emptyset$. Hence $P_{m-1} \cap T = \emptyset$. There exists a translate $S_{m-1} \subset P_m$ of P_{m-1} such that $S_{m-1} \cap T \neq \emptyset$. Let $x \in S_{m-1} \cap T$. Let S be a half-plane of P_m which is bounded by S_{m-1} and contains P_{m-1}. By Theorem 18 and Remark 3 the cone S is $\mathscr{C}$-convex. Obviously, the cones S and T have x as a common vertex, are closed and $\mathscr{C}$-convex. Moreover, $\operatorname{ri} S \cap \operatorname{ri} T = \emptyset$, $S \supset P_{m-1} \supset \operatorname{aff} C \supset C$ and $T \supset D$.

3. Domain finite families. Definition 1 of $\mathscr{C}$-hull is correct for any multiplicative family $\mathscr{C}$ of subsets of an arbitrary set X (not necessarily $X \subset R^n$). It is known ([12], see also Theorem 1.2 in [3], p. 45) that for such a family $\mathscr{C}$ condition (U*) defined at the beginning of this paper is equivalent to the condition

(F) $\mathscr{C}\text{-conv}\, A = \bigcup \{\mathscr{C}\text{-conv}\, F;\ F \subset A$ and F is finite$\}$ for any $A \subset X$.

Any multiplicative family $\mathscr{C}$ of subsets of X fulfilling condition (F) or, which is equivalent, condition (U*) will be called (after [6]) a domain finite family.

LEMMA 2. *Let $\mathscr{C}$ be a multiplicative family of subsets of a set X. The family*

$$\mathscr{C}_F = \{A;\ A \subset X \text{ and } \mathscr{C}\text{-conv}\, G \subset A \text{ for any finite } G \subset A\}$$

is the smallest domain finite family containing $\mathscr{C}$. The equality $\mathscr{C}_F$-conv $K = \mathscr{C}$-conv K holds for any finite $K \subset X$. The family $\mathscr{C}$ is domain finite if and only if it satisfies the condition

(F$_1$) *if $A \subset X$ and if $\mathscr{C}$-conv $G \subset A$ for any finite $G \subset A$, then $A \in \mathscr{C}$.*

Proof. We first prove the last part of the lemma.

Assume that $\mathscr{C}$ is domain finite. Let $A \subset X$ and let the inclusion $\mathscr{C}$-conv $G \subset A$ hold for any finite subset $G \subset A$. Let $a \in \mathscr{C}$-conv A. Since $\mathscr{C}$ is domain finite, $a \in \mathscr{C}$-conv F for a finite $F \subset A$. Obviously, $\mathscr{C}$-conv $F \subset A$. Hence $a \in A$. Therefore $\mathscr{C}$-conv $A \subset A$, and consequently $A = \mathscr{C}$-conv $A \in \mathscr{C}$. Thus (F$_1$) holds for the family $\mathscr{C}$.

Assume that $\mathscr{C}$ satisfies condition (F$_1$). Let $G = \{g_1, \ldots, g_m\}$ be a finite subset of the set

$$\bigcup \{\mathscr{C}\text{-conv}\, F;\ F \subset A \text{ and } F \text{ is finite}\}.$$

There exists a finite set $F_i \subset A$ such that $g_i \in \mathscr{C}$-conv F_i, $i = 1, \ldots, m$. So

$$\mathscr{C}\text{-conv}\, G \subset \mathscr{C}\text{-conv} \bigcup_{i=1}^{m} F_i \subset \bigcup \{\mathscr{C}\text{-conv}\, F;\ F \subset A \text{ and } F \text{ is finite}\}.$$

Analogically, for $G = \emptyset$. From (F$_1$) we get that the set

$$\bigcup \{\mathscr{C}\text{-conv}\, F;\ F \subset A \text{ and } F \text{ is finite}\}$$

belongs to $\mathscr{C}$. Hence the set is equal to $\mathscr{C}$-conv A because it contains A and is contained in $\mathscr{C}$-conv A. Therefore $\mathscr{C}$ is domain finite.

Now, we consider the family $\mathscr{C}_F$. Let $K_\lambda \in \mathscr{C}_F$, $\lambda \in \Lambda$. Let G be a finite subset of $\bigcap_{\lambda \in \Lambda} K_\lambda$. Since $G \subset K_\lambda$ and $K_\lambda \in \mathscr{C}_F$, we have $\mathscr{C}$-conv $G \subset K_\lambda$, $\lambda \in \Lambda$. Hence $\mathscr{C}$-conv $G \subset \bigcap_{\lambda \in \Lambda} K_\lambda$. Consequently, $\bigcap_{\lambda \in \Lambda} K_\lambda \in \mathscr{C}_F$. Therefore the family $\mathscr{C}_F$ is multiplicative and the $\mathscr{C}_F$-hull is defined.

If $K \subset X$ is finite, then by the definition of $\mathscr{C}_F$ and by the inclusion $K \subset \mathscr{C}_F$-conv K we get $\mathscr{C}$-conv $K \subset \mathscr{C}_F$-conv K. The inverse inclusion results from $\mathscr{C} \subset \mathscr{C}_F$. Hence $\mathscr{C}_F$-conv $K = \mathscr{C}$-conv K.

Let $A \subset X$ and let $\mathscr{C}_F$-conv $G \subset A$ for any finite $G \subset A$. Since G is finite, $\mathscr{C}_F$-conv $G = \mathscr{C}$-conv G. Hence $\mathscr{C}$-conv $G \subset A$. Therefore $A \in \mathscr{C}_F$. Thus the family $\mathscr{C}_F$ satisfies (F$_1$), and consequently it is domain finite.

Finally we shall show that $\mathscr{C}_F$ is the smallest domain finite family containing $\mathscr{C}$. Let $\mathscr{F} \supset \mathscr{C}$ be a domain finite family of subsets of the set X. Obviously, the family $\mathscr{F}$ satisfies condition (F$_1$). Therefore $\mathscr{F}_F \subset \mathscr{F}$. Moreover, $\mathscr{C}_F \subset \mathscr{F}_F$ because $\mathscr{C} \subset \mathscr{F}$. Hence $\mathscr{C}_F \subset \mathscr{F}$.

THEOREM 23. *Let $\mathscr{C}$ be a family of convex sets of the space R^n which fulfils conditions (M), (U) and (H). Then the following properties hold for the family $\mathscr{C}_F = \{A \subset R^n;\ \mathscr{C}\text{-conv}\, G \subset A \text{ for any finite } G \subset A\}$.*

1. $\mathscr{C}_F$ *is a family of convex sets fulfilling conditions* (M), (U), (H).

2. *For any denumerable set* $D \subset R^n$ *we have* $\mathscr{C}_F$-conv $D = \mathscr{C}$-conv D.

3. *The following kinds of* $\mathscr{C}$-*convex and* $\mathscr{C}_F$-*convex sets are identical: convex half-spaces, convex half-planes, planes, closed sets, relatively open sets, open sets.*

Proof. 1. Let A be an arbitrary set of the family $\mathscr{C}_F$. Let $a \in A$, $b \in A$. From the definition of $\mathscr{C}_F$ we get $A \supset \mathscr{C}$-conv $\{a, b\}$. Since all sets from $\mathscr{C}$ are convex,

$$\mathscr{C}\text{-conv}\,\{a, b\} \supset \{(1-\alpha)a+\alpha b;\ 0 \leqslant \alpha \leqslant 1\}.$$

Thus A is convex.

Condition (M) is shown in Lemma 2.

By Lemma 2 the family $\mathscr{C}_F$ is domain finite. Hence (U*) and consequently, (U) hold for the family $\mathscr{C}_F$.

Let $A \in \mathscr{C}_F$. Let G be a finite subset of $J_x^\lambda(A)$, where $\lambda > 0$. Then $J_x^{1/\lambda}(G)$ is a finite subset of A. Since $A \in \mathscr{C}_F$, so $\mathscr{C}$-conv $J_x^{1/\lambda}(G) \subset A$. By the equality in Theorem 3 we get

$$\mathscr{C}\text{-conv}\, G = J_x^\lambda\big(J_x^{1/\lambda}(\mathscr{C}\text{-conv}\, G)\big) = J_x^\lambda\big(\mathscr{C}\text{-conv}\, J_x^{1/\lambda}(G)\big) \subset J_x^\lambda(A).$$

Hence $J_x^\lambda(A) \in \mathscr{C}_F$. Consequently, (H) holds for the family $\mathscr{C}_F$.

2. Since $\mathscr{C} \subset \mathscr{C}_F$, we have $\mathscr{C}$-conv $D \supset \mathscr{C}_F$-conv D for any $D \subset R^n$. We shall show the inverse inclusion for the case where D is countable.

Let $D = \{x_1, x_2, \ldots\}$. Let

$$S = \bigcup_{i=1}^{\infty} \mathscr{C}_F\text{-conv}\,\{x_1, \ldots, x_i\}.$$

Since the set $\{x_1, \ldots, x_i\}$ is finite, from Lemma 2 we get

$$\mathscr{C}\text{-conv}\,\{x_1, \ldots, x_i\} = \mathscr{C}_F\text{-conv}\,\{x_1, \ldots, x_i\}, \qquad i = 1, 2, \ldots$$

Thus from (U) we get $S \in \mathscr{C}$. Since $\mathscr{C} \subset \mathscr{C}_F$, so $S \in \mathscr{C}_F$. Now, the inclusion $S \supset D$ implies that $S \supset \mathscr{C}_F$-conv D. The inverse inclusion is obvious. Hence $S = \mathscr{C}_F$-conv D. Since $S \in \mathscr{C}$, we have $\mathscr{C}_F$-conv $D \in \mathscr{C}$. Consequently, from $D \subset \mathscr{C}_F$-conv D we get $\mathscr{C}$-conv $D \subset \mathscr{C}_F$-conv D.

3. Since $\mathscr{C} \subset \mathscr{C}_F$, it is sufficient to show that $\mathscr{C}_F$-convex sets of the given form are $\mathscr{C}$-convex.

Any convex half-space is the convex hull of a sequence of points ([10], part 8 of Theorem 1). Hence any $\mathscr{C}$-convex convex half-space is $\mathscr{C}$-hull of a sequence of points. Consequently, by the preceding part of the theorem any $\mathscr{C}_F$-convex convex half-space is $\mathscr{C}$-convex.

Analogically, $\mathscr{C}_F$-convex half-planes and planes are $\mathscr{C}$-convex.

By property 1 of the theorem all theorems of the first part of the paper can be applied to $\mathscr{C}_F$-convexity. From Theorem 7 we conclude that any $\mathscr{C}_F$-convex closed set C is an intersection of $\mathscr{C}_F$-convex closed planes and half-planes. Since the planes and half-planes are $\mathscr{C}$-convex, C is $\mathscr{C}$-convex.

Let A be a relatively open and $\mathscr{C}_F$-convex set. By the first part of the theorem A is convex. Hence $A = \operatorname{ri} \bar{A}$ (see [11], Theorem 6.3, p. 46). By Theorem 1 applied to $\mathscr{C}_F$-convexity we get $\bar{A} \in \mathscr{C}_F$. Since $\bar{A}$ is closed, $\bar{A} \in \mathscr{C}$. It follows by Theorem 1 that $\operatorname{ri} \bar{A} \in \mathscr{C}$. Hence $A \in \mathscr{C}$.

Particularly, any open $\mathscr{C}_F$-convex set is $\mathscr{C}$-convex.

THEOREM 24. *Any family* $\mathscr{C}$ *of convex sets of* R^2 *satisfying conditions* (M), (U) *and* (H) *is domain finite.*

Proof. We shall first show that any $\mathscr{C}_F$-convex cone $S \subset R^2$ is $\mathscr{C}$-convex. As a convex cone with a vertex x (included or excluded) S is the convex hull of one ray, two rays, or a sequence of rays with the vertex x. It can also be empty or an one-point set. Any ray with the vertex x (included or excluded) is the convex hull of a sequence of points. Hence S is the convex hull of a countable set D. Since $\mathscr{C}_F$-convex sets are convex and since $S \in \mathscr{C}_F$, we have

$$S = \mathscr{C}_F\text{-conv}\, S = \mathscr{C}_F\text{-conv}\,(\operatorname{conv} D) = \mathscr{C}_F\text{-conv}\, D.$$

By Theorem 23, part 2, we get the equality $S = \mathscr{C}$-conv D. Consequently, the cone S is $\mathscr{C}$-convex.

From the first part of Theorem 23 it results that Theorem 5 can be applied to the family $\mathscr{C}_F$. Hence any $\mathscr{C}_F$-convex set $K \subset R^2$ is the intersection of a family of $\mathscr{C}_F$-convex cones. It was shown above that the cones are $\mathscr{C}$-convex. Therefore K is $\mathscr{C}$-convex. Hence $\mathscr{C} \supset \mathscr{C}_F$. Since the inverse inclusion is obvious, $\mathscr{C} = \mathscr{C}_F$. By Lemma 2 the family $\mathscr{C}_F$ is domain finite. Therefore $\mathscr{C}$ is domain finite.

The following example shows that in R^3 (and in R^n for $n \geqslant 3$) there exist no domain finite families of convex sets satisfying conditions (M), (U) and (H).

EXAMPLE 5. We shall construct a family $\mathscr{D}$ of convex sets which fulfils conditions (M), (U), (H) and does not satisfy condition (F_1). We use the notation of Example 4. Let $\mathscr{S}$ be a family of cones in R^3 which are unions of the cone S (as in Example 4) and of countably many rays, with the included vertex x, lying in the boundary of S. Let $\mathscr{S}_T$ be the family of all translates of cones from $\mathscr{S}$. Put $\mathscr{D} = \mathscr{C} \cup \mathscr{S}_T$, where $\mathscr{C}$ is the family defined in Example 4.

Obviously, the sets from $\mathscr{D}$ are convex.

Now, we shall show that (U) holds for the family $\mathscr{D}$. Let $V_i \in \mathscr{D}$, where $V_i \subset V_{i+1}$ for $i = 1, 2, \ldots$ A subsequence V_{i_j}, $j = 1, 2, \ldots$, can be selected such that all its sets belong to the family $\mathscr{C}$ or that all its sets belong to the family $\mathscr{S}_T$. If $V_{i_j} \in \mathscr{C}$, $j = 1, 2, \ldots$, then

$$V = \bigcup_{i=1}^{\infty} V_i = \bigcup_{j=1}^{\infty} V_{i_j} \in \mathscr{C} \subset \mathscr{D}.$$

Let $V_{i_j} \in \mathcal{S}_T$, $j = 1, 2, \ldots$ If almost all cones of the subsequence have a common vertex, then

$$V = \bigcup_{i=1}^{\infty} V_i = \bigcup_{j=1}^{\infty} V_{i_j} \in \mathcal{S}_T \subset \mathcal{D}.$$

In the opposite case

$$V = \bigcup_{j=1}^{\infty} V_{i_j} = \bigcup_{j=1}^{\infty} (V_{i_j} \setminus \{x_{i_j}\}),$$

where x_{i_j} is the vertex of V_{i_j}, $j = 1, 2, \ldots$ From the form of the $\mathcal{C}$-basis $\mathcal{A}$ (in Example 4) we get $V_{i_j} \setminus \{x_{i_j}\} \in \mathcal{C}$, $j = 1, 2, \ldots$ Therefore $V \in \mathcal{C} \subset \mathcal{D}$. Hence condition (U) holds for the family $\mathcal{D}$.

We shall show that the family $\mathcal{D}$ satisfies condition (M). Let $\mathcal{K} \subset \mathcal{D}$. If all sets from $\mathcal{K}$ are cones which belong to $\mathcal{S}_T$ and have a common vertex, then $\bigcap \mathcal{K} \in \mathcal{S}_T \subset \mathcal{D}$. We now assume the contrary case. By the preceding case we can assume that those sets from $\mathcal{K}$ which are cones from $\mathcal{S}_T$ have different vertices. We replace in $\mathcal{K}$ any cone $K \in \mathcal{K} \cap \mathcal{S}_T$ which contains another cone from $\mathcal{K} \cap \mathcal{S}_T$ by the cone $K \setminus \{x\}$, where x is the vertex of K. Obviously, $K \setminus \{x\} \in \mathcal{C}$. Denote the new family by $\mathcal{L}$. From the construction we infer that $\mathcal{L}$ contains (besides some sets from $\mathcal{C}$) only those cones from $\mathcal{S}_T$ which have different vertices and that no inclusions hold between the cones. Moreover, $\bigcap \mathcal{K} = \bigcap \mathcal{L}$. Let $\mathcal{L} \cap \mathcal{S}_T = \{S_\lambda, \lambda \in \Lambda\}$. If $\|\Lambda\| = 0$, then $\bigcap \mathcal{K} = \bigcap \mathcal{L} \in \mathcal{C} \subset \mathcal{D}$. If $\|\Lambda\| > 1$, then

$$\bigcap_{\lambda \in \Lambda} S_\lambda = \bigcap_{\lambda \in \Lambda} (S_\lambda \setminus \{x_\lambda\}) \in \mathcal{C},$$

where x_λ is the vertex of S_λ, $\lambda \in \Lambda$. So $\bigcap \mathcal{L} \in \mathcal{C} \subset \mathcal{D}$, and $\bigcap \mathcal{K} \in \mathcal{D}$. For $\|\Lambda\| = 1$ the family $\mathcal{L} \cap \mathcal{S}_T$ contains only one cone S. Let $K = \bigcap(\mathcal{L} \setminus \{S\})$. Obviously, $K \in \mathcal{C}$, $S \in \mathcal{S}_T$, $\bigcap \mathcal{L} = K \cap S$. If $S \subset K$, then $\bigcap \mathcal{L} = S \in \mathcal{S}_T \subset \mathcal{D}$. Suppose the inclusion $S \subset K$ does not hold. Since $K \in \mathcal{C}$, so K is an intersection of sets of the intersection basis $\mathcal{A}$ (as in Example 4). Therefore, for a set $Z \in \mathcal{A}$ such that $Z \supset K$, the inclusion $S \subset Z$ does not hold. Consequently, from the forms of $S \in \mathcal{S}_T$ and $Z \in \mathcal{A}$ we get $S \cap Z = (S \setminus \{t\}) \cap Z$, where t is the vertex of S. Hence

$$\bigcap \mathcal{L} = S \cap K = (S \setminus \{t\}) \cap K.$$

Since $S \setminus \{t\} \in \mathcal{C}$ and $K \in \mathcal{C}$, we have $\bigcap \mathcal{L} \in \mathcal{D}$. So $\bigcap \mathcal{K} \in \mathcal{D}$. Therefore $\mathcal{D}$ is multiplicative.

For any homothety with a positive coefficient the image of an arbitrary ray is a translate of the ray. Hence it is also true for an arbitrary cone. Therefore (H) holds for the family $\mathcal{S}_T$. Obviously, (H) holds for the family $\mathcal{C}$. Thus (H) is satisfied for the family $\mathcal{D}$.

Finally we shall show that (F_1) does not hold for the family $\mathcal{D}$. Let $A = \bar{S} \setminus L$, where $L \subset \operatorname{bd} S$ is a ray with the vertex 0 excluded. For any $M \in \mathcal{A}$ such that $M \supset A$ we have $M \supset L$. Hence $A \notin \mathcal{C}$. Moreover, $A \notin \mathcal{S}_T$. Therefore $A \notin \mathcal{D}$. On the other hand, for any finite $G \subset A$ the set $\mathcal{D}$-conv G is a subset of a cone from the family $\mathcal{S} \subset \mathcal{S}_T \subset \mathcal{D}$. So $\mathcal{D}$-conv $G \subset A$. Consequently, condition (F_1) does not hold for the family $\mathcal{D}$. By Lemma 2 the family $\mathcal{D}$ is not domain finite.

Lemma 3. *Let S be a dense subset of a convex relatively open set $T \subset R^n$. If, for a convex half-space A, the inclusion $A \supset S$ holds, then the inclusion $A \supset T$ also holds.*

Proof. For $n = 1$ the lemma is obvious. We assume that it is true in the spaces $R^1, \ldots, R^{n-1}$ and consider the space R^n.

If $\operatorname{aff} T = R^n$, then T is an open body, and consequently $T = \operatorname{int} T = \operatorname{int} \bar{T}$ (see [13], Theorem 1.16, p. 13). Analogically, $\operatorname{int} A = \operatorname{int} \bar{A}$. Moreover, $\bar{S} = \bar{T}$ because S is dense in T. Consequently,

$$A \supset \operatorname{int} A = \operatorname{int} \bar{A} \supset \operatorname{int} \bar{S} = \operatorname{int} \bar{T} = T.$$

If $\operatorname{aff} T \neq R^n$, then $A \cap \operatorname{aff} T$ is a convex half-plane of the plane $\operatorname{aff} T$. By the inductive hypothesis we get $A \cap \operatorname{aff} T \supset T$. Hence $A \supset T$.

Theorem 25. *The smallest family $\mathcal{C}$ of sets which contains a given family of convex half-spaces and fulfils conditions* (M), (U) *and* (H) *is domain finite.*

Proof. It is sufficient to prove the equality $\mathcal{C} = \mathcal{C}_F$ because by Lemma 2 the family $\mathcal{C}_F$ is domain finite. Obviously, $\mathcal{C} \subset \mathcal{C}_F$. We shall show the inverse inclusion.

In the trivial case $\mathcal{C} = \{R^n\}$ we have $\mathcal{C}_F = \{R^n\}$. Below we assume that $\mathcal{C}$ contains a set different from the set R^n.

Let $C \in \mathcal{C}_F$. By Theorem 2 the sets $\emptyset$ and R^n are $\mathcal{C}$-convex, and so we consider only the case where $C \neq R^n$ and $C \neq \emptyset$. To prove $C \in \mathcal{C}$ it is sufficient to show that for any $x \notin C$ there exists a $\mathcal{C}$-convex set $V \supset C$ such that $x \notin V$.

We shall first prove that $x \notin \mathcal{C}_F$-$\operatorname{cone}_x C$. Suppose the contrary. Owing to the first part of Theorem 23 we can apply Theorem 4, part 4, to the family $\mathcal{C}_F$. We get

$$\mathcal{C}_F\text{-}\operatorname{cone}_x C = \mathcal{C}_F\text{-conv}\, \operatorname{cone}_x C.$$

Since (F) holds for $\mathcal{C}_F$ and since $x \in \mathcal{C}_F$-conv $\operatorname{cone}_x C$, there exist points $x_1, \ldots, x_m \in \operatorname{cone}_x C$ such that $x \in \mathcal{C}_F$-conv $\{x_1, \ldots, x_m\}$. From Lemma 2 we get

$$\mathcal{C}_F\text{-conv}\, \{x_1, \ldots, x_m\} = \mathcal{C}\text{-conv}\, \{x_1, \ldots, x_m\}.$$

Therefore $x \in \mathcal{C}$-conv $\{x_1, \ldots, x_m\}$. Since $x_1, \ldots, x_m \in \operatorname{cone}_x C$, there exist $\lambda_1 > 0, \ldots, \lambda_m > 0$ such that

$$x_1' = x + \lambda_1 (x_1 - x) \in C, \ \ldots, \ x_m' = x + \lambda_m (x_m - x) \in C.$$

Obviously, $\mathrm{cone}_x\{x_1, \ldots, x_m\} = \mathrm{cone}_x\{x'_1, \ldots, x'_m\}$. Hence Theorem 17 implies $x \in \mathscr{C}\text{-conv}\{x'_1, \ldots, x'_m\}$. By Lemma 2 we have

$$\mathscr{C}\text{-conv}\{x'_1, \ldots, x'_m\} = \mathscr{C}_F\text{-conv}\{x'_1, \ldots, x'_m\}.$$

Consequently, $x \in \mathscr{C}_F\text{-conv}\{x'_1, \ldots, x'_m\}$. Since C is $\mathscr{C}_F$-convex and $\{x'_1, \ldots, x'_m\} \subset C$, we have $\mathscr{C}_F\text{-conv}\{x'_1, \ldots, x'_m\} \subset C$. Thus $x \in C$. This contradicts the assumption $x \notin C$. Hence $x \notin \mathscr{C}_F\text{-cone}_x C$.

We now denote the cone $\mathscr{C}_F\text{-cone}_x C$ by G.

Recurrently, we shall define finite or infinite sequences of $\mathscr{C}$-convex blunt convex half-spaces B_i and blunt convex cones G_i.

We define B_1 and G_1. From Theorem 1 applied to the family $\mathscr{C}_F$ we get $\mathrm{ri}\, G \in \mathscr{C}_F$. Consequently, from the last part of Theorem 23 we get $\mathrm{ri}\, G \in \mathscr{C}$. Since $x \notin \mathrm{ri}\, G \in \mathscr{C}$, by Theorem 15 there exists a $\mathscr{C}$-convex blunt convex half-space $B_1 \supset \mathrm{ri}\, G$ with the vertex x. If $B_1 \supset G$, then B_1 is the required set V, which ends the construction. In the opposite case we put $G_1 = G \setminus B_1$. So $G_1 = (R^n \setminus B_1) \cap G$. Since B_1 is a convex half-space with the vertex x, $R^n \setminus B_1$ is also a convex half-space with the vertex x. Hence $R^n \setminus B_1$ is a convex cone with the vertex x. Since G is a blunt convex cone with the vertex x, $G_1 = (R^n \setminus B_1) \cap G$ is also a blunt convex cone with the vertex x. From $G_1 = G \setminus B_1$ and from $B_1 \supset \mathrm{ri}\, G$ we get $G_1 \cap \mathrm{ri}\, G = \emptyset$, and consequently G_1 lies in the relative boundary of G.

We define B_i and G_i. We assume that the sets $B_1, \ldots, B_{i-1}$ and $G_1, \ldots, G_{i-1}$ are defined. Let

$$Z_{i_1 \ldots i_k} = \mathrm{ri}\,(G_{i_1} \cap \ldots \cap G_{i_k}),$$

where $1 \leqslant k \leqslant i-1$ and $1 \leqslant i_1 < \ldots < i_k \leqslant i-1$. Since the space R^n is separable, in any nonempty set $Z_{i_1 \ldots i_k}$, there exists a dense countable set $S_{i_1 \ldots i_k}$. Let S be a countable dense subset of $\mathrm{ri}\, G$. Let T_i denote the union of S and of all sets $S_{i_1 \ldots i_k}$. Since T_i is countable, by Theorem 23 the equality $\mathscr{C}\text{-conv}\, T_i = \mathscr{C}_F\text{-conv}\, T_i$ holds. From $G \in \mathscr{C}_F$ we get $G \supset \mathscr{C}_F\text{-conv}\, T_i$, i.e. $G \supset \mathscr{C}\text{-conv}\, T_i$. Since $x \notin G$, we have $x \notin \mathscr{C}\text{-conv}\, T_i$. By Theorem 15 there exists a $\mathscr{C}$-convex blunt convex half-space $B_i \supset \mathscr{C}\text{-conv}\, T_i$ with the vertex x. If $B_i \supset G$, then B_i is the required set V. In the opposite case we put $G_i = G \setminus B_i$. Since $G_i = (R^n \setminus B_i) \cap G$, we can show analogically (as for G_1) that G_i is a blunt convex cone with the vertex x. Since $B_i \supset T_i$, we infer from Lemma 3 that B_i contains $\mathrm{ri}\, G$ and all sets $Z_{i_1 \ldots i_k}$. From $G_i = G \setminus B_i$ and from $B_i \supset \mathrm{ri}\, G$ we get $G_i \cap \mathrm{ri}\, G = \emptyset$, and consequently G_i lies in the relative boundary of G. Moreover, $G_i \cap Z_{i_1 \ldots i_k} = \emptyset$ for any $Z_{i_1 \ldots i_k}$.

If in the above construction $B_i \setminus G \neq \emptyset$, $i = 1, 2, \ldots$, then the sequences $B_1, B_2, \ldots$ and $G_1, G_2, \ldots$ are defined.

We shall show that the intersection of any n cones from the sequence $G_1, G_2, \ldots$ is empty. We suppose the contrary, i.e. we suppose $G_{i_1} \cap \ldots \cap G_{i_n} \neq \emptyset$, where $1 \leqslant i_1 < \ldots < i_n$. Let d_j denote the dimension of the set $G_{i_1} \cap \ldots \cap G_{i_j}$, $j = 1, \ldots, n$. Since the convex set G_{i_1} lies in the relative boundary of the convex set G, we have $d_1 < n$. From the construction of the sets $G_1, G_2, \ldots$ it follows that the convex set $G_{i_1} \cap \ldots \cap G_{i_j}$ lies in the relative boundary of the convex set $G_{i_1} \cap \ldots \cap G_{i_{j-1}}$. Hence $d_j < d_{j-1}$ for $j = 2, \ldots, n$. Moreover, $d_n \geqslant 0$ because $G_{i_1} \cap \ldots \cap G_{i_n} \neq \emptyset$. Therefore $n > d_1 > \ldots > d_n \geqslant 0$. Hence $d_n = 0$, i.e. $\dim G_{i_1} \cap \ldots \cap G_{i_n} = 0$. Since the set $G_{i_1} \cap \ldots \cap G_{i_n}$ is convex, it is a one-point set. But, on the other hand, $G_{i_1} \cap \ldots \cap G_{i_n}$ cannot be one-point because it is the intersection of blunt convex cones with the common vertex x. The contradiction implies that the intersection of any n cones from the sequence $G_1, G_2, \ldots$ is empty.

Thus from $G_i = G \setminus B_i$, $i = 1, 2, \ldots$, we conclude that any point of G belongs to all sets $B_1; B_2, \ldots$, with the exception of at most $n-1$ of them.

Hence for any point $a \in G$ there exists an integer i_a such that $a \in B_i$ for $i \geqslant i_a$. Therefore $G \subset \operatorname{Lim\,inf}_{i \to \infty} B_i$. Consequently, $C \subset \operatorname{Lim\,inf}_{i \to \infty} B_i$. Since $B_i \in \mathscr{C}$, $i = 1, 2, \ldots$, by Theorem 1 the set $\operatorname{Lim\,inf}_{i \to \infty} B_i$ is $\mathscr{C}$-convex. Moreover, from $x \notin B_i$, $i = 1, 2, \ldots$, we get $x \notin \operatorname{Lim\,inf}_{i \to \infty} B_i$. Hence $\operatorname{Lim\,inf}_{i \to \infty} B_i$ is the required set V.

The proof is complete.

Finally, we get a theorem concerning B-convexity (see Example 2). The last part of the theorem gives the solution of a problem from [9].

THEOREM 26. *The following families of subsets of R^n are identical with the family $\mathscr{C}_B$ of all B-convex sets*: 1) *the smallest family fulfilling conditions* (M), (U), (H) *and containing the unit ball B*, 2) *the smallest family satisfying conditions* (M), (U), (H) *and containing all closed half-spaces of the form* $B(x)$ [*analogically*: *all open half-spaces of the form* $\mathrm{int}\, B(x)$], 3) *the smallest family fulfilling conditions* (M), (U), (H) *and containing all closed balls.*

Proof. 1. The family of all B-convex sets fulfils conditions (M), (U), (H) and contains the unit ball B. Moreover, the intersection of any number of families of sets fulfilling conditions (M), (U), (H) and containing B is also such a family. Thus there exists the smallest family $\mathscr{K}$ satisfying conditions (M), (U), (H) and containing B.

Obviously, $\mathscr{K}$ is a subfamily of the family of B-convex sets.

On the other hand, let A be a B-convex set. Owing to Theorem 21 we can apply Theorem 25 to the family $\mathscr{K}$. Therefore $\mathscr{K}$ is domain finite. Thus by Lemma 2 condition (F_1) holds for $\mathscr{K}$. Let G be a finite subset of A. Since any closed ball belongs to $\mathscr{K}$, the set $\mathscr{K}\text{-conv}\, G$ is a subset of the intersection of all closed balls containing G. Thus by the B-convexity of A we have $\mathscr{K}\text{-conv}\, G \subset A$. From (F_1) we conclude that $A \in \mathscr{K}$.

2. This results from the first property and from the proof of Theorem 21.

3. Let $\mathcal{G}$ denote the family of all open B-convex half-spaces. By the preceding part of the theorem $\mathcal{C}_B$ is the smallest family which satisfies conditions (M), (U), (H) and the inclusion $\mathcal{C}_B \supset \mathcal{G}$. From Theorem 13 we get $\mathcal{C}_B = \mathcal{G}_{TL'M}$. Since $\mathcal{G}_T = \mathcal{G}$, $\mathcal{G}_{L'}$ is an intersection basis of the family $\mathcal{C}_B$. Consequently, we can repeat the proof of Theorem 18 in [9] for any natural number n. We get the last conclusion of Theorem 26.

The proof is complete.

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Solenoids and inverse limits of sequences of arcs with open bonding maps

by

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Abstract. The class $\mathcal{K}$ of inverse limits of sequences of arcs with open bonding maps is characterized as the class of chainable continua with property K, with one or two end-points and with arcs as proper subcontinua. Also it is proved that each monotone image of X, where X is from $\mathcal{K}$ or from the class of solenoids, is homeomorphic to X.

Introduction. Let us denote by $\mathcal{K}$ the class of inverse limits of sequences of arcs with open bonding maps.

In this paper we establish some analogies between solenoids and class $\mathcal{K}$. Next, we give a characterization of continua from $\mathcal{K}$ as chainable continua with property K, with one or two end-points and with arcs as proper subcontinua. This answers Problem 2 in [7] and corresponds to the characterization of solenoids in [7]. Finally, it is shown that a monotone image of X from $\mathcal{K}$ is homeomorphic to X and that the same holds for solenoids. Thus both of these classes provide examples of the continua which J. J. Charatonik asks about in [4].

Preliminaries. By a continuum we mean a compact, connected, metric nondegenerate space. Denote by I the interval $[0, 1]$. For each integer $s \geqslant 1$ let w_s denote the map of I onto I such that $w_s(i/s) = 0$ if i is even, $w_s(i/s) = 1$ if i is odd, where $0 \leqslant i \leqslant s$ and w_s is linear on each interval $[i/s, (i+1)/s)$ for $0 \leqslant i < s$.

It is known from [9, Lemma 1, page 453 and Theorem 7, page 455] that class $\mathcal{K}$ is topologically equal to the class of inverse limits of sequences $\{I, f_i\}$, where, for each i, $f_i = w_s$ for some s. So, each continuum $K \in \mathcal{K}$ is determined by a sequence of natural numbers $(s_1, s_2, \ldots)$ such that $K = \operatorname{inv\,lim} \{I, w_{s_k}\}$. We will denote such K by $K(s_1, s_2, \ldots)$.

A chain (circular chain) is a finite collection of open sets $\{U_1, \ldots, U_m\}$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$ ($|i-j| \leqslant 1$ or $i = 1$ and $j = m$). A subchain of a chain $\mathcal{U}$ between links U_i and U_j will be denoted by $\mathcal{U}(i, j)$.

A chain $\mathcal{U}^2 = \{U_1^2, \ldots, U_m^2\}$ refines a chain $\mathcal{U}^1 = \{U_1^1, \ldots, U_k^1\}$ if there is a function α: $\{1, \ldots, m\} \to \{1, \ldots, k\}$ such that $U_i^2 \subset U_{\alpha(i)}^1$ for every i.

A chain $\mathcal{U}^2$ is of *type* s in a chain $\mathcal{U}^1$ if $\mathcal{U}^2$ refines $\mathcal{U}^1$ and if there is