[3] K. Morita, *Products of normal spaces with metric spaces*, Math. Ann. 154 (1964), pp. 365–382.
[4] R. Telgársky, *Spaces defined by topological games*, Fund. Math. 88 (1975), pp. 193–223.
[5] — *A characterization of P-spaces*, Proc. Japan Acad. 51 (1975), pp. 802–807.
[6] — *Spaces defined by topological games, II*, Fund. Math. 116 (1983), pp. 189–207.
[7] Y. Yajima, *Topological games and products III*, Fund. Math. 117 (1983), pp. 223–238.

DEPARTMENT OF MATHEMATICS
KANAGAWA UNIVERSITY
Yokohama, 221
Japan

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Fixed point index for open sets in euclidean spaces*

by

G. S. Skordev (Sofia)

Abstract. Using chain approximations of multi-valued mappings a fixed point index for a large class of such mappings of open sets in euclidean spaces is constructed. This fixed point index satisfies all usual properties of fixed point index for single valued maps including commutativity and mod-p property.

Introduction. The aim of the present note is to give an unified approach to fixed point theory for single-valued as well as for certain classes of multi-valued mappings on locally compact polyhedra and in particular on open sets in euclidean spaces. A fixed point index with all usual properties (additivity, homotopy invariance, normalization, commutativity and mod-p property) is constructed. In particular for single-valued maps on open sets in euclidean spaces we obtain the classical theory [5, 9]. The main idea is to use certain chain approximations of a given map and to localize the Lefschetz number of these chain approximations. In the global Lefschetz fixed point theory of multi-valued maps the chain approximations are used in [2, 8, 23, 28–32]. In the case of single-valued maps the fixed point index is defined as local Lefschetz number of chain approximation in [4, 10, 18, 22]. In the case of multi-valued maps on compact polyhedra these two approaches were used in [27] to define a fixed point index with all properties. For a review of results, applications and problems of the fixed point index theory see [12, 13, 15, 25].

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§ 0. Notations. We consider maps $\Phi\colon X \to Y$ for which the sets $\Phi(x)$ are not empty and compact for every $x \in X$. The map Φ is called *upper-semi-continuous* (u.s.c) if for every open set U in Y the set

$$\Phi^{-1}(U) = \{x \in X\colon \Phi(x) \subset U\}$$

is open, [16], ch. 4, p. 32.

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Let $X_1 \subset X$, $Y_1 \subset Y$ and $\Phi(X_1) \subset Y_1$. By $\Phi_{X_1Y_1}\colon X_1 \to Y_1$ we denote the map defined as $\Phi_{X_1Y_1}(x) = \Phi(x)$ for every x in X_1.

With $\operatorname{Int} Y_1$ we denote the interior of the set Y_1 in Y, and by $\bar{Y}$ – the closure of Y_1 in Y.

By N we denote the natural numbers and by F – given field.

Suppose K is a polyhedron with a triangulation τ. By τ^k we denote the kth barycentric subdivision of τ. By $C_*(K, k)$ denote the chain group of τ^k with coefficients F, [33].

Let $\varphi\colon C_*(K, k) \to C_*(L, l)$ be a chain map, and L – polyhedron. If K_1 is a subpolyhedron in τ^k by $\varphi|K_1$ we denote the chain map

$$\varphi|C_*(K_1, k)\colon C_*(K_1, k) \to C_*(L, l).$$

For a given space X by $\check{H}_*(X)$ we denote Čech homology with coefficients F, [33], ch. 6. We say that the u.s.c. map Φ is *F-acyclic* (or *acyclic*) if for every $x \in X$ the set $\Phi(x)$ is connected and $\check{H}_i(\Phi(x)) = 0$ for $i > 0$. In this note we consider F-acyclic maps.

§ **1. Approximation systems.** Here we use the notion of A-system for a given u.s.c. map (with respect to a given and fixed field F). For the definition and properties of A-systems see [27], ch. 2. By K and L we denote locally compact polyhedra.

(1.1) Definition. Let $\Phi\colon K \to L$ be an u.s.c. map. Suppose that for every compact polyhedra M and N with $M \subset K$, $\Phi(M) \subset \operatorname{Int} N \subset N \subset L$ is given an A-system $A(\Phi_{MN})$ for Φ_{MN}. Then we say that $\mathscr{A}(\Phi) = \{A(\Phi_{MN})\}$ is *F-$\mathscr{A}$-system* (or simple *$\mathscr{A}$-system*) for Φ provided: for every compact polyhedra M_i, N_i with $M_1 \subset M_2 \subset K$, $N_1 \subset \operatorname{Int} N_2 \subset N_2 \subset L\Phi(M_i) \subset \operatorname{Int} N_i$ there is k_0 such that for all $k > k_0$ there is φ in $A(\Phi_{M_2N_2})_k$ with $\varphi|M_1 \in A(\Phi_{M_1N_1})_k$.

(1.2) Remarks. (i) Let $\Psi\colon P \to Q$ be an u.s.c. map and P, Q – compact polyhedra. Then every A-system $A(\Psi)$ for Ψ induces an $\mathscr{A}$-system for Ψ. For M and N – compact polyhedra with $M \subset P$, $\Psi(M) \subset \operatorname{Int} N \subset N \subset Q$ denote by $A(\Psi_{MN})_k$ the set

$$\{\varphi \in A(\Psi)_k\colon \varphi(C_*(M, k)) \subset C_*(N, k)\}.$$

Since M is a compact set and Ψ – u.s.c. map and $\Psi(M) \subset \operatorname{Int} N$, then $A(\Psi_{MN}) = \{A(\Psi_{MN})_k\}$ is an A-system for the map Ψ_{MN}. It is easy to see that $\{A(\Psi_{MN})\}$ is an $\mathscr{A}$-system for Ψ. We denote this $\mathscr{A}$-system also by $A(\Psi)$.

(ii) Let $\Phi\colon K \to L$ be an acyclic map. For compact polyhedra M and N with $M \subset K$, $\Phi(M) \subset \operatorname{Int} N \subset N \subset L$ consider $A^*(\Phi_{MN})$ – the A-system induced by Φ_{MN}, [27], ch. 3, § 1, (3.7). It is easy to check that $\mathscr{A}^*(\Phi) = \{A^*(\Phi_{MN})\}$ is an $\mathscr{A}$-system. We call this $\mathscr{A}$-system *$\mathscr{A}$-system induced by Φ*.

(1.3) Restriction of $\mathscr{A}$-systems. Let $\Phi\colon K \to L$ be an u.s.c. map and $\mathscr{A}(\Phi)$ – an $\mathscr{A}$-system for Φ. If K_1 is a subpolyhedron in K and L_1 – subpolyhedron in L with $\Phi(K_1) \subset L_1$, then

$$\{A(\Phi_{MN}) \in \mathscr{A}(\Phi)\colon M \subset K_1,\ N \subset L_1\}$$

is $\mathscr{A}$-system for the map $\Phi_{K_1L_1}$. We denote this $\mathscr{A}$-system by $\mathscr{A}(\Phi)_{K_1L_1}$.

In case $L = L_1$ we write $\mathscr{A}(\Phi)_{K_1}$ instead of $\mathscr{A}(\Phi)_{K_1L}$.

(1.4) Homotopy of $\mathscr{A}$-systems. Let Φ_1, $\Phi_2\colon K \to L$ be u.s.c. maps and $\mathscr{A}(\Phi_1)$, $\mathscr{A}(\Phi_2)$ – $\mathscr{A}$-systems for Φ_1, and Φ_2. Let $H\colon K \times I \to L$ be u.s.c. and $H(x, 0) = \Phi_1(x)$, $H(x, 1) = \Phi_2(x)$, x in K, i.e., H is a homotopy between the maps Φ_1 and Φ_2, $I = [0, 1]$. We say that the $\mathscr{A}$-systems $\mathscr{A}(\Phi_1)$ and $\mathscr{A}(\Phi_2)$ are *H-homotop* if for every compact polyhedra M, N with $M \subset K$, $H(M \times I) \subset \operatorname{Int} N \subset N \subset L$ the A-systems $A(\Phi_{1MN})$ and $A(\Phi_{2MN})$ are $H_{M \times I, N}$-homotop, [27], ch. 2, § 3, (2.12).

Here $A(\Phi_{iMN}) \in \mathscr{A}(\Phi_i)$, $i = 1, 2$.

(1.5) Remarks. (i) Let Ψ_1, $\Psi_2\colon P \to Q$ be u.s.c. maps, P, Q – compact polyhedra and $A(\Psi_1)$, $A(\Psi_2)$ – A-systems for Ψ_1, Ψ_2. Let $h\colon P \times I \to Q$ be u.s.c. homotopy between Ψ_1, Ψ_2, and $A(\Psi_1)$, $A(\Psi_2)$ are h-homotop A-system. Then $A(\Psi_1)$ and $A(\Psi_2)$ are h-homotop $\mathscr{A}$-systems, see remark (1.2) (i).

(ii) Let Φ_1, $\Phi_2\colon K \to L$ be acyclic maps and $H\colon K \times L \to L$ acyclic map, homotopy between Φ_1 and Φ_2. Using Lemma (4.3), § 2, ch. 4, [27] we obtain that the induced $\mathscr{A}$-systems $\mathscr{A}^*(\Phi_1)$ and $\mathscr{A}^*(\Phi_2)$ are H-homotop.

(1.6) Composition of $\mathscr{A}$-systems. Let K_1, K_2, K_3, be locally compact polyhedra and $\Phi_1\colon K_1 \to K_2$, $\Phi_2\colon K_2 \to K_3$ – u.s.c. maps. Let $\mathscr{A}(\Phi_i)$ be $\mathscr{A}$-systems for Φ_i, $i = 1, 2$. Using $\mathscr{A}(\Phi_i)$ we shall define $\mathscr{A}$-system for $\Phi_2 \circ \Phi_1$.

Consider the composition $A(\Phi_{2NP}) \circ A(\Phi_{1MN})$, here $A(\Phi_{1MN}) \in \mathscr{A}(\Phi_1)$ and $A(\Phi_{2NP}) \in \mathscr{A}(\Phi_2)$. From Lemma (2.16), § 4, ch. 2, [27] this composition is an A-system for the map $(\Phi_2 \cdot \Phi_1)_{MP}$. It is easy to check that

$$\{A(\Phi_{2NP}) \circ A(\Phi_{1MN})\colon A(\Phi_{2NP}) \in \mathscr{A}(\Phi_2),\ A(\Phi_{1MN}) \in \mathscr{A}(\Phi_1)\}$$

is an $\mathscr{A}$-system for $\Phi_2 \circ \Phi_1$. We call this $\mathscr{A}$-system – *composition of the $\mathscr{A}$-systems $\mathscr{A}(\Phi_1)$ and $\mathscr{A}(\Phi_2)$* and denote it by $\mathscr{A}(\Phi_2) \circ \mathscr{A}(\Phi_1)$.

(1.7) Remarks. (i) Let $\Phi\colon P \to Q$ be an u.s.c. map and $\Phi = \varphi_n \circ \ldots \circ \varphi_1$ where $\varphi_i\colon P_i \to P_{i+1}$ is F-acyclic map and P_i – compact polyhedron, $i = 1, \ldots, n+1$, $P_1 = P$, $P_{n+1} = Q$. From [27], Example 1, § 1, ch. 3 we know that the decomposition $\varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1$ of the map Φ induces an A-system $A^*(\Phi)$ for $\Phi\colon A^*(\varphi_n) \circ \ldots \circ A^*(\varphi_1)$, where $A^*(\varphi_i)$ are induced A-systems. The A-system $A^*(\Phi)$ is called *induced by* Φ. More general, if $\Psi\colon P \to Q$, we say that Ψ admits F-acyclic (acyclic) decomposition if $\Psi = \psi_n \circ \ldots \circ \psi_1$ where $\psi_i\colon C_i \to C_{i+1}$ are F-acyclic maps and X_i – compact Hausdorff spaces, $C_1 = P$, $C_{n+1} = Q$, [26, 28]. Using Vietoris–Begle construction of the induced chain homomorphism for acyclic maps ψ_i, [1, 28, 36], we construct an A-system $A(\Psi)$ as follows: let τ be a triangulation of P and μ triangulation of Q. Let $A^*(\Psi)_k$ be the set of all chain maps

$$\varphi\colon C_*(P, k) \to C_*(Q, k)$$

constructed for the composition of acyclic maps $\psi_n \circ \ldots \circ \psi_1$ as in Lemma 1,

[28]. Then it is easy to check that

$$A^*(\Psi) = \{A^*(\Psi)_k\}$$

is an A-system for Ψ. Obviously this A-system depends on the decomposition $\psi_n \circ \ldots \circ \psi_1$. We call it A-system for Φ induced by the decomposition $\psi_n \circ \ldots \circ \psi_1$. Denote by $\mathscr{P}$ the class of all maps which admit F-acyclic decomposition.

(ii) Let $\Psi\colon K \to L$ be an u.s.c. map. We say that Ψ has an acyclic decomposition if $\Psi = \psi_n \circ \ldots \circ \psi_1$ where

(a) $\psi_i\colon X_i \to X_{i+1}$ — acyclic maps,

(b) X_i — Hausdorff spaces, $X_1 = K$, $X_{n+1} = L$.

Let M be a compact polyhedron in K, N compact polyhedron in L such that $\Psi(M) \subset \operatorname{Int} N$. Denote by $C_i = \psi_i \psi_{i-1} \ldots \psi_1(M)$ and by $\psi_i'\colon C_i \to C_{i+1}$ the map for which $\psi_i'(x) = \psi_i(x)$ for every $x \in C_i$. Then the map $\Psi_{MN}\colon M \to N$ belongs to $\mathscr{P}$ ($\psi_n' \psi_{n-1}' \ldots \psi_1'$ is a F-acyclic decomposition for Ψ_{MN}). Let $A^*(\Psi_{MN})$ be the A-system for Ψ_{MN} induced by the decomposition $\psi_n \circ \ldots \circ \psi_1$. Then

$$\mathscr{A}(\Psi) = \{A^*(\Psi_{MN})\}$$

is $\mathscr{A}$-system for Ψ. We call it — $\mathscr{A}$-system induced by the decomposition $\psi_n \circ \ldots \circ \psi_1$. Denote by $\mathscr{P}_1$ the class of all pairs $(\Psi, \psi_n \circ \ldots \circ \psi_1)$ where $\Psi\colon K \to L$ and $\psi_n \circ \ldots \circ \psi_1$ is the acyclic decomposition for the map Ψ. Remark that a given map could have different acyclic decompositions.

(iii) There are different possibilities to define homotopy in the class $\mathscr{P}_1$. We shall consider the following one.

Let $\varphi_i\colon X_i \to X_{i+1}$, $1 \leqslant i \leqslant n$, $i \neq s$, $H\colon X_s \times I \to X_{s+1}$, be acyclic maps, X_1, X_{n+1} — locally compact polyhedra. The homotopy H induces the family of maps $H_t\colon X_s \to X_{s+1}$, defined as $H_t(x) = H(x, t)$, $x \in X$, $t \in I$.

Let $\varphi_s' = H_0$, $\varphi_s'' = H_1$. Then we say that the decompositions $\varphi_n \circ \ldots \circ \varphi_{s+1} \circ \varphi_s' \circ \varphi_{s-1} \circ \ldots \circ \varphi_1$ and $\varphi_n \circ \ldots \circ \varphi_{s+1} \circ \varphi_s'' \circ \varphi_{s-1} \circ \ldots \circ \varphi_1$ are elementary homotop in $\mathscr{P}_1$ (obviously the maps are homotop with the homotopy $F_t = \varphi_n \ldots \varphi_{s+1} H_t \varphi_{s-1} \ldots \varphi_1$).

Let $\varphi_n \circ \ldots \circ \varphi_1$ and $\psi_n \circ \ldots \circ \psi_1$ be in $\mathscr{P}_1$. We say that they are homotop in $\mathscr{P}_1$ if there is a finite number elements $\vartheta_1, \ldots, \vartheta_m$ in $\mathscr{P}_1$ such that ϑ_i is an elementary homotopic in $\mathscr{P}_1$ with ϑ_{i+1} and $\vartheta_1 = \varphi_n \circ \ldots \circ \varphi_1$ and $\vartheta_m = \psi_n \circ \ldots \circ \psi_1$.

(1.8) Homotopy of compositions of $\mathscr{A}$-systems. Using the same technique as in the proof of Lemma (2.16), § 4, ch. 2, [27] we prove

(1.9) Lemma. *Let $\Phi_1, \Phi_1'\colon P_1 \to P_2$, $\Phi_2, \Phi_2'\colon P_2 \to P_3$, $H_1\colon P_1 \times I \to P_2$, $H_2\colon P_2 \times I \to P_3$ be u.s.c. maps, P_i — compact polyhedra and $A(\Phi_i)$, $A(\Phi_i')$ — A-systems for Φ_i, Φ_i', $i = 1, 2$. If $A(\Phi_i)$ and $A(\Phi_i')$ are H_i-homotop, then $A(\Phi_2) \circ A(\Phi_1)$ and $A(\Phi_2') \circ A(\Phi_1')$ are H-homotop where $H_t = H_{2t} \circ H_{1t}$, $t \in I$.*

Now from Lemma (1.9) we obtain

(1.10) Lemma. *Let $\Phi_1, \Phi_1'\colon K_1 \to K_2$, $\Phi_2, \Phi_2'\colon K_2 \to K_3$, $H_1\colon K_1 \times I \to K_2$, $H_2\colon K_2 \times I \to K_3$ be u.s.c. maps and K_i — locally compact polyhedra. Let $\mathscr{A}(\Phi_i)$, $\mathscr{A}(\Phi_i')$ be $\mathscr{A}$-systems for Φ_i, Φ_i', $i = 1, 2$. If $\mathscr{A}(\Phi_i)$, $\mathscr{A}(\Phi_i')$ are H_i-homotop, then $\mathscr{A}(\Phi_2) \circ \mathscr{A}(\Phi_1)$ and $\mathscr{A}(\Phi_2') \circ \mathscr{A}(\Phi_1')$ are H-homotop where $H_t = H_{2t} \circ H_{1t}$, $t \in I$.*

(1.11) Remark. Let $(\Phi, \varphi_n \circ \ldots \circ \varphi_1)$, $(\Psi, \psi_n \circ \ldots \circ \psi_1) \in \mathscr{P}_1$ and $\mathscr{A}^*(\Phi)$ be $\mathscr{A}$-system induced by acyclic decomposition $\varphi_n \circ \ldots \circ \varphi_1$ an $\mathscr{A}^*(\Psi)$ — $\mathscr{A}$-system induced by the acyclic decomposition $\psi_n \circ \ldots \circ \psi_1$. Then straightforward from the definitions we obtain: if $\varphi_n \circ \ldots \circ \varphi_1$ and $\psi_n \circ \ldots \circ \psi_1$ are homotop in $\mathscr{P}_1$ with a homotopy H, then $\mathscr{A}^*(\Phi)$ and $\mathscr{A}^*(\Psi)$ are H-homotopic.

(1.12) Special $\mathscr{A}$-systems. Let $\Phi\colon K \to L$ be an u.s.c. map and $\mathscr{A}(\Phi)$ an $\mathscr{A}$-system for Φ. Suppose $K \subset L$. $\mathscr{A}$-system is called *special* if for every open and polyhedral sets $O_1, \ldots, O_n$ in L with

$$\bar{O}_i \cup \Phi(\bar{O}_i) \subset \operatorname{Int} O_{i+1} \qquad 1 \leqslant i \leqslant n-1$$

and $\bar{O}_n$ — compact there is $k_0 \in N$ such that for all $k \in N$, $k \geqslant k_0$ there is $\varphi \in A(\Phi_{\bar{O}_{n-1}\bar{O}_n})_k$ with following property

$$\varphi \,|\, \bar{O}_i \in A(\Phi_{\bar{O}_i \bar{O}_{i+1}})_k \quad \text{for} \quad i = 1, \ldots, n.$$

Here $A(\Phi_{\bar{O}_i \bar{O}_{i+1}}) \in \mathscr{A}(\Phi)$, $i = 1, \ldots, n$.

(1.13) Remarks. (i) Let $\Psi\colon M \to N$ be an u.s.c. map, M and N — compact polyhedra, $M \subset N$ and $A(\Psi)$ an A-system for Ψ. Then $A(\Psi)$ is a special $\mathscr{A}$-system.

(ii) Let $(\Phi, \varphi_m \circ \ldots \circ \varphi_1) \in \mathscr{P}_1$, (1.7) (ii), and $\mathscr{A}^*(\Phi)$ be the $\mathscr{A}$-system induced by the acyclic decomposition $\varphi_m \circ \ldots \circ \varphi_1$. Then from the definition follows that $\mathscr{A}^*(\Phi)$ is a special $\mathscr{A}$-system.

§ 2. A category appropriate for fixed point index. Let F be a fixed field and K, K_i, L, L_i — locally compact polyhedra.

(2.1) Definition. Consider the following category $\mathscr{K} = \mathscr{K}(F)$: the elements of $\mathscr{K}$ are pairs $(\Phi, \mathscr{A}(\Phi))$ where $\Phi\colon K \to L$ is an u.s.c. map and $\mathscr{A}(\Phi)$ an $\mathscr{A}$-system for Φ. The composition of two objects in $\mathscr{K}$ is defined naturally. Let $(\Phi_i, \mathscr{A}(\Phi_i)) \in \mathscr{K}$ and $\Phi_1\colon K_1 \to K_2$, $\Phi_2\colon K_2 \to K_3$. Then the composition $(\Phi_2, \mathscr{A}(\Phi_2)) \circ (\Phi_1, \mathscr{A}(\Phi_1))$ is defined as $(\Phi_2 \circ \Phi_1, \mathscr{A}(\Phi_2) \circ \mathscr{A}(\Phi_1))$, see (1.6).

We call the category $\mathscr{K}$ — category of all F-$\mathscr{A}$-systems.

Let $\mathscr{K}_0$ be the class of objects in $\mathscr{K}$ such that $(\Phi, \mathscr{A}(\Phi)) \in \mathscr{K}_0$ iff $\Phi \in \mathscr{P}_1$, (1.7) (ii) and $\mathscr{A}^*(\Phi)$ is an $\mathscr{A}$-system induced by some acyclic decomposition of Φ. Obviously $\mathscr{K}_0$ is a subcategory of $\mathscr{K}$.

An element $(\Phi, \mathscr{A}(\Phi))$ in $\mathscr{K}$ we call *admissible* if $\Phi\colon U \to K$, where U is an open set in K and the set

$$\mathrm{Fix}(\Phi) = \{x \in U\colon\ x \in \Phi(x)\}$$

is compact. Denote by $\mathfrak{A}$ the class of all admissible objects in $\mathscr{K}$.

(2.2) Homotopy in $\mathscr{K}$. Let $H\colon K \times I \to L$ be an u.s.c. map and $\Phi_1, \Phi_2\colon K \to L$ are defined as $\Phi_1(x) = H(x, 0)$, $\Phi_2(x) = H(x, 1)$ for $x \in K$. Suppose $(\Phi_i, \mathscr{A}(\Phi_i)) \in \mathscr{K}$. We say that $(\Phi_1, \mathscr{A}(\Phi_1))$ and $(\Phi_2, \mathscr{A}(\Phi_2))$ are H-homotop in $\mathscr{K}$ if $\mathscr{A}(\Phi_1)$ and $\mathscr{A}(\Phi_2)$ are H-homotop. We say that these two elements are *admissible homotop in* $\mathscr{K}$ if K is an open set in L and

$$\mathrm{Fix}(H) = \bigcup \{\mathrm{Fix}(H_t)\colon\ t \in I\}$$

is compact set (H_t is defined as $H_t(x) = H(x, t)$).

(2.3) Extension of the functor of singular homology with coefficients F on $\mathscr{K}$. By H_*^s we denote the functor of singular homology with coefficients F, [33], ch. 4. Here for every $(\Phi, \mathscr{A}(\Phi)) \in \mathscr{K}$, $\Phi\colon K \to L$, we define a homomorphism

$$\mathscr{A}(\Phi)_*\colon\ H_*^s(K) \to H_*^s(L)$$

such that

(i) If $(\Phi_i, \mathscr{A}(\Phi_i)) \in \mathscr{K}$ and $(\Phi_2, \mathscr{A}(\Phi_2)) \circ (\Phi_1, \mathscr{A}(\Phi_1))$ is defined, then

$$\mathscr{A}(\Phi_2)_* \circ \mathscr{A}(\Phi_1)_* = (\mathscr{A}(\Phi_2) \circ \mathscr{A}(\Phi_1))_*.$$

(ii) If $(\Phi, \mathscr{A}(\Phi)) \in \mathscr{K}_0$ and Φ is single-valued, then

$$\mathscr{A}^*(\Phi)_* = \Phi_*$$

where Φ_* is the homomorphism induced by Φ.

(iii) If $(\Phi_1, \mathscr{A}(\Phi_1))$ and $(\Phi_2, \mathscr{A}(\Phi_2))$ are homotop in $\mathscr{K}$, then

$$\mathscr{A}(\Phi_1)_* = \mathscr{A}(\Phi_2)_*.$$

We call the homomorphism $\mathscr{A}(\Phi)_*$ *induced* by $(\Phi, \mathscr{A}(\Phi))$.

Let $A(\Phi_{MN}) \in \mathscr{A}(\Phi)$. From Definition (2.3), § 1, ch. 1, [27] it follows that for sufficiently large k and every $\varphi \in A(\Phi_{MN})_k$ the homomorphism

$$\varphi_*\colon\ H_*(M) \to H_*(N)$$

does not depend on k and φ (φ_* – is induced by φ). Denote this homomorphism by $A(\Phi_{MN})_*$. From Definition (1.1) we obtain: if $M_1 \subset M_2$ are compact polyhedra in K and $N_1 \subset N_2$ – compact polyhedra in L such that $\Phi(M_i) \subset \mathrm{Int}\, N_i$, $i = 1, 2$, then

$$j_* A(\Phi_{M_1N_1})_* = A(\Phi_{M_2N_2})_* i_*,$$

here $i\colon M_1 \to M_2$, $j\colon N_1 \to N_2$ are inclusions and i_*, j_* – homomorphisms

in homology, induced by i and j. This gives us that the family of homomorphisms

$$\underline{\mathscr{A}(\Phi)_*} = \{A(\Phi_{MN})_*\colon\ A(\Phi_{MN}) \in \mathscr{A}(\Phi)\}$$

is homomorphism of the direct system of groups

$$\underline{H_*^c(K)} = \{H_*(M)\colon\ M\ -\ \text{compact subpolyhedra in } K\}$$

into the direct system of groups

$$\underline{H_*^c(L)} = \{H_*(P)\colon\ P\ -\ \text{compact subpolyhedra in } L\}$$

(see [20], Appendix, § A.2., p. 381).

Denote by

$$\overline{\mathscr{A}(\Phi)}_*\colon = \varinjlim \underline{\mathscr{A}(\Phi)_*}$$

the limit of the direct system of homomorphisms $\underline{\mathscr{A}(\Phi)_*}$ and by

$$H_*^c(K) = \varinjlim \underline{H_*^c(K)} \quad \text{and} \quad H_*^c(L) = \varinjlim \underline{H_*^c(L)}$$

direct limits of $\underline{H_*^c(K)}$ and $\underline{H_*^c(L)}$.

By definition H_*^c are the homology with compact supports and coefficients F ([20], ch. 9, § 9.1, p. 269).

Therefore we have a homomorphism

$$\overline{\mathscr{A}(\Phi)}_*\colon\ H_*^c(K) \to H_*^c(L).$$

It is known that homology with compact supports and singular homology are isomorph on the category of locally compact polyhedra and continuous mappings, [20], ch. 9, § 9.6.

Let $\mu(X)\colon H_*^s(X, F) \to H_*^c(X, F)$ be the natural isomorphism for X – locally compact polyhedron.

Then defining

$$\mathscr{A}(\Phi)_* = \mu^{-1}(L)\overline{\mathscr{A}(\Phi)}_*\mu(K)$$

we obtain a homomorphism

$$\mathscr{A}(\Phi)_*\colon\ H_*^s(K) \to H_*^s(L);$$

we call this homomorphism – *homomorphism in singular homology, induced by the $\mathscr{A}$-system $\mathscr{A}(\Phi)$*.

Properties (i), (ii), (iii) follow from the definitions (for the definition of Φ_* in case Φ single-valued and continuous see [20], ch. 9, p. 270).

(2.4) Remarks. (i) Let $(\Phi, \mathscr{A}^*(\Phi)) \in \mathscr{K}$, $\Phi\colon K \to L$ and Φ be an acyclic map. Let $A^*(\Phi_{MN}) \in \mathscr{A}^*(\Phi)$. Consider the map $\Phi_{MN}\colon M \to N$. This is an acyclic map and M, N are compact spaces. There is well known definition of the induced homomorphism in homology of the map Φ_{MN}:

$$(\Phi_{MN})_*\colon H_*(M) \to H_*(N),$$

[13, 28]. From the arguments in [28], Lemma 1 we have that for sufficiently large $k \in N$ and every $\varphi \in A(\Phi_{MN})_k$ follows $\varphi_* = (\Phi_{MN})_*$; here φ_* is the homomorphism in homology, induced by the chain mapping φ. Therefore if

$$\bar{\Phi}_* = \varinjlim (\Phi_{MN})_* \quad \text{and} \quad \Phi_* = \mu^{-1}(L)\,\bar{\Phi}_*\mu(K),$$

then $\mathscr{A}^*(\Phi)_* = \Phi_*$.

(ii) Let $(\Phi, \mathscr{A}^*(\Phi)) \in \mathscr{K}_0$, $\Phi\colon K \to L$ and $\mathscr{A}^*(\Phi)$ is induced by the acyclic decomposition $\varphi_n \circ \ldots \circ \varphi_1$, X_i — Hausdorff spaces, $X_1 = K$ and $X_{n+1} = L$. Let M and N be compact polyhedra with $M \subset K$, $\Phi(M) \subset \operatorname{Int} N$, and $C_i = \varphi_{i-1} \ldots \varphi_1(M)$. The map $\varphi'_i = \varphi_i|C_i\colon C_i \to C_{i+1}$ is acyclic and φ'_{i*} a homomorphism in Čech homology with coefficients F, [13]:

$$\varphi'_{i*}\colon \check{H}_*(C_i) \to \check{H}_*(C_{i+1}).$$

Consider the homomorphism

$$\varphi'_{n*} \circ \varphi'_{n-1*} \circ \ldots \circ \varphi'_{1*}\colon H_*(M) \to H_*(N).$$

Since M and N are compact polyhedra then from the arguments in Lemma 1, [28] we obtain: for k sufficiently large and $\varphi \in \mathscr{A}(\Phi_{MN})_k$, $\varphi_* = \varphi'_{n*} \circ \ldots \circ \varphi'_{1*}$ and therefore

$$\mathscr{A}(\Phi)_* = \varinjlim \{\varphi'_{n*} \circ \ldots \circ \varphi'_{1*}\}.$$

We denote this homomorphism by $\varphi_{n*} \circ \ldots \circ \varphi_{1*}$.

(2.5) Lefschetz number of the elements of $\mathscr{K}$. *Let $(\Phi, \mathscr{A}(\Phi)) \in \mathscr{K}$, $\Phi\colon K \to K$. If Φ is compact map then the induced homomorphism $\mathscr{A}(\Phi)$ is a Leray endomorphism of $H^s_*(K)$ and if M is a compact subpolyhedron in K with $\Phi(K) \subset \operatorname{Int} M$, then*

$$\Lambda(\mathscr{A}(\Phi)_*) = \Lambda(A(\Phi_{MM})_*).$$

Here $\Lambda(\mathscr{A}(\Phi)_*)$ and $\Lambda(A(\Phi_{MN})_*)$ are the Lefschetz numbers of the endomorphisms $\mathscr{A}(\Phi)_*$ and $A(\Phi_{MM})_*$, [14, 18].

Proof. Let M be a compact subpolyhedra in K such that $\Phi(K) \subset \operatorname{Int} M$. Consider the maps $\Phi_{MM}\colon M \to M$ and $\Phi_{KM}\colon K \to M$. We have the following $\mathscr{A}$-systems: $\mathscr{A}(\Phi_{KM}) = \mathscr{A}(\Phi)_{KM}$, and $\mathscr{A}(\Phi_{MM}) = \mathscr{A}(\Phi)_M$, see (1.3). Denote by $i\colon M \to K$ the inclusion. Then from (2.3) we have

$$\mathscr{A}(\Phi)_* = i_* \mathscr{A}(\Phi_{KM})_* \quad \text{and} \quad A(\Phi_{MM})_* = \mathscr{A}(\Phi_{KM})_* i_*.$$

Since M is a compact polyhedron, then $\dim_F H_*(M) < \infty$ and $\Lambda(A(\Phi_{MM})_*)$

exist. Therefore $\mathscr{A}(\Phi)_*$ is a Leray endomorphism and $\Lambda(\mathscr{A}(\Phi)_*) = \Lambda(A(\Phi_{MM})_*)$, [14, 18].

§ 3. Fixed point index on $\mathscr{K}$. Let $(\Phi, \mathscr{A}(\Phi)) \in \mathfrak{A}$, (2.1) and $\Phi\colon U \to K$. Since the set $\operatorname{Fix}(\Phi)$ is compact, then there is an open and polyhedral set V in K such that $\operatorname{Fix}(\Phi) \subset V \subset \bar{V} \subset U$ and $\bar{V}$ is compact. Let W be an open and polyhedral set in K with $\bar{V} \cup \Phi(\bar{V}) \subset W$ and $\bar{W}$ — compact.

Now the set $\bar{W}$ is compact polyhedron and V — open and polyhedral set in $\bar{W}$. Consider $A(\Phi_{\bar{V}\bar{W}}) \in \mathscr{A}(\Phi)$. Since $\operatorname{Fix}(\Phi_{\bar{V}\bar{W}}) \subset V$, then $A(\Phi_{\bar{V}\bar{W}})$ is an I-system on V, Lemma (2.5), § 1, ch. 2, [27]; for the definition of an I-system see (1.1), § 1, ch. 1, [27].

(3.1) Definition. The fixed point index $I(\Phi, \mathscr{A}(\Phi))$ of the element $(\Phi, \mathscr{A}(\Phi))$ is defined as follows

$$I(\Phi, \mathscr{A}(\Phi)) = I(A(\Phi_{\bar{V}\bar{W}})).$$

Here $I(A(\Phi_{\bar{V}\bar{W}}))$ is the index of the I-system $A(\Phi_{\bar{V}\bar{W}})$, (1.2), § 1, ch. 1, [27].

It seems that this definition depends not only on the $\mathscr{A}$-system $\mathscr{A}(\Phi)$ but also on V and W. Using only the definitions one prove immediately that Definition (2.1) is independent on V and W.

(3.2) Remark. Let $f\colon U \to K$ be an admissible and single-valued map. Then the fixed point index $I(f, \mathscr{A}^*(f))$ coincides with the fixed point index $i(K, f, U)$ of f on U, see [4, 5, 7, 9, 10, 22].

§ 4. Additivity, homotopy invariance and normalization property of the fixed point index.

(4.1) Additivity. *Let U be an open set in K, $\Phi\colon U \to K$, and $(\Phi, \mathscr{A}(\Phi)) \in \mathfrak{A}$. Let U_1, U_2 be open and disjoint subsets in U. If $\operatorname{Fix}(\Phi) \subset U_1 \cup U_2$, then*

$$I(\Phi, \mathscr{A}(\Phi)) = I(\Phi_1, \mathscr{A}(\Phi)_{U_1}) + I(\Phi_2, \mathscr{A}(\Phi)_{U_2}).$$

Here $\Phi_i = \Phi|U_i\colon U_i \to K$, and $\mathscr{A}(\Phi)_{U_i}$ is the restriction $\mathscr{A}(\Phi)$ on U_i, $i = 1, 2$.

Proof is a straightforward application of the definitions and (2.8), § 2, ch. 2, [27].

(4.2) Homotopy invariance. *Let U be an open set in K, Φ_1, $\Phi_2\colon U \to K$, $H\colon U \times I \to K$ — homotopy between Φ_1 and Φ_2. If $(\Phi_1, \mathscr{A}(\Phi_1))$ and $(\Phi_2, \mathscr{A}(\Phi_2))$ are admissible H-homotopic, then*

$$I(\Phi_1, \mathscr{A}(\Phi_1)) = I(\Phi_2, \mathscr{A}(\Phi_2)).$$

Proof. Immediate consequence from (1.4) and Lemma (2.14), § 3, ch. 2, [27], Proposition (1.5), § 1, ch. 1, [27].

(4.3) Normalization. *Let $(\Phi, \mathscr{A}(\Phi)) \in \mathfrak{A}$, $\Phi\colon K \to K$ and Φ-compact map. Then*

$$\Lambda(\mathscr{A}(\Phi)_*) = I(\Phi, \mathscr{A}(\Phi)).$$

Proof. Let M be a compact subpolyhedron in K such that $\Phi(K)\subset M$. Then from (2.5)

$$\Lambda\big(\mathscr{A}(\Phi)_*\big) = \Lambda\big(A(\Phi_{MM})_*\big).$$

From H. Hopf trace formula, [5], p. 6 we have

$$\Lambda\big(A(\Phi_{MM})_*\big) = I\big(A(\Phi_{MM})\big).$$

From (3.1)

$$I\big(A(\Phi_{MM})\big) = I\big(\Phi, \mathscr{A}(\Phi)\big).$$

§ 5. Commutativity. Let U be an open set in K, V — open set in L, $\Phi_1\colon U\to L$, $\Phi_2\colon V\to K$, and $(\Phi_i, \mathscr{A}(\Phi_i))\in\mathscr{K}$. Let

$$\Phi_1' = \Phi_1|\Phi_1^{-1}(V)\colon \Phi_1^{-1}(V)\to L,$$
$$\Phi_2' = \Phi_2|\Phi_2^{-1}(U)\colon \Phi_2^{-1}(U)\to K,$$

and

$$\mathscr{A}(\Phi_1') = \mathscr{A}(\Phi_1)_{\Phi_1^{-1}(V)}, \qquad \mathscr{A}(\Phi_2') = \mathscr{A}(\Phi_2)_{\Phi_2^{-1}(U)}$$

then

$$\big(\Phi_2\circ\Phi_1',\ \mathscr{A}(\Phi_2)\circ\mathscr{A}(\Phi_1')\big) \quad\text{and}\quad \big(\Phi_1\circ\Phi_2',\ \mathscr{A}(\Phi_1)\circ\mathscr{A}(\Phi_2')\big)$$

belong to $\mathscr{K}$. In case they belong to $\mathfrak{A}$ the indices

$$I\big(\Phi_2\circ\Phi_1',\ \mathscr{A}(\Phi_2)\circ\mathscr{A}(\Phi_1')\big) \quad\text{and}\quad I\big(\Phi_1\circ\Phi_2',\ \mathscr{A}(\Phi_1)\circ\mathscr{A}(\Phi_2')\big)$$

are defined. Here we shall find conditions under which these two indices are equal. For this we need some notations. Let

$$\mathrm{Fix}(\Phi_2\Phi_1) = \{x\in U\colon \text{there is } y\in\Phi_1(x)\cap V \text{ such that } x\in\Phi_2(y)\},$$

and similar

$$\mathrm{Fix}(\Phi_1\Phi_2) = \{y\in V\colon \text{there is } x\in\Phi_2(y)\cap U \text{ such that } y\in\Phi_1(x)\}.$$

(5.1) Assumptions.

(i) $\mathrm{Fix}(\Phi_2\Phi_1)$ *and* $\mathrm{Fix}(\Phi_1\Phi_2)$ *are compact sets.*

(ii) $\mathrm{Fix}(\Phi_2\Phi_1')$ *and* $\mathrm{Fix}(\Phi_1\Phi_2')$ *are compact sets.*

(iii) $\Phi_1\big(\mathrm{Fix}(\Phi_2\Phi_1)\backslash\Phi_1^{-1}(V)\big)\cap\mathrm{Fix}(\Phi_1\Phi_2') = \emptyset$.

(iv) $\Phi_2\big(\mathrm{Fix}(\Phi_1\Phi_2)\backslash\Phi_2^{-1}(U)\big)\cap\mathrm{Fix}(\Phi_2\Phi_1') = \emptyset$.

(5.2) Lemma. *Under the conditions* (5.1) *we have*

$$I\big(\Phi_2\Phi_1',\ \mathscr{A}(\Phi_2)\circ\mathscr{A}(\Phi_1)'\big) = I\big(\Phi_1\Phi_2',\ \mathscr{A}(\Phi_1)\circ\mathscr{A}(\Phi_2)'\big).$$

Before proving this lemma we make some remarks.

(5.3) Remarks. (i) In case Φ_1 is single-valued then $\mathrm{Fix}(\Phi_2\Phi_1) = \mathrm{Fix}(\Phi_2\Phi_1')$ and the condition (5.1) (iii) is satisfied.

(ii) If Φ_2 is single valued, then $\mathrm{Fix}(\Phi_1\Phi_2) = \mathrm{Fix}(\Phi_1\Phi_2')$ and (5.1) (iv) is satisfied.

(iii) In case Φ_1 and Φ_2 are single-valued, then the assumptions (5.1) reduce to: one of the sets $\mathrm{Fix}(\Phi_1\Phi_2)$ or $\mathrm{Fix}(\Phi_2\Phi_1)$ is compact, see [7].

(iv) If $V=L$, then $\Phi_1^{-1}(V) = U$ and the condition (5.1) (iii) is satisfied.

(5.4) Proof of Lemma (5.2). The proof is similar to the proof of Lemma (1.6), § 2, ch. 1, [27].

Denote

$$B_1 = \mathrm{Fix}(\Phi_2\Phi_1)\backslash\Phi_1^{-1}(V), \qquad B_2 = \mathrm{Fix}(\Phi_2\Phi_1'),$$
$$B_3 = \mathrm{Fix}(\Phi_1\Phi_2)\backslash\Phi_2^{-1}(U), \qquad B_4 = \mathrm{Fix}(\Phi_1\Phi_2').$$

From (5.1) (i), (ii) we have that the sets B_i are compact, $i=1,\ldots,4$.

Let W_1 be an open and polyhedral set in K such that

$$\Phi_2(B_3)\cap\bar{W}_1 = \emptyset, \qquad B_2\subset W_1\subset\bar{W}_1\subset\Phi_1^{-1}(V)$$

and $\bar{W}_1$ — compact set.

Let O_1 be open and polyhedral set in L and such that

$$B_4\cup\Phi_1(\bar{W}_1)\subset O_1\subset\bar{O}_1\subset V$$

and $\bar{O}_1$ — compact set.

Let O_2 be an open and polyhedral set in L and such that

$$B_4\subset O_2\subset\bar{O}_2\subset\Phi_2^{-1}(U), \qquad \bar{O}_2\subset O_1, \qquad \bar{O}_2\cap\Phi_1(B_1) = \emptyset$$

and $\bar{O}_2$ — compact set.

Let W_2 be an open and polyhedral set in K and

$$\Phi_2(\bar{O}_2)\cup\bar{W}_1\subset W_2\subset\bar{W}_2\subset U$$

and $\bar{W}_2$ — compact set.

By K_1 denote a compact subpolyhedra in K with $\Phi_2(\bar{O}_1)\cup\bar{W}_2\subset\mathrm{Int}\,K_1$ and by L_1 — compact polyhedra in L with $\bar{O}_1\cup\Phi_1(\bar{W}_2)\subset\mathrm{Int}\,L_1$.

Now consider the following I-system on W_1

$$A(\Phi_{2\bar{O}_1K_1})A(\Phi_{1\bar{W}_1\bar{O}_1}). \tag{5.5}$$

By the definition — the index of this I-system is

$$I\big(\Phi_2\circ\Phi_1',\ \mathscr{A}(\Phi_2)\circ\mathscr{A}(\Phi_1')\big).$$

Here $A(\Phi_{2\bar{O}_1K_1})$ belongs to $\mathscr{A}(\Phi_2)$ and $A(\Phi_{1\bar{W}_1\bar{O}_1})$ belongs to $\mathscr{A}(\Phi_1)$.

Consider also the I-system on O_2

$$A(\Phi_{1\bar{W}_2L_1})A(\Phi_{2\bar{O}_2\bar{W}_2}). \tag{5.6}$$

By the definition — the index of this I-system is

$$I\big(\Phi_1\circ\Phi_2',\ \mathscr{A}(\Phi_1)\circ\mathscr{A}(\Phi_2')\big).$$

Here $A(\Phi_{1\bar{W}_2L_1})\in\mathscr{A}(\Phi_1)$ and $A(\Phi_{2\bar{O}_2\bar{W}_2})\in\mathscr{A}(\Phi_2)$.

In order to prove Lemma (5.2) we check that the indices of I-systems (5.5) and (5.6) are equal.

Consider $A(\Phi_{1\bar{W}_2L_1})$ and $A(\Phi_{1\bar{W}_1\bar{O}_1})$. Since these two A-systems belong to $\mathscr{A}(\Phi_1)$, then by Definition (1.1) for every sufficiently large $k\in N$ there is a chain map $\varphi\in A(\Phi_{1\bar{W}_2L_1})_k$ such that

$$\varphi|\bar{W}_1\in A(\Phi_{1\bar{W}_1\bar{O}_1})_k. \tag{5.7}$$

Consider $A(\Phi_{2\bar{O}_2\bar{W}_2})$ and $A(\Phi_{2\bar{O}_1K_1})$. Since these two A-systems belong to $\mathscr{A}(\Phi_2)$, then from Definition (1.1) we have: for every sufficiently large k, there is a chain map $\psi\in A(\Phi_{2\bar{O}_1K_1})_k$ such that

$$\psi|O_2\in A(\Phi_{2\bar{O}_2\bar{W}_2})_k. \tag{5.8}$$

Let $\pi_1\colon C_*(K_1,k)\to C_*(\bar{W}_1,k)$, $\pi_2\colon C_*(L_1,k)\to C_*(\bar{O}_2,k)$ are the F-linear projections of the chain groups of the triangulations τ^k and μ^k. Here τ is a triangulation of K and μ – of L.

By definition of the index of I-system, (1.2), § 1, ch. 1, [27] we have

$$I\big(A(\Phi_{2\bar{O}_1K_1})A(\Phi_{1\bar{W}_1\bar{O}_1})\big)=\Lambda(\pi_1\psi\varphi|\bar{W}_1)$$

and

$$I\big(A(\Phi_{1\bar{W}_2L_1})A(\Phi_{2\bar{O}_2W_2})\big)=\Lambda(\pi_2\varphi\psi|\bar{O}_2)$$

in case k is sufficiently large and φ, ψ have the properties (5.7), (5.8).

Consider the Lefschetz number $\Lambda(\pi_2\varphi\psi|\bar{O}_2)$. Using Assumption (5.1) (iii) and Lemma (2.7), § 2, ch. 2, [27], we obtain: for sufficiently large $k\in N$ one has: if σ is a simplex in μ^k and $\sigma\subset\bar{O}_2$, $\sigma\in\varphi\psi(\sigma)$, then for every simplex $\sigma'\in\tau^k$ with $\sigma'\in\psi(\sigma)$ and $\sigma'\in\bar{W}_2\backslash W_1$ holds $\sigma\notin\psi(\sigma')$. This gives – for sufficiently large $k\in N$

$$\Lambda(\pi_2\varphi\psi|\bar{O}_2)=\Lambda(\pi_2\varphi\pi_1\psi|\bar{O}_2)$$

and therefore

$$\Lambda(\pi_2\,\varphi\psi|\bar{O}_2)=\Lambda(\pi_1\varphi|\bar{O}_2\circ\pi_2\psi|\bar{W}_2).$$

Similar

$$\Lambda(\pi_1\psi\pi_2\varphi|\bar{W}_2)=\Lambda(\pi_1\psi|\bar{O}_2\circ\pi_2\psi|\bar{W}_2).$$

Now using commutativity of the trace of composition of F-linear maps we obtain

$$\Lambda(\pi_2\varphi|\bar{W}_2\circ\pi_1\psi|\bar{O}_2)=\Lambda(\pi_1\psi|\bar{O}_2\circ\pi_2\varphi|\bar{W}_2).$$

Thus Lemma (5.2) is proved.

(5.9) Corollary. *Let $f\colon U\to L$ be single-valued continuous map, $\Phi\colon V\to K$ and $(\Phi,\mathscr{A}(\Phi))\in\mathscr{K}$. Let* $\mathrm{Fix}(\Phi f)$, $\mathrm{Fix}(f\Phi)$ *and* $\mathrm{Fix}(f\Phi')$ *be compact. If*

$$\Phi\big(\mathrm{Fix}(f\Phi)\backslash\Phi^{-1}(U)\big)\cap\mathrm{Fix}(\Phi f)=\emptyset,$$

then

$$I\big(\Phi f',\mathscr{A}(\Phi)\circ\mathscr{A}^*(f)\big)=I\big(f\Phi',\mathscr{A}^*(f)\circ\mathscr{A}(\Phi)\big).$$

Proof. Lemma (5.2) and remark (5.3) (i).

(5.10) Corollary. *Let $\Phi\colon L\to K$ and $(\Phi,\mathscr{A}(\Phi))\in\mathscr{K}$. Let $f\colon U\to L$ be single-valued and continuous. If* $\mathrm{Fix}(f\Phi)$, $\mathrm{Fix}(\Phi f)$ *and* $\mathrm{Fix}(f\Phi')$ *are compact set, then*

$$I\big(f\Phi',\mathscr{A}^*(f)\circ\mathscr{A}(\Phi')\big)=I\big(\Phi f',\mathscr{A}(\Phi)\circ\mathscr{A}^*(f')\big).$$

(5.11) Reduction property. *Let $\Phi\colon U\to K$, $(\Phi,\mathscr{A}(\Phi))$ and $L\subset K$. If $\Phi(U)\subset L$, then*

$$I\big(\Phi,\mathscr{A}(\Phi)\big)=I\big(\Phi|L,\mathscr{A}(\Phi)_L\big).$$

§ 6. mod-p property. Suppose U is an open set in K and $\Phi\colon U\to K$ is an u.s.c. map. Here we consider the composition Φ^p of the map Φ. The map Φ^p is defined as follows

$$\Phi^p(x)=\Phi\big(\ldots\big(\Phi(\Phi(x))\big)\ldots\big),$$

therefore Φ^p is defined for those x for which $\Phi(x)$, $\Phi^2(x),\ldots,\Phi^{p-1}(x)$ are subsets in U. Denote by $D(\Phi^k)$ the set where Φ^k is defined, then

$$D(\Phi^k)=\Phi^{-1}\big(D^{k-1}(\Phi)\big)$$

for $k\in N$ and therefore $D(\Phi^k)$ is an open subset in U (possible empty).

Here we suppose that the set $D(\Phi^p)$ is not empty. Let V be an open subset of $D(\Phi^p)$ and $\Phi'=\Phi|V\colon V\to K$, then the map $(\Phi)^p\colon V\to K$ is defined.

Now suppose that $(\Phi',\mathscr{A}(\Phi'))$ and $(\Phi'^p,\mathscr{A}(\Phi)\ldots\mathscr{A}(\Phi'))$ are admissible elements in $\mathscr{K}$, i.e. the sets $\mathrm{Fix}(\Phi')$ and $\mathrm{Fix}(\Phi'^p)$ are compact. Then the indices $I(\Phi',\mathscr{A}(\Phi'))$ and $I(\Phi'^p,\mathscr{A}(\Phi)\ldots\mathscr{A}(\Phi'))$ are defined, here $\mathscr{A}(\Phi')=\mathscr{A}(\Phi)_V$. We shall find conditions under which these indices are equal.

(6.1) Assumptions. Let $p\in N$ be a prime number, $F=Z_p$ and $\mathscr{A}(\Phi)$ – Z_p-$\mathscr{A}$-system.

(i) $\mathscr{A}(\Phi)$ is a special $\mathscr{A}$-system, (1.12).

(ii) $D(\Phi^p)$ is not empty.

(iii) $\mathrm{Fix}(\Phi^p)$ and $\mathrm{Fix}(\Phi'^p)$ are compact set.

(iv) $\Phi^i\big(\mathrm{Fix}(\Phi^p)\backslash V\big)\cap\mathrm{Fix}(\Phi'^p)=\emptyset$ for every $1\leqslant i\leqslant p-1$.

(6.2) Lemma. *Let $\Phi\colon U\to K$, $(\Phi,\mathscr{A}(\Phi))\in\mathscr{K}$ and U – open in K. If assumptions (6.1) are satisfied, then*

$$I\big(\Phi',\mathscr{A}(\Phi')\big)=I\big(\Phi'^p,\mathscr{A}(\Phi)\circ\ldots\circ\mathscr{A}(\Phi')\big).$$

Proof. Let O_1 be an open and polyhedral set in K with $\mathrm{Fix}(\Phi'^p)\subset O_1\subset\bar{O}_1\subset V$ and $\bar{O}_1$ – compact and

$$\bar{O}_1\cap\Phi^s\big(\mathrm{Fix}(\Phi^p)\backslash V\big)=\emptyset\quad\text{ for }\quad 1\leqslant s\leqslant p-1.$$

Suppose O_i is defined. Define O_{i+1} as an open and polyhedral set in K with $\bar{O}_i\cup\Phi(\bar{O}_i)\subset O_{i+1}\subset\bar{O}_{i+1}\subset D(\Phi^{p-i+1})$ and $\bar{O}_{i+1}$ – compact for $1\leqslant i\leqslant p-1$.

Consider $\bar{O}_p$ – it is compact and $\bar{O}_p\subset U$. Let K_1 be a compact subpolyhedron in K with $\bar{O}_p\cup\Phi(\bar{O}_p)\subset\mathrm{Int}\,K_1$.

If $A(\Phi_{\bar{O}_i\bar{O}_{i+1}})$ and $A(\Phi_{\bar{O}_pK_1})$ belong to $\mathscr{A}(\Phi)$, then

$$I(\Phi', \mathscr{A}(\Phi')) = I(A(\Phi_{\bar{O}_1\bar{O}_2}))$$

and

$$I(\Phi'^p, \mathscr{A}(\Phi)\circ \ldots \circ \mathscr{A}(\Phi')) = I(A(\Phi_{\bar{O}_pK_1})\circ A(\Phi_{\bar{O}_{p-1}\bar{O}_p})\circ \ldots \circ A(\Phi_{\bar{O}_1\bar{O}_2})).$$

Consider $A(\Phi_{\bar{O}_pK_1})$ and $A(\Phi_{\bar{O}_i\bar{O}_{i+1}})$ for $1 \leqslant i \leqslant p-1$. Since $\mathscr{A}(\Phi)$ is a special $\mathscr{A}$-system, then for every sufficiently large $k \in N$ there is $\varphi \in A(\Phi_{\bar{O}_pK_1})_k$ such that

(6.3) $$\varphi_i = \varphi|\bar{O}_i \in A(\Phi_{\bar{O}_i\bar{O}_{i+1}})_k \quad \text{for} \quad i = 1, \ldots, p-1.$$

Let $\pi\colon C_*(K_1, k) \to C_*(\bar{O}_1, k)$ be the linear projection of the chain group of the triangulation τ^k on K_1 on the chain group of τ^k on $\bar{O}_1$ (chain groups are with coefficients Z_p, and τ is a triangulation of K).

Then by Definition (1.2), § 1.2, ch. 1, [27],

$$I(A(\Phi_{\bar{O}_1\bar{O}_2})) = \Lambda(\pi\varphi_1)$$

and

$$I(A(\Phi_{\bar{O}_pK_1})\circ A(\Phi_{\bar{O}_{p-1}\bar{O}_p})\circ \ldots \circ A(\Phi_{\bar{O}_1\bar{O}_2})) = \Lambda(\pi\varphi_p\varphi_{p-1} \ldots \varphi_2\varphi_1).$$

Here φ_i is defined in (6.3).

Consider the chain map $\pi\varphi_p\varphi_{p-1} \ldots \varphi_2\varphi_1$. From Lemma (2.7), § 2, ch. 2, [27] and Assumption (6.1) (iv) we obtain: if k is sufficiently large σ is simplex in $\bar{O}_1$, $\sigma \in \tau^k$, and $\sigma \in \varphi_p\varphi_{p-1} \ldots \varphi_2\varphi_1(\sigma)$, then for every simplex $\sigma_1 \in \tau^k$, $\sigma_1 \in \varphi_1(\sigma)$ and $\sigma_1 \in \bar{O}_2 \backslash O_1$ follows $\sigma \notin \varphi_p \ldots \varphi_2(\sigma_1)$. Therefore

$$\Lambda(\pi\varphi_p\varphi_{p-1} \ldots \varphi_2\varphi_1) = \Lambda(\pi\varphi_p\varphi_{p-1} \ldots \varphi_2\pi\varphi_1).$$

Since

$$\pi\varphi_p\varphi_{p-1} \ldots \varphi_2\pi\varphi_1 = \pi\varphi_{p-1}\varphi_{p-2} \ldots \varphi_2\varphi_1\varphi_1,$$

then

$$\Lambda(\pi\varphi_p\varphi_{p-1} \ldots \varphi_2\varphi_1) = \Lambda(\pi\varphi_{p-1}\varphi_{p-2} \ldots \varphi_2\varphi_1\varphi_1).$$

Using the same arguments we prove that

$$\Lambda(\pi\varphi_p\varphi_{p-1} \ldots \varphi_1) = \Lambda((\pi\varphi_1)^p)$$

where $(\pi\varphi_1)^p = \pi\varphi_1\pi\varphi_1 \ldots \pi\varphi_1$ – composition p-times. Since p is a prime number, then from "small Fermat" theorem [24], the Lemma, p. 441, follows $\Lambda((\pi\varphi_1)^p) = \Lambda(\pi\varphi_1)$ in Z_p. Thus Lemma (6.2) is proved.

§ 7. Summary of results.

(7.1) Theorem. *Suppose F is a fixed field and $\mathscr{K} = \mathscr{K}(F)$ is a category of all F-$\mathscr{A}$-systems. On the class of all admissible elements $\mathfrak{A}$ of $\mathscr{K}$ is defined a function $I\colon \mathfrak{A} \to F$, a fixed point index, with following properties:*

I. Additivity. *Let U_1, U_2 be open disjoint sets in U and U – open in K. If $(\Phi, \mathscr{A}(\Phi)) \in \mathfrak{A}$, $\Phi\colon U \to K$ and $\mathrm{Fix}(\Phi) \subset U_1 \cup U_2$, then*

$$I(\Phi, \mathscr{A}(\Phi)) = I(\Phi_1, \mathscr{A}(\Phi)_{U_1}) + I(\Phi_2, \mathscr{A}(\Phi)_{U_2});$$

here $\Phi_i = \Phi|U_i\colon U_i \to K$, $i = 1, 2$.

II. Homotopy invariance. *Let $(\Phi_1, \mathscr{A}(\Phi_1))$ and $(\Phi_2, \mathscr{A}(\Phi_2))$ belong to $\mathscr{K}$ and be admissible H-homotop in $\mathscr{K}$, then*

$$I(\Phi_1, \mathscr{A}(\Phi_1)) = I(\Phi_2, \mathscr{A}(\Phi_2)).$$

III. Normalization. *Let $\Phi\colon K \to K$ be a compact map and $(\Phi, \mathscr{A}(\Phi)) \in \mathscr{K}$. Then the homomorphism $\mathscr{A}(\Phi)_*\colon H_*^s(K) \to H_*^s(K)$ is Leray endomorphism and*

$$I(\Phi, \mathscr{A}(\Phi)) = \Lambda(\mathscr{A}(\Phi)_*).$$

IV. Commutativity. *Let U be an open set in K and V – open in L, $\Phi_1\colon U \to L$, $\Phi_2\colon V \to K$, and $(\Phi_i, \mathscr{A}(\Phi_i))$ belongs to $\mathscr{K}$, $i = 1, 2$. Let*

$$\Phi_1' = \Phi_1|\Phi_1^{-1}(V)\colon \Phi_1^{-1}(V) \to L, \qquad \Phi_2' = \Phi_2|\Phi_2^{-1}(U)\colon \Phi_2^{-1}(U) \to K.$$

If

(i) $\mathrm{Fix}(\Phi_1\Phi_2)$, $\mathrm{Fix}(\Phi_2\Phi_1)$ *are compact sets,*

(ii) $\mathrm{Fix}(\Phi_1\Phi_2')$, $\mathrm{Fix}(\Phi_2\Phi_1')$ *are compact sets,*

(iii) $\Phi_1(\mathrm{Fix}(\Phi_2\Phi_1)\backslash\Phi_1^{-1}(V)) \cap \mathrm{Fix}(\Phi_1\Phi_2') = \emptyset$,

(iv) $\Phi_2(\mathrm{Fix}(\Phi_1\Phi_2)\backslash\Phi_2^{-1}(U)) \cap \mathrm{Fix}(\Phi_2\Phi_1') = \emptyset$,

then

$$I(\Phi_2\Phi_1', \mathscr{A}(\Phi_2)\circ\mathscr{A}(\Phi_1')) = I(\Phi_1\Phi_2', \mathscr{A}(\Phi_1)\circ\mathscr{A}(\Phi_2')).$$

V. mod-P property. *Let U be an open set in K, $\Phi\colon U \to K$, V – non-empty set in $D(\Phi^p)$ and $\Phi' = \Phi|V\colon V \to K$. Let $(\Phi, \mathscr{A}(\Phi)) \in \mathscr{K}$. If*

(i) *p is a prime number, $F = Z_p$ and $\mathscr{A}(\Phi)$ is a special $\mathscr{A}$-system,*

(ii) $\mathrm{Fix}(\Phi^p)$, $\mathrm{Fix}(\Phi'^p)$ *are compact sets*

(iii) $\Phi^i(\mathrm{Fix}(\Phi^p)\backslash V) \cap \mathrm{Fix}(\Phi'^p) = \emptyset$ *for every* $1 \leqslant i \leqslant p-1$, *then*

$$I(\Phi', \mathscr{A}(\Phi')) = I(\Phi'^p, \mathscr{A}(\Phi)\circ \ldots \circ \mathscr{A}(\Phi')).$$

(7.2) Corollary. *Restricting the fixed point index on the subcategory $\mathscr{K}_0$ we obtain fixed point index with properties (I)–(V). Homotopy invariance is formulated as follows:*

(II') Homotopy invariance in $\mathscr{K}_0$. *Let Φ_1 and Φ_2 be admissible homotop in $\mathscr{P}_1$, then*

$$I(\Phi_1, \mathscr{A}^*(\Phi_1)) = I(\Phi_2, \mathscr{A}^*(\Phi_2)).$$

(7.3) Reduction property. *Let U be an open set in K, L locally compact polyhedron in K, $\Phi\colon U \to K$ and $(\Phi, \mathscr{A}(\Phi)) \in \mathfrak{A}$. If $\Phi(U) \subset L$, then*

$$I(\Phi, \mathscr{A}(\Phi)) = I(\Phi|L \cap U, \mathscr{A}(\Phi)_{UL}).$$

§ 8. Concluding remarks.

(8.1) Let Φ: $K \to L$ be a continuous (u.s.c. and l.s.c.) map, and $n \in N$. If for every point $x \in L$ the set $\Phi(x)$ is entweder F-acyclic compactum or sum of n F-acyclic compacta, then Φ has an $\mathscr{A}$-system — $\mathscr{A}^*(\Phi)$; it is constructed using [23]. Therefore $(\Phi, \mathscr{A}^*(\Phi)) \in \mathscr{K}$.

(8.2) Let Φ: $K \to L$ be "admissible", [13], multivalued map, i.e. there exist two single-valued, continuous maps p: $X \to K$ and q: $X \to L$ such that

(i) p is a proper and F-Vietoris map,

(ii) $qp^{-1}(x) \subset \Phi(x)$ for every $x \in X$,

(iii) X is a Hausdorff topological space (see [1, 13, 36]).

We know that there is an induced F-$\mathscr{A}$-system for the map qp^{-1}. Denote this $\mathscr{A}$-system by $\mathscr{A}(qp^{-1})$, (1.7) (i). Obviously $\mathscr{A}(qp^{-1})$ is an $\mathscr{A}$-system for Φ. Therefore $(\Phi, \mathscr{A}(qp^{-1})) \in \mathscr{K}$.

(8.3) Let Φ: $K \to L$ be an u.s.c. map and suppose that for every compact subpolyhedron M in K, and compact subpolyhedron N in L with $\Phi(M) \subset N$ the map Φ_{MN}: $\Phi|M$: $M \to N$ can be approximated with a single-valued map f_{MN}: $M \to N$ arbitrary good. This means that for every $k \in N$ there is single-valued and continuous map f_{MN}: $M \to N$ such that $f_{MN}(x) \subset \mathrm{St}(\Phi(x), \mu^k)$ for every $x \in M$. Here μ is a triangulation of L and μ^k — kth barycentric subdivision of μ, $\mathrm{St}(\Phi(x), \mu^k) = \bigcup\{\sigma \in \mu^k\colon \sigma \cap \Phi(x) \neq \emptyset\}$. In this case we say that Φ admits approximations by single-valued maps. Then these approximations induce an $\mathscr{A}$-system — $\mathscr{A}^*(\Phi)$. We call this $\mathscr{A}$-system induced by single-valued approximations. Then $(\Phi, \mathscr{A}^*(\Phi)) \in \mathscr{K}$.

If Φ_1: $U \to L$, and Φ_2: $V \to K$ — admit single-valued approximations, U — open in K, V — open in L. Then $(\Phi_i, \mathscr{A}^*(\Phi_i)) \in \mathscr{K}$. In this case commutativity property for $(\Phi_1, \mathscr{A}^*(\Phi_1))$ and $(\Phi_2, \mathscr{A}^*(\Phi_2))$ is simple:

(IV′) Commutativity. *Let Φ_1: $U \to L$ and Φ_2: $V \to K$ admit approximations by single-valued maps and*

$$\Phi_1' = \Phi_1|\Phi_1^{-1}(V)\colon\ \Phi_1^{-1}(V) \to K,$$

$$\Phi_2' = \Phi_2|\Phi_2^{-1}(U)\colon\ \Phi_2^{-1}(U) \to L.$$

If $\mathrm{Fix}(\Phi_1\Phi_2')$ *and* $\mathrm{Fix}(\Phi_2\Phi_1')$ *are compact sets, then*

$$I(\Phi_2\Phi_1', \mathscr{A}^*(\Phi_2) \circ \mathscr{A}^*(\Phi_1')) = I(\Phi_1\Phi_2', \mathscr{A}^*(\Phi_1) \circ \mathscr{A}^*(\Phi_2')).$$

For maps admitting approximations by single-valued continuous maps mod-p property is also simpler.

(V′) mod-P property. *Let U be an open set in K, Φ: $U \to K$, V — nonempty open set in $D(\Phi^p)$ and $\Phi' = \Phi|V$: $V \to K$. Let Φ admits single valued approximations. If*

(i) *p is a prime number, $F = Z_p$,*

(ii) $\mathrm{Fix}(\Phi'^p)$ *is a compact set,*

then

$$I(\Phi', \mathscr{A}(\Phi')) = I(\Phi'^p, \mathscr{A}(\Phi) \circ \ldots \circ \mathscr{A}(\Phi')).$$

For examples of maps admitting approximation by single-valued maps see [3, 6, 11, 19, 21]. For mod-p property see [10, 34, 37].

(8.4) Since every open set in a finite dimensional euclidean space is a locally compact polyhedron, we obtain a fixed point index for a large class of multi-valued maps Φ: $U \to R^n$ where U is open in R^n, and R^n is n-dimensional euclidean space.

(8.5) The results of the paper were proved for mappings of locally compact polyhedra. Using these results and the method proposed by A. Granas, [14], it is possible to define a fixed point index with all properties for a large class of multi-valued maps, including compositions of acyclic maps, on metric ANR spaces. It should be possible to define a fixed point index for maps of Browder's semicomplexes [4, 35]. For this is necessary to find an appropriate notion of an $\mathscr{A}$-system for u.s.c. maps Φ: $X \to Y$ in terms of Čech or Vietoris complexes of the spaces X and Y.

References

[1] E. Begle, *The Vietoris mapping theorem for bicompact spaces*, Ann. Math. 51 (1950), pp. 534–543.

[2] — *A fixed point theorem for bicompact spaces*, Ann. Math. 51 (1950), pp. 544–550.

[3] Yu. G. Borisovic, E. D. Gel'man, A. D. Myškis and V. V. Obuhovskii, *Topological methods in fixed point theory for multi-valued maps*, Uspechi Mat. Nauk 35 (1980), pp. 59–126 (in Russian).

[4] F. E. Browder, *On the fixed point index for continuous mappings of locally connected spaces*, Summa Brasil. Math. 4 (1960), pp. 253–293.

[5] R. Brown, *The Lefschetz fixed point theorem*, Scot, Foresman and Co, Glenview, Ill., 1971.

[6] A. Cellina and A. Lasota, *A new approach to the definition of topological degree for multi-valued mappings*, Lincei Rend. Sci. Mat. Nat. 47 (1969), pp. 434–440.

[7] A. Dold, *Fixed point index and fixed point theorem for euclidean neighborhood retracts*, Topology 4 (1965), pp. 1–8.

[8] S. Eilenberg and D. Montgomery, *Fixed point theorems for multi-valued transformations*, Amer. J. Math. 58 (1946), pp. 214–222.

[9] G. Eisenack and C. Fenske, *Fixpunkttheorie*, Bibliographisches Inst., Mannheim–Wien–Zürich 1978.

[10] G. Fournier, *A simplicial approach to the fixed point index*, Springer Lecture Notes 886 (1981), pp. 73–102.

[11] Yu. E. Glinklih, *Fixed points of multi-valued maps with nonconvex images and rotation of multi-valued vector fields*, Sborn. Trudov Aspirantov Mat. Fac. VGU, Voronej 1 (1971), pp. 30–38 (in Russian).

[12] L. Górniewicz, *A review of various results and problems of the fixed point theory of multi-valued mappings*, preprint, Univ. Gdańsk, Inst. Math. 1 (1978).

[13] — *Homological methods in fixed point theory of multi-valued maps*, Dissertations Math. 129 (1979), 71 pp.

[14] A. Granas, *The Leray–Schauder index and the fixed point theory for arbitrary* ANR's, Bull. Soc. Math. France 100 (1972), pp. 209–228.

[15] Z. Kucharski, *A coincidence index*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 4 (1976), pp. 245–252.

[16] C. Kuratowski, *Topologie*, II, PWN, Warszawa 1961.

[17] J. Leray, *Theorie des points fixes: indice total et nombre de Lefschetz*, Bull. Soc. Math. France 87 (1959), pp. 221–233.
[18] — *Sur la position d'un ensemble ferme de points d'un espace topologique. Sur les equationes et les transformations*, J. Math. Pures et Appl. 24 (1945), pp. 169–248.
[19] A. Mas-Colell, *A note on a theorem of F. Browder*, Math. Progr. 6 (1974), pp. 229–233.
[20] W. Massey, *Homology and cohomology theory*, Marcel Dekker Inc., N. Y., 1978.
[21] A. D. Myškis, *Generalization of the theorem on a fixed point of a dynamical system inside of a closed trajectory*, Mat. Sbornik 34 (1954), pp. 525–540 (in Russian).
[22] B. O'Neil, *Essential sets and fixed points*, Amer. J. Math. 75 (1953), pp. 497–509.
[23] — *Induced homology homomorphism for set maps*, Pacific J. Math. 7 (1957), pp. 1179–1184.
[24] H. O. Peitgen, *On the Lefschetz number for iterates of continuous mappings*, Proc. Amer. Math. Soc. 54 (1976), pp. 441–444.
[25] — *Methoden der topologische Fixpunkttheorie in der nichtlinearen Funktionalanalysis*, Habilitationsschrift, Univ. Bonn, 1976.
[26] M. Powers, *Lefschetz fixed point theorem for a new class of multi-valued maps*, Pacific J. Math. 42 (1972), pp. 211–220.
[27] H. W. Siegberg and G. S. Skordev, *Fixed point index and chain approximations*, Pacific J. Math. 102 (1982), pp. 455–486.
[28] G. S. Skordev, *Incidence and coincidence of acyclic maps*, Serdica, Bulgarian Math. J. 1 (1975), pp. 29–41 (in Russian).
[29] — *Equivariant multi-valued maps of the sphere*, Ann. de l'Univ. Sofia, Fac. Math. Mech. 67 (1972/73), pp. 113–137 (in Russian).
[30] — *Fixed points of acyclic maps*, Ann. de l'Univ. Sofia, Fac. Math. Mech. 67 (1972/73), pp. 7–18 (in Russian).
[31] — *On a coincidence of mappings of compact spaces in topological groups*, Fund. Math. 104 (1979), pp. 111–125.
[32] — *Coincidence of maps in Q-simplicial spaces*, Fund. Math. 103 (1981), pp. 67–79.
[33] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York 1966.
[34] H. Steinlein, *Ein Satz über der Leray-Schuder Abbildungsgrad*, Math. Z. 126 (1972), pp. 176–208.
[35] R. Thompson, *A unified approach to local and global fixed point indices*, Adv. Math. 3 (1969), pp. 1–72.
[36] L. Vietoris, *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen*, Math. Ann. 97 (1927), pp. 454–472.
[37] P. Zabreiko and M. Krasnosel'skii, *Iterations of operators and fixed points*, Dokl. Akad. Nauk. USSR 196 (1971), pp. 1006–1009.

UNIVERSITY OF BREMEN, FACHBEREICH MATHEMATIK
FORSCHUNGSSCHWERPUNKT "DYNAMISCHE SYSTEME"
Permanent Address:
CENTER OF MATHEMATICS AND MECHANICS
1090 Sofia, P. O. 373, Bulgaria

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Remarks on the *n*-dimensional geometric measure of compacta

by

K. Borsuk, S. Nowak and S. Spież (Warszawa)

Abstract. By the *n-dimensional geometric measure* of a compactum X lying in the Hilbert space E^ω, one understands the lower bound $\mu_n(X)$ of all positive numbers α such that for every $\varepsilon > 0$ there is an ε-translation $f\colon X \to E^\omega$ such that $f(X)$ lies in a polyhedron $P \subset E^\omega$ for which the n-dimensional measure $|P|_n$ (in the elementary sense) is $\leqslant \alpha$. If $\dim P < n$, we assume $|P|_n = 0$, and if $\dim P > n$, we assume $|P|_n = \infty$.

Some relations between geometric measures of two compacta $X, Y \subset E^\omega$ and the pseudo-measures of $X \cup Y$, $X \cap Y$ and $X \times Y$ are studied.

1. Introduction. In the elementary geometry one assigns to every n-dimensional polyhedron P the number $|P|_n$, defined as the sum of the volumes of all n-dimensional simplices belonging to a triangulation of P. If $\dim P < n$, then one assumes that $|P|_n = 0$, and if $\dim P > n$, then $|P|_n = \infty$.

One knows that $|P|_n$ does not depend on the choice of the triangulation of P. Moreover, one sees easily that

(1.1) If $P_1, P_2, \ldots, P_k$ are polyhedra, then $|P_1 \cup \ldots \cup P_k|_n \leqslant \sum_{i=1}^{k} |P_i|_n$.

Let E^ω denote the usual Hilbert space, i.e. the space consisting of all real sequences $(x_1, x_2, \ldots)$, such that $\sum_{i=1}^{\infty} x_i^2 < \infty$, metrized by the formula

$$\varrho\big((x_1, x_2, \ldots)(y_1, y_2, \ldots)\big) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}.$$

We may consider the Euclidean m-space E^m as the subset of E^ω consisting of all points $(x_1, x_2, \ldots, x_m, 0, 0, \ldots)$ denoted also by $(x_1, x_2, \ldots, x_m)$.

By the *n-dimensional geometric measure* of a compactum $X \subset E^\omega$, we understand the number $\mu_n(X)$ (finite or ∞) defined as the lower bound of all numbers $\alpha > 0$ such that for every $\varepsilon > 0$, there is an ε-translation $f_\varepsilon\colon X \to E^\omega$ (i.e. a map f_ε satisfying the condition $\varrho(x, f_\varepsilon(X)) < \varepsilon$ for every $x \in X$) such that $f_\varepsilon(X)$ is a subset of a polyhedron $P \subset E^\omega$ with $|P|_n < \alpha$. It is known (see [2]) that: