

Subopen multifunctions and selections

by

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Abstract. Let $m: X \to Y$ be a subopen (e.g., open graph) multifunction with infinitely connected values. If (X, A) is a relative CW complex it is proved that any continuous partial selection $g: A \to Y$ for m can be extended to a continuous selection $f: X \to Y$ for m. A fixed point corollary is also given.

Let X and Y be topological spaces. A set valued function $m: X \to Y$ will be called a *multifunction* if $m(x) \neq \emptyset$, all x in X. $G(m) = \{(x, y) | y \in m(x)\} \subset X \times Y$ is the graph of m. Let $p: G(m) \to X$ be the projection.

DEFINITION. *m* is a subopen multifunction if for each *x* in *X* there is an open nbhd *U*, a Serre fibration $q: T \to U$, and an embedding $e: p^{-1}(U) \to T$ such that $e(p^{-1}(U))$ is open in *T* and qe = p.

Recall that m is an open-graph multifunction if G(m) is an open subset of $X \times Y$. Use U = X and $T = X \times Y$ to see that every open graph multifunction is a subopen multifunction.

The selection theorem referred to in the title is the following.

1.1. THEOREM. Let (X,A) be a relative CW complex and $m: X \to Y$ a subopen multifunction with m(x) infinitely connected for all x in X. Then any continuous partial selection $g: A \to Y$ for m can be extended to a continuous selection $f: X \to Y$ for m. Furthermore, any two such extensions are homotopic by a homotopy $\{f_i\}$ with each f_i a selection for m extending g.

This will be deduced from the following main result:

1.2. Theorem. Let $T \rightarrow B$ be a Serre fibration and E open in T. Assume that each E(b) is N-connected and non-empty. Then $E \rightarrow B$ is an N-fibration.

Theorem 1.1 generalizes the result of [2] and [3] in that there are no metric hypotheses on the domain, there are no conditions on the codomain at all, and "open-graph" is replaced by "sub-open".

The method of proof of 1.1 is to first use Theorem 1.2 to prove that $G(m) \to X$ is a Serre fibration and then apply obstruction theory.

Theorem 1.2 is related to results of Wong [5], however, the metric hypotheses used in [5] are avoided here. As a corollary to 1.2 (or 1.1) we obtain a lifting theorem which can be viewed as a generalization of [5, Th. 2.1, Cor. 2.2]. In Section 2 an ANR version of Theorem 1.1 is proved and a

multifunction fixed point theorem is given which generalizes slightly a result of [3].

If it is not assumed that m has contractible values then more elaborate hypotheses are required for selection theorems. This will be discussed in a separate paper.

1. Notation, proof of Theorem 1.1 using Theorem 1.2. A single valued (continuous) function $f: X \to Y$ is a (continuous) selection for $m: X \to Y$ if $f(x) \in m(x)$, all x in X. If $A \subset X$ and $g: A \to Y$ is a single valued function such that $g(x) \in m(x)$, all x in A, then g is called a partial selection for m (A may be empty).

A map is a continuous function. I is the unit interval. If $c\colon C\to B$ and $d\colon D\to B$ are given maps then a map $f\colon C\to D$ is over B if df=c. Let $p\colon B\times Z\to B$ and $r\colon B\times Z\to Z$ be the projections. If $j\colon E\subset B\times Z$ then pj and rj will be written as p and r. If $q\colon T\to B$ is a map and E open in E the composite $E\subset T\to B$ will be denoted by E and E are E and E and E are E are E and E are E and E are E and E are E and E are E are E and E are E are E and E are E are E are E and E are E are E and E are E are E are E are E and E are E are E are E and E are E are E are E are E and E are E are E are E and E are E are E are E and E are E are E are E are E are E are E and E are E are E are E are E are E and E are E and E are E are E are E and E are E are E are E are E are E and E are E are E are E and E are E are E are E are E are E and E are E are E and E are E are E are E are E and E are E are E are E and E are E are E are E are E and E are E and E are E are

A space is called N-connected if every map $\partial I^{j+1} \to X$ can be extended to a map $I^{j+1} \to X$, $0 \le j \le N$. X is infinitely connected if it is N-connected for all N.

Proof of Theorem 1.1 assuming $G(m) \rightarrow X$ is a Serre fibration. Consider

$$\begin{array}{ccc}
A & \xrightarrow{g'} & G(m) \\
 & & & & \\
X & & & & X
\end{array}$$

where g'(a) = (a, g(a)). $G(m) \to X$ is a surjective Serre fibration with fiber m(x) which is infinitely connected — so standard obstruction theory (e.g. [4, p. 404, Th. 22, p. 416, Th. 9]) gives an extension $f' \colon X \to G(m)$ of g', over X, and shows that two such extentions are homotopic, rel(A), over X. Let f = rf'. Then f is the extension of g required in the theorem and if f_1 and f_2 are two such and $H \colon f'_1 \sim f'_2$ the given homotopy then $rH \colon f_1 \sim f_2$ is the homotopy required in the theorem.

We are left with the problem of proving that $G(m) \to X$ is a Serre fibration. Let us say that a map $E \to B$ is an N-fibration if it has the homotopy lifting property for CW complexes of dimension $\leq N$. A Serre fibration is map which is an N-fibration for all N. To prove $E \to B$ an N-fibration it suffices [4, p. 375, p. 416] to prove that it has the homotopy lifting property for cubes of dimension $\leq N$. By breaking a cube into smaller cubes it is easily shown that $E \to B$ is an N-fibration if B has an open cover



 $\{U\}$ such that each $E(U) \to U$ is an N-fibration. In our case $G(m) \to X$ is locally an open subset of a Serre fibration so that the theorem will follow from Theorem 1.2 below.

1.2. THEOREM. Let $T \rightarrow B$ be a Serre fibration and E open in T. Assume each E(b) is N-connected and non-empty. Then $E \rightarrow B$ is an N-fibration.

In proving 1.2 the following lemma will be useful:

1.3. Lemma. Suppose $T \to B$ a Serre fibration, E open in T. Let (X, A) be a relative CW complex, $f: X \to E$ a map, $G: A \times I \to E$ a homotopy with $G_0 = f|A$ and pG(a, t) = pf(a), all a, t. Then G extends to a homotopy $H: X \times I \to E$ with $H_0 = f$ and pH(x, t) = pf(x), all x, t.

Proof of Lemma 1.3. Let $Q = X \times O \cup A \times I \subset X \times I$ and $G' = f \cup G$: $Q \to E$. So we must find a map $H: X \times I \to E$ extending G' and over B (where $X \times I \to B$ is $pf\pi$, π the projection). Let $j: E \to T$ be the inclusion. Because $T \to B$ is a Serre fibration jG' has an extension to $g: X \times I \to T$ over B [4, p. 416, proof of Th. 9]. Since E is open $V = g^{-1}(E)$ is an open nbhd of Q and if F is the restriction of G to G then G to G then G is an extension of G' over G. Because G is closed in G which is normal there is a map G is an extension of G' over G is defined and G in G in G in G in G in G is compact, so get G is G if G if G if G in G in G in G if G in G in

2. Proof of Theorem 1.2 and corollaries. The proof is by induction on N. First take N=0. Consider:

$$\begin{array}{ccc}
0 & \xrightarrow{g} & E & \subset T \\
\downarrow i & & \downarrow p \\
I & \xrightarrow{f} & B
\end{array}$$

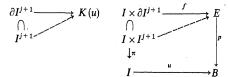
Given that pg = fi it is necessary to find an extension $F: I \to E$ of g over B. For $t \in I$ choose $e(t) \in p^{-1}f(t) \subset E$. Because $T \to B$ is a 0-fibration there is an $h: I \to T$, over B, with $h(t) = e(t) \in E$. $h^{-1}(E)$ is open in I so t has an open nbhd W in I and a map $u: W \to E$ over B (u is a restriction of h). Since I is compact there are $0 = a_0 < a_1 < \ldots < a_n = 1$ with $F_i: [a_{i-1}, a_i] \to E$, $pF_i = f$.

It is now necessary to modify each F_i to F_i' so that F_1' extends g and $F_i'(a_i) = F_{i+1}'(a_i)$. Then the F_i' fit together to give the desired F. Let $F_1(0) = c$, g(0) = d. Since E(f(0)) is path connected there is a homotopy $G: 0 \times I \to E$ of $F_1[0]$ which is over B from C to C. By Lemma 1.3 (with C = C = C 0, C = C 0 the homotopy extends to a homotopy of C 1, which is over C 1. Let C 1 be the other end of the homotopy so that C 1 from C 1 is modified to C 2 so that C 2 from C 2 from C 3 to C 3.

 $0 \le i \le N-1$

Now assume 1.2 for $0 \le M < N$ and prove it for N. Consider

To prove $E \to B$ an N-fibration it is necessary to show the dashed arrow exists in the first diagram. Now let E^I by the space of maps I to E with the compact-open topology. By adjointness (replacing maps $P \times Q \to R$ by maps $P \to R^Q$) this is equivalent to finding the dashed arrow in the second diagram, i.e., showing $E^I \to B^I$ is an (N-1)-fibration. Now E is open in T so E^I is open in T^I and also, $T^I \to B^I$ is a Serre fibration. By the induction hypothesis it will suffice to prove K(u) is (N-1)-connected and non-empty for each $u \in B^I$ where $K(u) = \{h: I \to E | ph = u\}$. The fact that K(u) is non-empty follows easily from the fact that $E \to B$ is a 0-fibration. Consider



It suffices to find the dashed arrow in the first diagram – but, again by adjointness, this is equivalent to finding F in the second diagram. Taking the part of the second diagram over t and u(t) gives

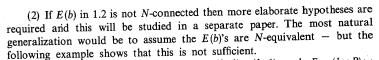
$$t \times \partial(I^{j+1}) \xrightarrow{f_t} E(u(t))$$

$$t \times I^{j+1} \xrightarrow{e_t} E(u(t))$$

and e_t exists and is unique up to homotopy rel (∂I^{j+1}) since E(u(t)) is N-connected. Let $Q(t) = I \times \partial (I^{j+1}) \cup t \times I^{j+1}$ and $f_t = f \cup e_t$: $Q(t) \to E$. Because $T \to B$ is a Serre fibration jf_t has an extension H_t : $I \times I^{j+1} \to T$ over B. $H_t^{-1}(E)$ is an open nbhd of Q(t) and by restriction we obtain a map F_t : $W \times I^{j+1} \to E$ extending f_t over f_t where f_t is a nbhd of f_t in f_t . Since f_t is compact there are $f_t = a_t < a_t$

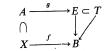
Now the F_i must be modified. Let $F_1' = F_1$ and $F_1'|a_1 \times I^{j+1} = c$, $F_2|a_1 \times I^{j+1} = d$. Because $E(u(a_1))$ is N-connected d and c are homotopic $\operatorname{rel}(a_1 \times \partial I^{j+1})$ as maps to $E(u(a_1)) \subset E$. Now Lemma 1.3 can be applied (with $X = [a_1, a_2] \times I^{j+1}$, $A = a_1 \times I^{j+1}$) to give F_2' extending f and over B with $F_2'|a_1 \times I^{j+1} = c$. The other F_i are modified by the same procedure and they then fit together to give the required F.

Notes. (1) The proof shows that if $T \to B$ is an (N+1)-fibration then $E \to B$ is an N-fibration.



Let B = I, $P = (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2})$, $Z = (0, \frac{4}{2}) \cup (\frac{1}{2}, 1)$ and $E = (I \times P) \cup ((\frac{1}{2}, 1] \times Z) \subset I \times Z$. Then E is open in $I \times Z$. Let $p: E \to I$ be the composition $E \subset I \times Z \to I$. All of the fibers E(t) are homotopically equivalent (to the discrete space with two points) but $E \to I$ is not a 0-fibration since the identity path: $I \to I$ in the base can not be lifted with given end point $(1, \frac{3}{4})$.

Now consider the following situation



2.1. COROLLARY. Suppose $T \to B$ a Serre fibration, E open in T, each E(b) non-empty and infinitely connected, (X, A) a relative CW complex. Then there is a map $F \colon X \to E$ extending g over B and any two such extensions are homotopic.

Proof. By 1.2 $E \rightarrow B$ is a Serre fibration and the result follows by standard obstruction theory, e.g. [4, p. 404, Th. 22, p. 416, Th. 9].

2.2. COROLLARY (to the proof). Suppose X is a compact finite dimensional ANR and $m: X \to Y$ a subopen multifunction with each m(x) infinitely connected. Then m has a continuous selection and any two such are homotopic by a homotopy $\{f_t\}$ with each f_t a selection for m.

Proof. Theorem 1.2 and the paragraph preceding it show that $G(m) \to X$ is a Serre fibration. By [1, p. 122] X is a retract of a finite dimensional polyhedron, say $i: X \to P$, $u: P \to X$, ui = id. As in the first paragraph of the proof of Theorem 1.1 there is a lifting $f': P \to G(m)$, pf' = u, and any two such liftings are homotopic. Let f = rf'i. Then f is the desired selection and the needed homotopy is similarly defined.

2.3. COROLLARY. Suppose X a compact finite dimensional contractible ANR and $m: X \to X$ a subopen multifunction with contractible values. Then m has a fixed point (i.e., $x \in m(x)$).

Proof. Corollary 2.2 gives a selection and the Brouwer fixed point theorem applies to the selection to give a fixed point.

Notes. (1) Corollary 2.1 can also be deduced directly from Theorem 1.1. Corollary 2.1 generalizes [5, 2.1, 2.2].

(2) Corollary 2.3 generalizes the fixed point theorem of [3]. The results of the present paper will be used in a separate paper to obtain a more general fixed point theorem of the Lefschetz type.

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Notes on topological games

by

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Abstract. The topological game G(K, X) is in the sense of R. Telgársky. Let K and K' be classes of spaces with $K \subset K'$. It is studied in this paper when Player I has a winning strategy in G(K, X) if he has one in G(K', X). We will discuss three questions of this kind.

§ 1. Each space considered here means a topological space and no separation axioms are assumed unless otherwise stated. Throughout this paper, K denotes a class of spaces which are hereditary with respect to closed subspaces. We need no other assumptions of K. When we consider such two classes of spaces, they are denoted by K_1 and K_2 .

The topological game G(K, X) is introduced and studied by R. Telgársky [4]. The detail is seen in it. For a class K, I(K) denotes the class of all spaces X for which Player I has a winning strategy in G(K, X), and the class I(I(K)) is abbreviated to $I^2(K)$. Moreover, DK, LK and SK denote the classes of all spaces being discrete unions of spaces from K, locally K and K-scattered, respectively.

The purpose of this paper is to study the following three questions:

- (A) What kind of a space X, does $X \in I^2(K)$ imply $X \in I(K)$ for?
- (B) What kind of a space X, does $X \in I(SK)$ imply $X \in I(DK)$ for?
- (C) What kind of a space X, does $X \in I(K_1) \cap I(K_2)$ imply $X \in I(K_1 \cap K_2)$ for?

In § 2, § 3 and § 4, the questions (A), (B) and (C) are answered, respectively. Though the question (B) has been already studied by R. Telgársky [6], we give here another result and the improvement of his ones.

Concerning the topological game G(K, X), we use the notations in [7] rather than in [4]. Here, we do not restate them except the following. Let s be a strategy of Player I in G(K, X). A finite sequence $\langle F_0, F_1, \ldots, F_n \rangle$ of closed sets in X is said to be an admissible choice of Player II for s in G(K, X) (ad. ch. for II (s, K, X)) if $F_0 = X$ and the sequence $\langle E_1, F_1, \ldots, E_n, F_n \rangle$, such that $E_i = s(F_0, F_1, \ldots, F_{i-1})$ for $1 \le i \le n$, is admissible for G(K, X). If s is a winning strategy of Player I in G(K, X), then it should be noted that each infinite sequence $\langle F_0, F_1, \ldots \rangle$ of closed sets in X, such that $\langle F_0, F_1, \ldots, F_n \rangle$ is an ad. ch. for II (s, K, X) for each $n \ge 1$, has