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Almost maximal ideals

by

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Abstract. An ideal in a distributive lattice is said to be almost maximal if it is prime and satisfies a first-order closure condition which, in the presence of the axiom of choice, is equivalent to saying that it is an intersection of maximal ideals. Assuming the axiom of choice, we show that the almost maximal ideals correspond to points of the soberification of the maximal ideal space of the lattice; in the absence of the axiom of choice, we investigate the strength of the "almost maximal ideal theorem" that every nontrivial distributive lattice has an almost maximal ideal. Our two main results are that this assertion implies the Tychonoff theorem for products of compact sober spaces, and that it does not imply the axiom of choice.

Introduction. It is well known that J. L. Kelley [17] proved that the Tychonoff product theorem is logically equivalent (in any reasonable set theory) to the axiom of choice. However, elsewhere in topology and analysis it is far commoner to encounter theorems which are equivalent not to the full axiom of choice but to the prime ideal theorem, i.e., the assertion that every nontrivial Boolean algebra (or equivalently, every nontrivial distributive lattice) has a prime ideal. Among examples of such theorems, let us cite:

(i) The Stone representation theorem for Boolean algebras (or for distributive lattices).

(ii) The Stone–Čech compactification theorem.

(iii) Tychonoff's theorem for products of compact Hausdorff spaces.

(iv) Alaoglu's theorem on compactness of the unit ball of the dual of a Banach space.

(v) The theorem that the hyperspace of a compact Hausdorff space (i.e., the space of closed subsets with the Vietoris topology) is compact.

There is of course a family resemblance between these theorems: each of them asserts the compactness of some space which may be *constructed* without any use of choice, but which will not have its expected properties unless it is compact. What is more striking is that in each case the space in question occurs naturally as the space of points of a certain locale (or "pointless space"; see [15]), and that the compactness of this locale can be proved constructively. (For the appropriate locale-theoretic construction, see [14] in case (i), [2] or [12] in cases (ii) and (iii), [21] in case (iv) and [16] in case (v).) Thus the non-constructive input of each of these five theorems reduces to one or other of the assertions.

(vi) Every coherent locale has enough points (i.e., is isomorphic to the open-set locale of a space).

(vii) Every compact regular locale has enough points.

Of these, (vi) is virtually a direct translation of the prime ideal theorem (since the spaces of points of coherent locales are just the prime ideal spaces of distributive lattices), and (vii) is easily proved equivalent to it (see [13]).

On the other hand, there does not appear to be any hope of deducing the Tychonoff theorem for arbitrary compact spaces in a similar fashion from an existence-of-points theorem for locales; although the Tychonoff theorem for locales can be proved without choice [12], the most that we can expect to deduce from it in this way is the Tychonoff theorem for compact sober spaces (i.e., those spaces which occur as the spaces of points of locales). And Kelley's proof of (Tychonoff $\Rightarrow$ AC) makes what appears to be unavoidable use of non-sober spaces. (Specifically, Kelley's method requires the ability to impose on an arbitrary set X a compact topology which does not discriminate between the points of X — so that every permutation of X must be continuous for it. It is easy to see that the only such topologies are the indiscrete topology and the cofinite topology, neither of which is sober when X is infinite.)

It therefore becomes of interest to ask: what is the strength of the Tychonoff theorem for compact sober spaces? Although we do not (yet) have a definitive answer to this question, the purpose of this paper is to present evidence for the conjecture that it is equivalent to a choice principle which we have christened the "almost maximal ideal theorem", and which is in an obvious sense intermediate between the prime and maximal ideal theorems for distributive latices. (It is known that the latter is equivalent to the full axiom of choice — see [20].) We also discuss a number of other locale-theoretic equivalents and consequences of the almost maximal ideal theorem, and provide a model to show that it is strictly weaker than the axiom of choice. (The problem of showing that it is inequivalent to the prime ideal theorem remains open.)

In the first sentence of this Introduction, we used the phrase "in any reasonable set theory". It is perhaps appropriate to emphasize here that the formal properties which we shall require of our underlying set theory are fairly naive — in particular, we shall not require the axioms of foundation or of replacement, except in one instance where we need to justify a proof by transfinite induction. For the reader who is familiar with categorical logic, let us say that our arguments could be carried out in the internal logic of a Boolean topos.

1. The soberification of $\max A$. Let A be a distributive lattice (with 0 and 1). The space spec A of prime ideals of A (with the Stone–Zariski toplogy, i.e., that based by the sets $\{I \in \operatorname{spec} A \mid a \notin I\}$) is well known to be sober; it is (almost by definition) the space of points of the locale $\operatorname{Idl}(A)$ of all ideals of A. On the other

hand, the subspace $\max A$ consisting of the closed points of spec A (i.e., the maximal ideals of A) need not be sober; it is always compact (if we assume the maximal ideal theorem) and T_1, but Wallman [27] showed that any compact T_1-space can occur as $\max A$ for some A. It is therefore of interest to determine the soberification of $\max A$.

If I is an ideal of A, consider the set

$$j(I) = \{a \in A \mid (\forall b \in A)(a \vee b = 1 \Rightarrow (\exists c \in I)(b \vee c = 1))\}.$$

LEMMA 1.1. *$j(I)$ is an ideal of A.*

Proof. It is clearly a lower set. If $a_1, a_2 \in j(I)$ and we have $a_1 \vee a_2 \vee b = 1$, then there exists $c_1 \in I$ with $c_1 \vee a_2 \vee b = 1$, and hence there exists $c_2 \in I$ with $c_1 \vee c_2 \vee b = 1$. Since I is closed under finite joins, we deduce $a_1 \vee a_2 \in j(I)$. ■

LEMMA 1.2. *The map j: $\operatorname{Idl}(A) \to \operatorname{Idl}(A)$ is a nucleus (in the sense of [24]).*

Proof. The inclusion $I \subseteq j(I)$ is clear from the form of the definition. Suppose $a \in j(j(I))$; then for any b with $a \vee b = 1$ we can find $c \in j(I)$ with $c \vee b = 1$, and hence we can find $d \in I$ with $d \vee b = 1$, so $a \in j(I)$. Thus j is idempotent. It is also clear that j is order-preserving. Suppose $a \in j(I_1) \cap j(I_2)$; then if $a \vee b = 1$ we can find $c_1 \in I_1$, $c_2 \in I_2$ with $c_1 \vee b = c_2 \vee b = 1$, whence by distributivity we have $(c_1 \wedge c_2) \vee b = 1$. But $c_1 \wedge c_2 \in I_1 \cap I_2$, so $a \in j(I_1 \cap I_2)$; thus j preserves finite intersections. ■

LEMMA 1.3. *If $j(I) = A$, then $I = A$.*

Proof. If $1 \in j(I)$, then there exists $c \in I$ with $c \vee 0 = 1$, i.e., $1 \in I$. ■

COROLLARY 1.4. *The sublocale $\operatorname{Idl}(A)_j$ of j-fixed ideals is compact.*

Proof. By Lemma 1.3, any covering of the top element of $\operatorname{Idl}(A)_j$ is actually a covering of the top element of $\operatorname{Idl}(A)$. But $\operatorname{Idl}(A)$ is well known to be compact. ■

COROLLARY 1.5. *Maximal (proper) ideals are j-fixed.*

Proof. Immediate from Lemma 1.3. ■

PROPOSITION 1.6. *If we assume the maximal ideal theorem, then $j(I)$ is the intersection of all the maximal ideals which contain I. In particular, an ideal is j-fixed if and only if it is an intersection of maximal ideals.*

Proof. By Corollary 1.5, $j(I)$ is included in the intersection of the maximal ideals which contain I. Suppose $a \notin j(I)$; then there exists $b \in A$ with $a \vee b = 1$ but $c \vee b \neq 1$ for all $c \in I$. So $I \cup \{b\}$ generates a proper ideal, which we can enlarge to a maximal ideal M; and since $b \in M$ we have $a \notin M$. So every element of the intersection is in $j(I)$. The second assertion follows immediately from the first. ■

COROLLARY 1.7. *If we assume the maximal ideal theorem, then $\operatorname{Idl}(A)_j$ is isomorphic to the locale of open sets of $\max A$.*

Proof. To determine $\Omega(\max A)$ as a sublocale of $\Omega(\operatorname{spec} A) \cong \operatorname{Idl}(A)$, we have to compute the nucleus k on $\Omega(\operatorname{spec} A)$, where

$$k(U) = \bigcup \{V \in \Omega(\operatorname{spec} A) \mid V \cap \max A = U \cap \max A\}.$$

In terms of ideals, this says that $k(I)$ is the largest ideal contained in exactly the same maximal ideals as I; but by Proposition 1.6 this is just $j(I)$. ■

It follows from Corollary 1.7 that the soberification of $\max A$ is simply the space of points of $\mathrm{Idl}(A)_j$. But points of $\mathrm{Idl}(A)_j$ correspond bijectively to prime elements of this lattice; and we have

LEMMA 1.8. *Let I be a j-fixed ideal of a distributive lattice A. Then I is prime as an element of $\mathrm{Idl}(A)_j$ if and only if it is a prime ideal in the usual sense.*

Proof. The first condition says that

$$J \cap K \subseteq I \quad \text{implies} \quad J \subseteq I \quad \text{or} \quad K \subseteq I$$

for j-fixed ideals J and K; the second says that

$$a \wedge b \in I \quad \text{implies} \quad a \in I \quad \text{or} \quad b \in I$$

for $a, b \in A$. To deduce the second condition from the first, take $J = j(\downarrow(a))$, $K = j(\downarrow(b))$, and note that $J \cap K = j(\downarrow(a) \cap \downarrow(b)) = j(\downarrow(a \wedge b))$; to deduce the first from the second, suppose $J \cap K \subseteq I$ and $a \in J - I$. Then for any $b \in K$ we have $a \wedge b \in J \cap K \subseteq I$ and so $b \in I$, i.e., $K \subseteq I$. ■

We define an ideal of a distributive lattice to be *almost maximal* if it is both prime and j-fixed; thus we can state

COROLLARY 1.9. *If we assume the maximal ideal theorem, then the soberification of* $\max A$ *is the space* $\mathrm{almax}\, A$ *of almost maximal ideals of A (with the Stone–Zariski topology).* ■

Clearly, every maximal ideal is almost maximal, but the converse is not true in general. For example, if A is the cofinite topology on an infinite set, then the trivial ideal $\{\emptyset\}$ is almost maximal in A (being the intersection of all the maximal ideals of A). Recalling Wallman's characterization [27] of spaces of the form $\max A$ as compact T_1-spaces, we can give a similar characterization of spaces of the form $\mathrm{almax}\, A$:

COROLLARY 1.10. *Assume the maximal ideal theorem. Then a space X is homeomorphic to* $\mathrm{almax}\, A$ *for some distributive lattice A if and only if it is compact, sober and a "Jacobson space" in the sense of* [4], 0 2.8.1 (*i.e., its subspace X^c of closed points is very dense*).

Proof. The given conditions are equivalent to saying that X is (homeomorphic to) the soberification of X^c, and that the latter is compact. Since X^c is clearly always a T_1-space, the result follows from Wallman's result quoted above and Corollary 1.9. ■

By Corollary 1.10, the almost maximal ideals are topologically "close" to the set of maximal ideals of A. It is natural, therefore, to ask whether they are order-theoretically close to the maximal ideals; that is, do they form an upper set in the poset of prime ideals of A? However, the answer is no:

EXAMPLE 1.11. Let A be the set of all subsets $S \subseteq N$ such that either $S = \emptyset$, or else S is cofinite and either $0 \in S$ or $1 \notin S$. A is a sublattice of PN (indeed, a topology on N) and therefore distributive; the ideals $\{S \in A \mid n \notin S\}$, $n > 0$, are all maximal, and their intersection is the prime (and hence almost maximal) ideal $\{\emptyset\}$. But the set $\{S \in A \mid 0 \notin S\}$ is a prime ideal which is not j-fixed; its j-closure is the maximal ideal $\{S \in A \mid 1 \notin S\}$.

Next we consider a particular class of distributive lattices. We shall say A is *semi-normal* if, whenever $a \vee b = 1$ in A, we can find elements c and d with $c \vee b = a \vee d = 1$ and $c \wedge d \in j(\{0\})$. This is weaker than the concept of normality (see [25]), which requires $c \wedge d = 0$; it is not hard to see that A is semi-normal if and only if $\max A$ is Hausdorff (cf. [14], II, 3.6). Since Hausdorff spaces are sober, we should expect every almost maximal ideal in a semi-normal lattice to be maximal. And this is indeed the case:

LEMMA 1.12. *In a semi-normal distributive lattice, almost maximal ideals are maximal.*

Proof. Let I be such an ideal, and suppose $a \notin I$. Then we can find $b \in A$ such that $a \vee b = 1$, but $c \vee b \neq 1$ for all $c \in I$. Apply the definition of semi-normality to obtain c and d with $a \vee d = c \vee b = 1$ and $c \wedge d \in j(\{0\})$; then $c \notin I$, but $c \wedge d \in I$ since I is j-fixed, and so $d \in I$ by primeness. Hence $I \cup \{a\}$ generates A as an ideal; so I is maximal. ■

To conclude this section, let us briefly mention the ring-theoretic versions of the concepts we have introduced, exploiting the well-known link between the spectral theories of commutative rings and distributive lattices (cf. [25]). They are already well known to ring-theorists: in particular the analogue of $j(I)$ is the Jacobson radical [9], [10] of a ring ideal I. Recall that this can be defined in first-order terms as the set of elements a such that $1+ab$ is invertible modulo I for all b, i.e.

$$j(I) = \{a \in A \mid (\forall b \in A)(\exists c \in A)(ab+c+abc \in I)\}.$$

It is straightforward (even in a non-commutative ring A) to verify the analogues of 1.1–1.3 for this definition of j, i.e., that $j(I)$ is an ideal, that $I \subseteq j(I) = j(j(I))$ and $j(I_1 \cap I_2) = j(I_1) \cap j(I_2)$, and that $1 \in j(I)$ implies $1 \in I$. (We leave the verification of these assertions to the reader.) Moreover, if we assume the maximal ideal theorem for rings (which, like that for lattices, is equivalent to the axiom of choice [7]), then any element not in $j(I)$ is excluded from some maximal ideal containing I, and so the analogue of 1.6 holds. The lattice of all ideals of a ring A is not in general a locale (since it fails to be distributive), but if (in the commutative case, at least) we cut down to the distributive lattice $\mathrm{RIdl}(A)$ of radical ideals (and note that $j(I)$ is a radical ideal for any I), then we may regard j as a nucleus on $\mathrm{RIdl}(A)$, and proceed to obtain the analogues of 1.7 and 1.9. Thus one can develop a theory of almost maximal ideals in commutative rings; however, it remains to be seen whether the theory has a useful generalization to non-commutative rings.

2. Feebly maximal ideals. This section is a digression from our main line of research: it began as an attempt to give a model-theoretic characterization of almost maximal ideals, but it turns out that the class of ideals characterized is strictly larger than that of almost maximal ideals. Nevertheless, it may be that this class (whose members we shall call feebly maximal ideals) will be useful for some future application, and so we here record the known facts about it. Throughout this section, we shall assume the axiom of choice.

Let I be an almost maximal ideal in a distributive lattice A, and denote by S the set of maximal ideals which contain I. By Proposition 1.6, we have $\bigcap S = I$; we now consider the set

$$\mathfrak{I} = \{S' \subseteq S \mid \bigcap S' \neq I\}.$$

LEMMA 2.1. *$\mathfrak{I}$ is a proper ideal of subsets of S.*

Proof. It is clearly a lower set in PS, and by the remarks above it does not contain S. It is closed under binary unions, for if $I = \bigcap (S_1 \cup S_2) = \bigcap S_1 \cap \bigcap S_2$ then by primeness we have either $I = \bigcap S_1$ or $I = \bigcap S_2$. ■

It follows from Lemma 2.1 (and the prime ideal theorem) that we can find an ultrafilter $\mathfrak{F}$ on S which is disjoint from $\mathfrak{I}$, i.e. such that $\bigcap S' = I$ for all $S' \in \mathfrak{F}$. We now consider the ultraproduct $(\prod_{M \in S} M)/\mathfrak{F}$ as an ideal in the ultrapower $(A^S)/\mathfrak{F}$; it is clearly maximal, since the property of being a maximal ideal is a first-order one. On the other hand, the inverse image of this ideal under the diagonal embedding $A \to (A^S)/\mathfrak{F}$ is $\bigcup \{(\bigcap S') \mid S' \in \mathfrak{F}\} = I$; thus we have shown

PROPOSITION 2.2. *Every almost maximal ideal in a distributive lattice A is expressible as the inverse image of a maximal ideal of some distributive lattice B under an elementary embedding $A \to B$.* ■

In the converse direction, suppose we have an elementary embedding $f\colon A \to B$ of distributive lattices and a maximal ideal M of B. What can we say about the ideal $I = f^{-1}(M)$ of A? It is clearly prime, since M is; also, if $a \in I$ and $b \notin I$, then we have

$$B \models (\exists y \in M)(y \vee f(b) = 1)$$

by maximality of M, whence

$$B \models (\exists y \in B)(y \vee f(b) = 1 \text{ and } y \vee f(a) \neq 1)$$

since $f(a) \in M$. Since f is elementary, we deduce

$$A \models (\exists x \in A)(x \vee b = 1 \text{ and } x \vee a \neq 1)$$

and so $b \notin j(\downarrow(a))$. Thus $j(\downarrow(a)) \subseteq I$ for every $a \in I$; equivalently, I is a union of j-fixed ideals. In particular if I is principal, then it must itself be j-fixed and hence almost maximal; but in general this is not so.

EXAMPLE 2.3. Let A be the set of all subsets S of N such that either (S is finite and $0 \notin S$) or ($N-S$ is finite), and let I be the ideal of all finite sets in A. A is a distributive lattice, since it is a sublattice of PN, and I is clearly a prime ideal. But I is not j-fixed, since $N-\{0\} \in j(I)$; however, it is easy to see that each $S \in I$ satisfies $j(\downarrow(S)) = \downarrow(S)$, and so I is a union of j-fixed ideals.

We shall say that an ideal in a distributive lattice is *feebly maximal* if it is prime and a union of j-fixed ideals. Now we can state

PROPOSITION 2.4. *An ideal I of a distributive lattice A is feebly maximal if and only if it can be expressed as $f^{-1}(M)$ for some elementary embedding $f\colon A \to B$ and maximal ideal M of B.*

Proof. One direction was proved in the remarks before Example 2.3; the converse is a straightforward strengthening of Proposition 2.2. Let I be a feebly maximal ideal of A; let S be the set of all maximal ideals of A, and define

$$\mathfrak{I} = \{S' \subseteq S \mid \bigcap S' \nsubseteq I\}$$

and $S_a = \{M \in S \mid a \in M\}$ for each $a \in I$. Since $S_a \cap S_b = S_{a \vee b}$ and $\bigcap S_a = j(\downarrow(a)) \subseteq I$ for all $a \in I$, it is clear that the S_a generate a filter in PS disjoint from $\mathfrak{I}$; also $\mathfrak{I}$ is an ideal by primeness of I. So we can find an ultrafilter $\mathfrak{F}$ on S which is disjoint from $\mathfrak{I}$ but contains every S_a, so that

$$I = \bigcup \{j(\downarrow(a)) \mid a \in I\} \subseteq \bigcup \{(\bigcap S') \mid S' \in \mathfrak{F}\} \subseteq I;$$

equivalently, I is the inverse image of $(\prod_{M \in S} M)/\mathfrak{F}$ under the diagonal embedding $A \to (A^S)/\mathfrak{F}$. ■

3. Topological consequences of AMIT. We now revert to our main line of development, by introducing the principle which we shall call the almost maximal ideal theorem (AMIT): Every nontrivial distributive lattice has an almost maximal ideal. It is clear that AMIT is implied by the maximal ideal theorem (i.e., by the axiom of choice) and implies the prime ideal theorem; we devote this section to studying results in locale theory and topology which are equivalent to AMIT or follow directly from it.

THEOREM 3.1. *The following assertions are equivalent:*

(i) AMIT.

(ii) *Every nontrivial compact locale has at least one point* (*cf.* [14], *Lemma* III, 1.9).

(iii) *The space of points of any compact locale is compact.*

Proof. (i) ⇒ (ii): Let A be a compact locale, I any ideal of A. First we observe that $\bigvee I \in j(I)$; for if $\bigvee I \vee b = 1$, then we have $\bigvee\{c \vee b \mid c \in I\} = 1$, and by compactness there exists $c \in I$ with $c \vee b = 1$. Thus any j-fixed ideal of A is principal, and hence any almost maximal ideal must be generated by a prime element of A. But prime elements correspond bijectively to points of a locale.

(ii) ⇒ (i): Let A be a distributive lattice. By Corollary 1.4, we know $\mathrm{Idl}(A)_j$ is compact; and Lemma 1.3 ensures that it is nontrivial whenever A is. But by Lemma 1.8 the points of $\mathrm{Idl}(A)_j$ correspond bijectively to almost maximal ideals of A.

(ii) ⇒ (iii): Let A be a compact locale, $a \in A$. If $a \neq 1$, then the closed sublocale $A_{c(a)}$ is compact ([14], III, 1.2) and nontrivial, and so has a point; that is, there exists a point p of A which is not in the open set $\varphi(a) \subseteq \mathrm{pt}(A)$ corresponding to a. So if $\{\varphi(s) \mid s \in S\}$ is a family of open subsets of $\mathrm{pt}(A)$ whose union $\varphi(\bigvee S)$ contains all the points of A, we deduce that $\bigvee S = 1$ in A; thus $\mathrm{pt}(A)$ inherits compactness from A.

(iii) ⇒ (ii): Suppose given a counterexample to (ii), i.e., a nontrivial compact locale A with no points. Let B be any locale, and form the ordinal sum $C = B \oplus A$, i.e., the poset obtained by identifying the top element of B with the bottom element of A. It is easily verified that C is a locale; and since A is nontrivial, any covering of the top element of C must contain a covering of the top element of A, so that C inherits compactness from A. But any prime element of C must be either a prime element of A or a prime element of B, so since A has no points we obtain a homeomorphism $\mathrm{pt}(C) \cong \mathrm{pt}(B)$. (Thus we can think of C as a "no-point compactification" of B.) In particular, on taking B to be a non-compact spatial locale, we obtain a counterexample to (iii). ■

THEOREM 3.2. AMIT *implies the Tychonoff theorem for products of compact sober spaces.*

Proof. Let $(X_\gamma \mid \gamma \in \Gamma)$ be a family of sober spaces. Then we have $X_\gamma \cong \mathrm{pt}(\Omega(X_\gamma))$ for each γ, and since the functor pt is right adjoint to Ω it preserves products; so $\prod_{\gamma \in \Gamma} X_\gamma \cong \mathrm{pt}(\prod_{\gamma \in \Gamma} \Omega(X_\gamma))$. Now the Tychonoff theorem for products of compact locales is valid without any use of choice ([12], Theorem 2.7), so compactness of all the X_γ (i.e., of the locales $\Omega(X_\gamma)$) implies compactness of $\prod \Omega(X_\gamma)$, and then from Theorem 3.1 (iii) we deduce compactness of $\prod X_\gamma$. ■

Before proceeding further, we should recall that the proof of Theorem 2.7 in [12] involves a transfinite induction, and so we cannot claim to have proved Theorem 3.2 in the "naive" set theory to which we alluded in the Introduction. However, as observed in [12], the transfinite induction reduces to a single step under the extra hypothesis of local compactness, and so we can say without any reservation that AMIT implies the Tychonoff theorem for compact and locally compact sober spaces.

It is not clear at present whether the converse of Theorem 3.2 is true. Since all Hausdorff spaces are sober, we know from the work of Łoś and Ryll-Nardzewski [18] that the Tychonoff theorem for sober spaces does at least imply the prime ideal theorem; and on the other hand, the prime ideal theorem implies Tychonoff's theorem for compact and stably locally compact sober spaces (cf. [13]). It seems plausible that we might be able to deduce condition (iii) of Theorem 3.1 from Tychonoff's theorem for compact sober spaces, by showing that any compact locale has enough closed maps into compact spatial locales to separate points; what is lacking at present is a satisfactory theory of closed maps of locales.

We now turn our attention to locally compact locales; recall that a locale is said to be *locally compact* if it is a continuous lattice [3]. Hofmann and Lawson [8] and Banaschewski [1] have given proofs, using the axiom of choice, that every locally compact locale is spatial, and that the category of locally compact locales is equivalent to the category of locally compact sober spaces. In what follows, we shall refer to Banaschewski's proof; this depends on two key lemmas, which we shall cite as "Banaschewski's lemma 1" and "Banaschewski's lemma 3", and which respectively assert

(BL1) In a continuous lattice, every Scott-open set is expressible as a union of Scott-open filters.

(BL3) In a locale, any Scott-open filter is expressible as an intersection of completely prime filters.

Here, as usual, a subset of a complete lattice is said to be *Scott-open* if it is an upper set and inaccessible by directed joins, and a filter is said to be *completely prime* if it is inaccessible by arbitrary joins (i.e., if its complement is a principal ideal). BL1 is not originally due to Banaschewski ([3] attributes it to J. D. Lawson), but BL3 does not seem to have appeared explicitly anywhere before [1]. From our point of view, its interest is given by

LEMMA 3.3. AMIT *is equivalent to* BL3.

Proof. Assume AMIT; let F be a Scott-open filter in a locale A, and a an element of $A - F$. By [16], Lemma 3.4, F is "admissible" for some nucleus j on A, and so by considering the sublocale A_j and the element $j(a)$ we may reduce to the case when A is compact and F is the trivial filter $\{1\}$. Then by the argument of the proof of (ii) ⇒ (iii) in Theorem 3.1, we may find a point p of A with $p^*(a) = 0$, so that $(p^*)^{-1}(1)$ is a completely prime filter not containing a. Conversely, if we assume BL3, then by applying it to the filter $\{1\}$ in a nontrivial compact locale A, we deduce that A has a completely prime filter and hence a point. ■

Now the proof of BL1 invokes no more than (countable) dependent choice (DC), in order to construct a descending sequence $a_1 \gg a_2 \gg a_3 \gg \ldots$ of instances of the way-below relation, and so we deduce

THEOREM 3.4. AMIT *plus* DC *implies that every locally compact locale has enough points.* ■

Once again, it is not clear whether the converse of Theorem 3.4 holds. The assertion "Every locally compact locale is spatial" is sufficient to imply the Tychonoff theorem for compact and locally compact sober spaces, as observed in [12]; but if we restrict ourselves to stably locally compact locales, then we obtain an assertion equivalent to the prime ideal theorem. The dependence on DC may be removed by redefining local compactness: let us say that a is *deeply way below* b (in a complete lattice A) if there exists a sequence $(c_1, c_2, c_3, \ldots)$ of elements of A with $a \ll c_{i+1} \ll c_i \ll b$ for all i, and that a complete lattice is *deeply continuous* (or that a locale is *deeply locally compact*) if every element is expressible as a join of elements deeply way below itself. Then AMIT implies that every deeply locally compact locale has enough points; the use of DC is in proving that every continuous lattice is deeply continuous. I do not know whether this last assertion (or even BL1) is sufficient to imply DC, but it seems not unlikely.

4. The strength of AMIT. In this section, we shall prove that the almost maximal ideal theorem does not imply the axiom of choice; we shall do this by showing that AMIT holds in the model used by Halpern [5] to show the independence of AC from the prime ideal theorem. Halpern's model is of Fraenkel–Mostowski type [19], and as such is determined by a single topological group G in the ground model, which in this particular case is the group of order-preserving permutations of the rationals (with topology defined by saying that the stabilizers of finite subsets of Q form a base of open neighbourhoods of the identity). Halpern showed that if B is a nontrivial internal Boolean algebra in this model (i.e. — essentially — a Boolean algebra in the ground model, equipped with a continuous action of some basic open subgroup G_0 of G), and I is an ideal of B maximal amongst those invariant under the action of G_0 (the existence of such an ideal being guaranteed by Zorn's lemma in the ground model), then I is prime. (Note that this is a stronger result than we actually need for the internal validity of PIT in the model; the latter merely asserts the existence of a prime ideal of B invariant under some open subgroup G_1 of G_0.)

Our first task is to modify Halpern's proof of this result so that it does not make any reference to complementation in B, and can therefore be applied directly to internal distributive lattices in the model. (Of course, we known that PIT holds for distributive lattices in the model, since it holds for Boolean algebras; but we are going to need the extra strength of Halpern's result, to which we alluded above.) We begin by stating two trivial lemmas which take the place of Halpern's Lemmas 2 and 3 ([5], p. 62).

LEMMA 4.1. *Let H be a group of automorphisms of a distributive lattice A, I an ideal of A invariant under H, a an element of A. Let J be the smallest H-invariant ideal of A which includes $I \cup \{a\}$. If $J = A$, then there exists a finite set $S \subseteq H$ and an element $b \in I$ such that*

$$1_A = \bigvee \{\varphi(a) \mid \varphi \in S\} \vee b\,. \ \blacksquare$$

LEMMA 4.2. *Let A be a distributive lattice, I an ideal of A and X a finite subset of $A \times A$ such that $a_1 \wedge a_2 \in I$ for each $(a_1, a_2) \in X$. Let P denote the set of all functions $X \to \{1, 2\}$; then*

$$\bigwedge \{\bigvee \{a_{f(a_1,a_2)} \mid (a_1, a_2) \in X\} \mid f \in P\} \in I\,. \ \blacksquare$$

THEOREM 4.3. *Let A be a distributive lattice equipped with a continuous action of a basic open subgroup G_0 of G, I an ideal of A maximal amongst proper G_0-invariant ideals. Then I is prime.*

Proof. Suppose we have $a_1 \wedge a_2 \in I$, $a_1 \notin I$, $a_2 \notin I$. By Lemma 4.1 we can find finite sets S_1, S_2 and elements $b_1, b_2 \in I$ such that

$$\bigvee \{\varphi(a_i) \mid \varphi \in S_i\} \vee b_i = 1_A$$

for $i = 1, 2$. Applying the combinatorial argument in Halpern's paper, we may now find a finite subset X of G_0 such that, for each f: $X \to \{1, 2\}$, the set $\{\varphi(a_{f(\varphi)}) \mid \varphi \in X\}$ contains the image under some $\psi_f \in G_0$ of either $\{\varphi(a_1) \mid \varphi \in S_1\}$ or $\{\varphi(a_2) \mid \varphi \in S_2\}$. Thus for each such f we have

$$\bigvee \{\varphi(a_{f(\varphi)}) \mid \varphi \in X\} \vee b_f = 1$$

for some $b_f \in I$, and hence

$$\bigwedge \{\bigvee \{\varphi(a_{f(\varphi)}) \mid \varphi \in X\} \mid f \in P\} \vee \bigvee \{b_f \mid f \in P\} = 1\,.$$

But the meet on the left is in I by Lemma 4.2, so we have a contradiction. ■

Given this result, it is now a triviality to show that any ideal satisfying the hypotheses of Theorem 4.3 is almost maximal.

LEMMA 4.4. *If A is a distributive lattice, and I an ideal of A invariant under some group H of automorphisms of A, then $j(I)$ is H-invariant.*

Proof. Suppose $a \in j(I)$, $\varphi \in H$. If $\varphi(a) \vee b = 1$, then we have

$$a \vee \varphi^{-1}(b) = \varphi^{-1}(1) = 1\,,$$

whence there exists $c \in I$ with $c \vee \varphi^{-1}(b) = 1$. So $\varphi(c) \vee b = 1$; but $\varphi(c) \in I$ since I is H-invariant. Hence $\varphi(a) \in j(I)$. ■

COROLLARY 4.5. *Let H be a group of automorphisms of a distributive lattice A, and I an ideal maximal amongst H-invariant proper ideals of A. Then I is j-fixed.*

Proof. Immediate from Lemmas 1.3 and 4.4. ■

THEOREM 4.6. AMIT *holds in Halpern's model; in particular it does not imply the axiom of choice.*

Proof. Let A be a nontrivial distributive lattice in the model, and G_0 a basic open subgroup of G which acts on A. By Zorn's Lemma applied in the ground model, we can find an ideal I which is maximal amongst G_0-invariant proper ideals of A. Then Theorem 4.3 tells us that I is prime, and Corollary 4.5 that it is j-fixed (note that both conditions on I, being first-order, have the same interpretation inside Halpern's model as in the ground model). ■

Combining Theorems 3.2 and 4.6, we see that the Tychonoff product theorem for compact sober spaces does not imply the axiom of choice. Thus the use of non-sober topologies in [17] was not merely an accidental consequence of Kelley's method of proof (as we observed in the Introduction), but an essential feature of any attempt to deduce the full axiom of choice from a Tychonoff-type theorem.

Subsequent to Halpern's paper, Halpern and Lévy [6] constructed a model of ZF in which PIT holds but AC fails; they achieved this essentially by forcing the counterexample to AC in Halpern's first model into the well-founded part of the universe, along the general lines laid down by Pincus [22], although their presentation was somewhat different. It seems clear that such a forcing extension will not disturb the validity of AMIT, and so we may obtain a ZF model for the independence of AC from AMIT.

A rather different model for the independence of AC from PIT was constructed by Pincus [23] using an iterated-forcing argument; Pincus's model satisfies DC as well as PIT, but still fails AC. I have not been able to determine whether AMIT holds in Pincus's model; but it is perhaps worth remarking that the conjunction of PIT and DC does imply something on the road to AMIT.

Remark 4.7. If PIT and DC both hold, then every nontrivial distributive lattice has a feebly maximal ideal.

Proof. Let A be such a lattice. If I_0 is a prime ideal of A, then $j(I_0)$ is a proper ideal by Lemma 1.3, so another application of PIT enables us to find a prime ideal $I_1 \supseteq j(I_0)$. Proceeding inductively, DC allows us to construct a chain of prime ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ with $j(I_n) \subseteq I_{n+1}$ for all n. Then $I = \bigcup_{n=0}^{\infty} I_n$ is prime because primeness is inherited by directed unions, but it is equal to $\bigcup_{n=0}^{\infty} j(I_n)$ and hence feebly maximal. ■

In view of Example 2.3, however, it seems unlikely that this argument can be extended any further; and so the problem of determining whether AMIT is independent of PIT, either with or without DC, remains open.

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