Free products of n-groups

by

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Abstract. We give two constructions of free products of n-groups and also a construction of free n-groups. We also prove a theorem on the form of free products of covering $(k+1)$-groups of $(n+1)$-groups (where $n = s \cdot k$).

1. Introduction. In [10] we described various classes of morphisms in the category of n-groups $\mathbf{Gr}_n$ and also two functors: the forgetful functor Ψ_s: $\mathbf{Gr}_{k+1} \to \mathbf{Gr}_{n+1}$ and its left adjoint Φ_s: $\mathbf{Gr}_{n+1} \to \mathbf{Gr}_{k+1}$ (where $n = sk$). We were interested mainly in the connections of these functors with projective and inductive limits. The preservation properties of Φ_s proved in [10] are used here to obtain a description of free products of n-groups and free n-groups.

In the definition of Φ_s (cf. [10]) it is essential that $\Phi_s(G)$ is a free covering $(k+1)$-group of an $(n+1)$-group G, but it is unimportant in which way that free covering $(k+1)$-group has been constructed, since distinct constructions yield naturally equivalent functors. In [10] we exploited the construction of a free covering $(k+1)$-group described in [9] and the definition of Φ_s based on that construction. In the present paper, whenever we need, we exploit two constructions of the free covering group: the construction of Post (cf. [12]) and that of [9]. We use Post's construction in Theorem 1, in which by $\Phi_n(G)$ we understand the free covering group as described in [12]. Otherwise, we adopt the definition of Φ_s as given in [10].

As is easy to check, the following theorem is true in any category with free products:

Let a family of objects $\{G_t\}_{t\in T}$ of a category $\mathscr{K}$ be given. If a subfamily $\{G_t\}_{t\in S}$ of the family $\{G_t\}_{t\in T}$ consists of noninitial objects in $\mathscr{K}$, then the free product $[\coprod_{t\in T} G_t;\ \{\alpha_t\colon G_t \to \coprod_{t\in T} G_t\}_{t\in T}]$ is isomorphic to the free product $[\coprod_{t\in S} G_t;\ \{\beta_t\colon G_t \to \coprod_{t\in S} G_t\}_{t\in S}]$, i.e., there exists a unique isomorphism $\eta\colon \coprod_{t\in T} G_t \to \coprod_{t\in S} G_t$ such that $\eta\alpha_t = \beta_t$ for $t \in S$.

This says that, as long as we are interested in free products, we may exclude initial objects from the category under consideration. Therefore we assume that all n-groups considered in the present paper are nonempty.

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The terminology of this paper is the same as in [9] and [10], where we also discussed relevant notions. Recall only that φ always denotes the $(k+1)$-group operation on the cyclic $(k+1)$-group $\mathfrak{C}_{s,k+1}$. In particular, if $k=1$, then φ denotes the group operation on the cyclic group $\mathfrak{C}_{n,2}$ of order n. The letters f and g will always denote, respectively, $(n+1)$-group and $(k+1)$-group operations in the $(n+1)$-groups and $(k+1)$-groups under consideration.

In [10] the symbol $f_{(s)}$ was defined for $s\geqslant 1$. Now it seems useful to extend that definition to the case of $s=0$ by assuming that $f_{(0)}$ is always the unary operation given by $f_{(0)}(x)=x$.

Throughout the paper we assume $n>1$.

Some of the results presented here were announced in [8].

2. Two constructions of free products of n-groups. The functor Φ preserves and reflects inductive limits (cf. [10]). This property enables us to extract a construction of free products of n-groups from the well-known construction of free products of groups. By a free product of groups we always mean a group constructed following [6], i.e., the set of reduced words together with an appriopriate group operation. The empty word is denoted by $\emptyset$.

As free covering groups play a central role in this construction, it depends on which construction of the free covering group we have in hand (cf. e.g. [12], [2], [5], [7], [9]).

LEMMA 1. *Let $[L'; \{\gamma_t: \Phi_n(G_t)\to L'\}_{t\in T}]$ be the free product of a nonempty family of free covering groups $\{\langle\Phi_n(G_t),\tau_t,\zeta_t\rangle\}_{t\in T}$ of $(n+1)$-groups $\{G_t\}_{t\in T}$. Then the morphism $\zeta\colon L'\to\mathfrak{C}_{n,2}$ given by $\zeta(a_1\ldots a_r)=\varphi_{(r-1)}(\zeta_{t_1}(a_1),\ldots,\zeta_{t_r}(a_r))$ (where $a_1\ldots a_r\neq\emptyset$, $a_i\in\Phi_n(G_{t_i})$ for $i=1,\ldots,r$) and $\zeta(\emptyset)=n-1$ is an epimorphism and the pair $\langle L',\tau\rangle$, where τ is the inclusion of $\zeta^{-1}(0)$ into L', is the free covering group of the $(n+1)$-group $L=\zeta^{-1}(0)$. Moreover, $[L; \{\alpha_t: G_t\to L\}_{t\in T}]$, where $\tau\alpha_t=\Psi_n(\gamma_t)\tau_t$, is the free product of $\{G_t\}_{t\in T}$.*

Proof. For the family $\{\zeta_t: \Phi_n(G_t)\to\mathfrak{C}_{n,2}\}_{t\in T}$ there exists a unique morphism $\zeta\colon L'\to\mathfrak{C}_{n,2}$ with $\zeta\gamma_t=\zeta_t$ for each $t\in T$, which implies that ζ is an epimorphism. A triple $\langle L',\tau,\zeta\rangle$, where τ is the inclusion of $\zeta^{-1}(0)$ into L', is the free covering group of the $(n+1)$-group $L=\zeta^{-1}(0)$ (cf. [9], Theorem 2, Definition 3 and Corollary 2). Furthermore, there exists an isomorphism $\eta\colon L'\to\Phi_n(L)$ such that $\eta\tau=\tau_L$ and $\zeta_L\eta=\zeta$, where $\langle\Phi_n(L),\tau_L,\zeta_L\rangle$ is also a free covering group of L. From the definition of a free product of groups (cf. [6]) it follows that the epimorphism ζ is given by $\zeta(a_1\ldots a_r)=\varphi_{(r-1)}(\zeta_{t_1}(a_1),\ldots,\zeta_{t_r}(a_r))$, where $a_i\in\Phi_n(G_{t_i})$, for $a_1\ldots a_r\neq\emptyset$ and $\zeta(\emptyset)=n-1$. In view of Theorem 4 of [9] there exist morphisms $\alpha_t\colon G_t\to L$ with $\tau\alpha_t=\Psi_n(\gamma_t)\tau_t$. Furthermore, $\Phi_n(\alpha_t)=\eta\gamma_t$ for $t\in T$. Since

$$[\Phi_n(L); \{\eta\gamma_t: \Phi_n(G_t)\to\Phi_n(L)\}_{t\in T}]$$

is the free product of $\{\Phi_n(G_t)\}_{t\in T}$, by Proposition 5 of [10] it follows that $[L; \{\alpha_t: G_t\to L\}_{t\in T}]$ is the free product of $\{G_t\}_{t\in T}$, which was to be proved.

Now we give two constructions of free products of n-groups. The first will be founded on the construction of a free covering group due to Post (cf. [12]).

Let a nonempty family $\{G_t\}_{t\in T}$ of a pairwise disjoint $(n+1)$-groups be given. Denote by B the set of all sequences $\langle a_1,\ldots,a_r\rangle$, where $a_i\in G_{t_i}$, such that $r\equiv 1 \pmod n$. Define a relation θ on B in the following way:

$\langle a_1,\ldots,a_r\rangle\,\theta\,\langle a_1,\ldots,a_l,b_{l+1},\ldots,b_{l+j+un},a_{l+j+1},\ldots,a_r\rangle$ (where $u=0,1,2,\ldots$) if and only if there exists $t\in T$ such that $a_{l+1},\ldots,a_{l+j},b_{l+1},\ldots,b_{l+j+un}\in G_t$ and there exist elements $c_1,\ldots,c_l,d_1,\ldots,d_m\in G_t$ (where $l+j+m\equiv 1 \pmod n$) such that

$$f_{(\cdot)}(c_1,\ldots,c_l,a_{l+1},\ldots,a_{l+j},d_1,\ldots,d_m)=f_{(\cdot)}(c_1,\ldots c_l,b_{l+1},\ldots,b_{l+j+un},d_1,\ldots,d_m).$$

Define on B an $(n+1)$-ary operation f by forming a sequence from given $n+1$ sequences by juxtaposing them.

THEOREM 1. *The relation θ is a congruence relation on the $(n+1)$-groupoid (B,f). The quotient $(n+1)$-groupoid B/θ together with the family $\{\alpha_t: G_t\to B/\theta\}_{t\in T}$, given by $\alpha_t(a)=\langle\langle a\rangle\rangle$ (where $\langle\langle a\rangle\rangle$ denotes the equivalence class represented by $\langle a\rangle$), is the free product of $(n+1)$-groups $\{G_t\}_{t\in T}$.*

Proof. We use the same notation as in Lemma 1. Recall that Post (cf. [12]) has constructed the free covering group $\Phi_n(G)$ of an $(n+1)$-group G from equivalence classes of polyads. An equivalence class represented by a polyad $\langle a_1,\ldots,a_r\rangle$ where $a_i\in G$ for $i=1,\ldots,r$, $r\leqslant n$, will be denoted by $\langle\langle a_1,\ldots,a_r\rangle\rangle$. Thus nonempty elements of the free product $L'=\coprod_{t\in T}\Phi_n(G_t)$, i.e., reduced words, are of the form of sequences $\langle\langle a_{11},\ldots,a_{1u_1}\rangle\rangle\langle\langle a_{21},\ldots,a_{2u_2}\rangle\rangle\ldots\langle\langle a_{r1},\ldots,a_{ru_r}\rangle\rangle$ where two adjacent polyads do not belong to the same group $\Phi_n(G_t)$ and the neutral elements do not occur in this sequence.

As we know from [9], Theorem 2, the free covering group $\langle\Phi_n(G),\tau_G\rangle$ of an $(n+1)$-group G determines a unique epimorphism $\zeta\colon\Phi_n(G)\to\mathfrak{C}_{n,2}$ such that $\zeta^{-1}(0)=\tau_G(G)$. This epimorphism is given by $\zeta(\langle\langle a_1,\ldots,a_r\rangle\rangle)\equiv r-1 \pmod n$. Hence, by Lemma 1, the morphism $\zeta\colon L'\to\mathfrak{C}_{n,2}$ given by

$$\begin{aligned}\zeta(\langle\langle a_{11},\ldots,a_{1u_1}\rangle\rangle\ldots\langle\langle a_{r1},\ldots,a_{ru_r}\rangle\rangle)&\\ &=\Phi_{(r-1)}(\zeta(\langle\langle a_{11},\ldots,a_{1u_1}\rangle\rangle),\ldots,\zeta(\langle\langle a_{r1},\ldots,a_{ru_r}\rangle\rangle))\\ &\equiv(u_1-1)+\ldots+(u_r-1)+(r-1) \pmod n\end{aligned}$$

and $\zeta(\emptyset)=n-1$ is an epimorphism. Let $L=\zeta^{-1}(0)$ and let τ be the inclusion of L into L'. Then $\langle\langle a_{11},\ldots,a_{1u_1}\rangle\rangle\ldots\langle\langle a_{r1},\ldots,a_{ru_r}\rangle\rangle\in L$ if and only if $u_1+\ldots+u_r\equiv 1 \pmod n$. The $(n+1)$-group operation on L is simply the long product (cf. [3], [9]) formed from the binary group operation on L. It is done by juxtaposing $n+1$ sequences of equivalence classes of polyads and performing all possible $(n+1)$-group operations. Elements of L will simply be written

$$\langle\langle a_{11},\ldots,a_{1u_1},a_{21},\ldots,a_{2u_2},\ldots,a_{r1},\ldots,a_{ru_r}\rangle\rangle \quad\text{or briefly } \langle\langle a_1,\ldots,a_r\rangle\rangle$$

where $r\equiv 1 \pmod n$. It may happen that adjacent elements a_i, a_{i+1} belong to the same $(n+1)$-group G_t. However, there is no sequence of length greater than n of such elements. The $(n+1)$-group operation on L is therefore given by juxtaposition of $n+1$ sequences and performing an appropriate $(n+1)$-group operation if a long enough sequence of elements of the same $(n+1)$-group appeared in the resulting long sequence. The embeddings $\alpha_t\colon G_t \to L$ are given by $\tau\alpha_t(a) = \Psi_n(\gamma_t)\tau_t(a) = \tau(\langle\langle a\rangle\rangle)$, i.e., $\alpha_t(a) = \langle\langle a\rangle\rangle$. This completes the proof of Theorem 1.

An alternative construction of a free product of n-groups can be extracted from the construction of a free covering group given in [9].

Let a nonempty family of pairwise disjoint nonempty $(n+1)$-groups $\{G_t\}_{t\in T}$ be given. Fix an element c_t in each $(n+1)$-group G_t. By a word we now mean any sequence of the form $a_1\overset{l_1}{c_{t_1}}\ldots a_r\overset{l_r}{c_{t_r}}$, where for each $i = 1, \ldots, r$ we have $a_i \in G_{t_i}$, $t_i \neq t_{i+1}$, $l_i = 0, 1, \ldots, n-1$ and $\varphi_{(r-1)}(l_1, \ldots, l_r) = 0$, in which expressions of the form $\bar{c}_t\,\overset{n-1}{c_t}$ do not occur. In the set L of words define an $(n+1)$-ary operation f which is accomplished by forming the word obtained from $n+1$ words by juxtaposing them and performing the necessary operations, i.e., if in the just-formed "long word" there appear neighbouring expressions $a_i\overset{l_i}{c_{t_i}}$ and $a_j\overset{l_j}{c_{t_j}}$ of the same $(n+1)$-group G_t (thus $t_i = t_j = t$), then $a_i\overset{l_i}{c_{t_i}}a_j\overset{l_j}{c_{t_j}}$ should be replaced by

$$f_{(\cdot)}(a_i, \underbrace{c_t, \ldots, c_t}_{l_i}, a_j, \underbrace{c_t, \ldots, c_t}_{l_j}, \bar{c}_t, \underbrace{c_t, \ldots, c_t}_{n-1-\varphi(l_i, l_j)})\overset{\varphi(l_i, l_j)}{c_t}$$

where

$$f_{(\cdot)}(a_i, \underbrace{c_t, \ldots, c_t}_{l_i}, a_j, \underbrace{c_t, \ldots, c_t}_{l_j}, \bar{c}_t, \underbrace{c_t, \ldots, c_t}_{n-1-\varphi(l_i, l_j)})$$

is already the element of G_t obtained by performing the $(n+1)$-group operation f in G_t. If after proceeding in this way we obtain expressions of the form $\bar{c}_t\,\overset{n-1}{c_t}$, we just cancel them. We iterate the procedure until the reduced "long word" becomes a word.

THEOREM 2. *The $(n+1)$-groupoid (L, f) is an $(n+1)$-group. The $(n+1)$-group L together with the family $\{\alpha_t\colon G_t \to L\}_{t\in T}$ given by $\alpha_t(a) = a\overset{0}{c_t}$ is the free product of $(n+1)$-groups $\{G_t\}_{t\in T}$.*

Proof. We use the same notation as in Lemma 1. Recall that the free covering group $\Phi_n(G_t)$ of an $(n+1)$-group G_t can be treated (cf. [9], Theorem 1) as the set $G_t\times Z_n$ together with a group operation f^* defined by the formula

$$f^*((a_1, l_1), (a_2, l_2)) = (f_{(\cdot)}(a_1, \underbrace{c_t, \ldots, c_t}_{l_1}, a_2, \underbrace{c_t, \ldots, c_t}_{l_2}, \bar{c}_t, \underbrace{c_t, \ldots, c_t}_{n-1-\varphi(l_1, l_2)}), \varphi(l_1, l_2));$$

where c_t is an arbitrary but fixed element of G_t. By Theorem 2 of [9], the morphism $\zeta_t\colon \Phi_n(G_t) \to \mathfrak{C}_{n,2}$ given by $\zeta_t(a, l) = l$ is an epimorphism. Furthermore, $\zeta^{-1}(0)$ is a sub-$(n+1)$-group of the $(n+1)$-group $\Psi_n\Phi_n(G_t)$ and it is isomorphic to G_t. Elements of the free product $L' = \coprod_{t\in T}\Phi_n(G_t)$ are of the form $w = (a_1, l_1)\ldots(a_r, l_r)$ where $a_i \in G_{t_i}$, $t_i \neq t_{i+1}$, $l_i \in Z_n$ for $i = 1, \ldots, r$ and elements $(\bar{c}_t, n-1)$ (i.e., the neutral elements of the groups $\Phi_n(G_{t_i})$) do not occur in this sequence.

According to Lemma 1, the morphism $\zeta\colon L' \to \mathfrak{C}_{n,2}$ given by

$$\zeta(w) = \varphi_{(r-1)}(\zeta_{t_1}(a_1, l_1), \ldots, \zeta_{t_r}(a_r, l_r)) \equiv l_1 + \ldots + l_r + r - 1 \pmod{n}$$

for $w \neq \varnothing$ and $\zeta(\varnothing) = n-1$ is an epimorphism. Let $L = \zeta^{-1}(0)$ and τ be the inclusion of L into L'. Then $w \in L$ if and only if $l_1 + \ldots + l_r + r \equiv 1 \pmod{n}$, i.e., $\varphi_{(r-1)}(l_1, \ldots, l_r) = 0$. The $(n+1)$-group operation f on L is simply the long product formed from the group operation f^* on L'. To simplify words of the form $(a, 0)$ in the $(n+1)$-group L we write briefly a. Then $\tau(a) = (a, 0) \in L$. Let $w = (a_1, l_1)\ldots(a_r, l_r) \in L$. Thus

$$\begin{aligned} w &= f^*_{(\cdot)}((a_1, 0), \underbrace{(c_{t_1}, 0), \ldots, (c_{t_1}, 0)}_{l_1}, \ldots, (a_r, 0), \underbrace{(c_{t_r}, 0), \ldots, (c_{t_r}, 0)}_{l_r}) \\ &= f^*_{(\cdot)}(\tau(a_1), \underbrace{\tau(c_{t_1}), \ldots, \tau(c_{t_1})}_{l_1}, \ldots, \tau(a_r), \underbrace{\tau(c_{t_r}), \ldots, \tau(c_{t_r})}_{l_r}) \\ &= \tau(a_1\overset{l_1}{c_{t_1}}\ldots a_r\overset{l_r}{c_{t_r}}). \end{aligned}$$

Hence it seems convenient to define elements of the $(n+1)$-group L as expressions of the form $a_1\overset{l_1}{c_{t_1}}\ldots a_r\overset{l_r}{c_{t_r}}$ where $a_i \in G_{t_i}$, $t_i \neq t_{i+1}$, $\varphi_{(r-1)}(l_1, \ldots, l_r) = 0$. The $(n+1)$-ary operation f is then accomplished by forming the word obtained from $n+1$ words by juxtaposing them and performing the necessary operations. If in the just-formed "long word" there appear neighbouring expressions $a_i\overset{l_i}{c_{t_i}}$ and $a_{i+1}\overset{l_{i+1}}{c_{t_{i+1}}}$ where $t_i = t_{i+1} = t$, then

$$\begin{aligned} &\tau(\ldots a_i\overset{l_i}{c_t}a_{i+1}\overset{l_{i+1}}{c_t}\ldots) \\ &= f^*_{(\cdot)}(\ldots, \tau(a_i), \underbrace{\tau(c_t), \ldots, \tau(c_t)}_{l_i}, \tau(a_{i+1}), \underbrace{\tau(c_t), \ldots, \tau(c_t)}_{l_{i+1}}, \ldots) \\ &= f^*_{(\cdot)}(\ldots, (f(a_i, \underbrace{c_t, \ldots, c_t}_{l_i}, a_{i+1}, \underbrace{c_t, \ldots, c_t}_{l_{i+1}}, \bar{c}_t, \underbrace{c_t, \ldots, c_t}_{n-1-\varphi(l_i, l_{i+1})}), \varphi(l_i, l_{i+1})), \ldots) \\ &= f^*_{(\cdot)}(\ldots, \tau(f(a_i, \underbrace{c_t, \ldots, c_t}_{l_i}, a_{i+1}, \underbrace{c_t, \ldots, c_t}_{l_{i+1}}, \bar{c}_t, \underbrace{c_t, \ldots, c_t}_{n-1-\varphi(l_i, l_{i+1})})), \underbrace{\tau(c_t), \ldots, \tau(c_t)}_{\varphi(l_i, l_{i+1})}, \ldots) \\ &= \tau(\ldots f(a_i, \underbrace{c_t, \ldots, c_t}_{l_i}, a_{i+1}, \underbrace{c_t, \ldots, c_t}_{l_{i+1}}, \bar{c}_t, \underbrace{c_t, \ldots, c_t}_{n-1-\varphi(l_i, l_{i+1})})\overset{\varphi(l_i, l_{i+1})}{c_t}\ldots). \end{aligned}$$

In a similar way one can show that

$$\tau(\ldots a_i \overset{l_i}{c_{t_i}} \bar{c}_t \overset{n-1}{c_t}\, a_j \overset{l_j}{c_{t_j}} \ldots) = \tau(\ldots a_i \overset{l_i}{c_{t_i}} a_j \overset{l_j}{c_{t_j}} \ldots).$$

According to Lemma 1, $\tau\alpha_t(a) = \Psi_n(\gamma_t)(a, 0) = \tau(ac_t)$. This completes the proof of Theorem 2.

Henceforth by the free product of n-groups we always understand the n-group constructed as in Theorem 2.

The above construction allows us to prove the primitiveness of the free product of n-groups. Recall that the n-group is said to be primitive if it is not derived from an l-group for any $l<n$ (cf. [3]).

THEOREM 3. *The free product of at least two nonempty $(n+1)$-groups is a primitive $(n+1)$-group.*

Proof. We use the same notation as in Theorem 2.

Suppose that the $(n+1)$-group $\coprod_{t\in T} G_t$ is derived from a $(k+1)$-group. Let $w = a_1 \overset{l_1}{c_{t_1}} \ldots a_r \overset{l_r}{c_{t_r}}$ be the skew element to $\overset{0}{c_{t_0} c_{t_0}}$ (where $t_0 \in T$ is arbitrary but fixed) in the creating $(k+1)$-group (cf. [9]). In view of Corollary 2 of [11] w is the s-skew element to $\overset{0}{c_{t_0} c_{t_0}}$ (cf. [11], Definition). Take an arbitrary element of the form $\overset{0}{c_t c_t}$ where $t \neq t_1$. From the definition of an s-skew element it follows that

(1)
$$\underbrace{\overset{0}{c_{t_0}c_{t_0}} \ldots \overset{0}{c_{t_0}c_{t_0}}}_{k-1} w \underbrace{\overset{0}{c_t c_t} \ldots \overset{0}{c_t c_t}}_{n-k+1} = w \underbrace{\overset{0}{c_{t_0}c_{t_0}} \ldots \overset{0}{c_{t_0}c_{t_0}}}_{k-1} \underbrace{\overset{0}{c_t c_t} \ldots \overset{0}{c_t c_t}}_{n-k+1}.$$

Suppose $k = 1$. Then the reduced word on the left side in (1) would begin with $a_1 \overset{l_1}{c_{t_1}}$ and that on the right side of (1) would begin with $\overset{0}{c_t c_t}$, which is impossible in view of the assumption $t \neq t_1$. Hence $k>1$. Consider two cases: $r>1$ and $r = 1$.

First, let $r>1$. Then the reduced word on the right of (1) has to begin from $a_1 \overset{l_1}{c_{t_1}}$ (since $t \neq t_1$). Hence the reduced word on the left begins from $a_1 \overset{l_1}{c_{t_1}}$ as well. But in this case $t_0 = t_1$ and $\varphi_{(k-1)}(0, \ldots, 0, l_1) = l_1$, i.e., $0+\ldots+0+l_1+k-1 \equiv l_1 (\mathrm{mod}\, n)$, whence $k = 1$, which is impossible.

Now, let $r = 1$. Then $w = a_1 \overset{l_1}{c_{t_1}}$. Moreover $w = a_1 \overset{0}{c_{t_1}}$. Hence equality (1) may be given in the form

$$\underbrace{\overset{0}{c_{t_0}c_{t_0}} \ldots \overset{0}{c_{t_0}c_{t_0}}}_{k-1} a_1 c_{t_1} \underbrace{\overset{0}{c_t c_t} \ldots \overset{0}{c_t c_t}}_{n-k+1} = a_1 c_{t_1} \underbrace{\overset{0}{c_{t_0}c_{t_0}} \ldots \overset{0}{c_{t_0}c_{t_0}}}_{k-1} \underbrace{\overset{0}{c_t c_t} \ldots \overset{0}{c_t c_t}}_{n-k+1}.$$

After some necessary cancellations it becomes

$$\overset{k-2}{c_{t_0}}\, \overset{0}{c_{t_0}} a_1 c_{t_1} \overset{n-k}{c_t}\, c_t = a_1 c_{t_1} \overset{0}{c_{t_0}}\, \overset{k-2}{c_{t_0}}\, \overset{n-k}{c_t}\, c_t$$

Hence $t_0 = t_1$. On the other hand, from the definition of an s-skew element

$$a_1 c_{t_1} \underbrace{\overset{0}{c_{t_0}c_{t_0}} \ldots \overset{0}{c_{t_0}c_{t_0}}}_{k-1} \underbrace{\overset{0}{c_t c_t} \ldots \overset{0}{c_t c_t}}_{n-k+1} = \underbrace{\overset{0}{c_t c_t} \ldots \overset{0}{c_t c_t}}_{n-k+1} a_1 c_{t_1} \underbrace{\overset{0}{c_{t_0}c_{t_0}} \ldots \overset{0}{c_{t_0}c_{t_0}}}_{k-1}.$$

After performing all cancellations (exploiting the equality $t_0 = t_1$) we get

$$a_1 \overset{k-1}{c_{t_1}} \overset{n-k}{c_t}\, c_t = \overset{n-k}{c_t}\, c_t\, a_1 \overset{k-1}{c_{t_1}}.$$

By the uniqueness of the form of a reduced word we have $t_1 = t$. This contradicts our assumption that $t_1 \neq t$, which completes the proof of Theorem 3.

Theorem 2 of [10] presented a description of inductive limits of covering $(k+1)$-groups. That theorem can be strengthened in the case of the free product of covering $(k+1)$-groups.

THEOREM 4. *Let $\{\langle G'_t, \lambda_t, \zeta_t\rangle\}_{t\in T}$ be a nonempty family of covering $(k+1)$-groups of indices q_t of $(n+1)$-groups $\{G_t\}_{t\in T}$. Then the free product $\coprod_{t\in T} G'_t$ is a covering $(k+1)$-group of index $q = \text{g.c.d.}\{q_t\}_{t\in T}$ of the $(n+1)$-group $(\coprod_{t\in T} G_t)/\theta$, where θ is the congruence relation defined as follows:*

Let $a = a_1 \overset{l_1}{c_{t_1}} \ldots a_r \overset{l_r}{c_{t_r}}$, $b = b_1 \overset{m_1}{c_{t_1}} \ldots b_r \overset{m_r}{c_{t_r}}$ where $a_i, b_i \in G_{t_i}$. Then $a\,\theta\, b$ if and only if $l_i \equiv m_i (\mathrm{mod}\, k)$ and for each $i = 1, \ldots, r$

$$g_{(\cdot)}(\lambda_{t_i}(a_i), \underbrace{\lambda_{t_i}(c_{t_i}), \ldots, \lambda_{t_i}(c_{t_i})}_{\varepsilon_i k}) = g_{(\cdot)}(\lambda_{t_i}(b_i), \underbrace{\lambda_{t_i}(c_{t_i}), \ldots, \lambda_{t_i}(c_{t_i})}_{\mu_i k})$$

where $0 \leqslant l_i - \varepsilon_i k < k$ and $0 \leqslant m_i - \mu_i k < k$.

Proof. Let $[G; \{\alpha_t: G_t \to G\}_{t\in T}]$ and $[L'; \{\alpha'_t: G_t \to L'\}_{t\in T}]$ be the free products of $(n+1)$-groups $\{G_t\}_{t\in T}$ and $(k+1)$-groups $\{G'_t\}_{t\in T}$ respectively, where the element c'_t in G'_t are chosen in such a way that $c'_t = \lambda_t(c_t)$. The functor Φ preserves inductive limits (cf. [10]), and so $[\Phi_s(G); \{\Phi_s(\alpha_t): \Phi_s(G_t) \to \Phi_s(G)\}_{t\in T}]$ is (up to an isomorphism) the free product of the family $\{\langle \Phi_s(G_t), \tau_t\rangle\}_{t\in T}$ of free covering $(k+1)$-groups of $(n+1)$-groups. The pair $\langle \Phi_s(G), \tau\rangle$, where $\tau: G \to \Psi_s \Phi_s(G)$ is the morphism induced by $\{\Psi_s \Phi_s(\alpha_t)\tau_t\}_{t\in T}$, is the free covering $(k+1)$-group of G. Then there exists a unique morphism $\delta: G \to L'$ such that $\delta\alpha_t = \Psi_s(\alpha'_t)\lambda_t$ for $t \in T$. Furthermore, there exists a family $\{\lambda_t^*: \Phi_s(G_t) \to G'_t\}_{t\in T}$ such that $\Psi_s(\lambda_t^*)\tau_t = \lambda_t$ for $t \in T$ and there exists a morphism $\delta^*: \Phi_s(G) \to L'$ with $\Psi_s(\delta^*)\tau = \delta$. Since $\Psi_s(\delta^* \Phi_s(\alpha_t))\tau_t = \Psi_s(\alpha'_t \lambda_t^*)\tau_t$, we have $\delta^* \Phi_s(\alpha_t) = \alpha'_t \lambda_t^*$ for $t \in T$.

Take two elements $a, b \in G$ which satisfy $\delta(a) = \delta(b)$. Let $a = a_1 \overset{l_1}{c_{t_1}} \ldots a_r \overset{l_r}{c_{t_r}}$, $b = b_1 \overset{m_1}{c_{u_1}} \ldots b_p \overset{m_p}{c_{u_p}}$ (where $a_i \in G_{t_i}$, $b_j \in G_{u_j}$, $l_1 + \ldots + l_r + r \equiv 1 (\mathrm{mod}\, n)$, $m_1 + \ldots +$

$+m_p+p \equiv 1 (\mathrm{mod}\, n))$ be words in the free product G (cf. Theorem 2). Then

$$\delta(a) = \delta\big(f_{(\cdot)}(\alpha_{t_1}(a_1), \underbrace{\alpha_{t_1}(c_{t_1}), \ldots, \alpha_{t_1}(c_{t_1})}_{l_1}, \ldots, \alpha_{t_r}(a_r), \underbrace{\alpha_{t_r}(c_{t_r}), \ldots, \alpha_{t_r}(c_{t_r})}_{l_r})\big)$$

$$= g_{(\cdot)}(\lambda_{t_1}(a_1), \underbrace{\lambda_{t_1}(c_{t_1}), \ldots, \lambda_{t_1}(c_{t_1})}_{\varepsilon_1 k}) \overset{l_1-\varepsilon_1 k}{\lambda_{t_1}(c_{t_1})} \ldots g_{(\cdot)}(\lambda_{t_r}(a_r), \underbrace{\lambda_{t_r}(c_{t_r}), \ldots, \lambda_{t_r}(c_{t_r})}_{\varepsilon_r k}) \overset{l_r-\varepsilon_r k}{\lambda_{t_r}(c_{t_r})},$$

where $0 \leqslant l_i - \varepsilon_i k < k$, is a word in L'. Similarly,

$$\delta(b) = g_{(\cdot)}(\lambda_{u_1}(b_1), \underbrace{\lambda_{u_1}(c_{u_1}), \ldots, \lambda_{u_1}(c_{u_1})}_{\mu_1 k}) \overset{m_1-\mu_1 k}{\lambda_{u_1}(c_{u_1})} \ldots$$

$$\ldots g_{(\cdot)}(\lambda_{u_p}(b_p), \underbrace{\lambda_{u_p}(c_{u_p}), \ldots, \lambda_{u_p}(c_{u_p})}_{\mu_p k}) \overset{m_p-\mu_p k}{\lambda_{u_p}(c_{u_p})}$$

where $0 \leqslant m_j - \mu_j k < k$, is a word in L'. By assumption $\delta(a) = \delta(b)$. Since each element of a free product can be expressed in a unique way as a word, we have $r = p$ and $t_i = u_i$, $l_i \equiv m_i (\mathrm{mod}\, k)$,

$$g_{(\cdot)}(\lambda_{t_i}(a_i), \underbrace{\lambda_{t_i}(c_{t_i}), \ldots, \lambda_{t_i}(c_{t_i})}_{\varepsilon_i k}) = g_{(\cdot)}(\lambda_{t_i}(b_i), \underbrace{\lambda_{t_i}(c_{t_i}), \ldots, \lambda_{t_i}(c_{t_i})}_{\mu_i k}) \quad \text{for} \quad i = 1, \ldots, r.$$

Conversely, consider words $a = a_1 \overset{l_1}{c_{t_1}} \ldots a_r \overset{l_r}{c_{t_r}}$ and $b = b_1 \overset{m_1}{c_{t_1}} \ldots b_r \overset{m_r}{c_{t_r}}$ where $a_i, b_i \in G_{t_i}$, $0 \leqslant l_i - \varepsilon_i k < k$, $0 \leqslant m_i - \mu_i k < k$, $l_i \equiv m_i (\mathrm{mod}\, k)$,

$$g_{(\cdot)}(\lambda_{t_i}(a_i), \underbrace{\lambda_{t_i}(c_{t_i}), \ldots, \lambda_{t_i}(c_{t_i})}_{\varepsilon_i k}) = g_{(\cdot)}(\lambda_{t_i}(b_i), \underbrace{\lambda_{t_i}(c_{t_i}), \ldots, \lambda_{t_i}(c_{t_i})}_{\mu_i k}) \quad \text{for} \quad i = 1, \ldots, r.$$

One can verify that $\delta(a) = \delta(b)$. The homomorphism $\delta\colon G \to \Psi_s(L')$ determines a congruence relation θ on G such that G/θ is isomorphic to $\delta(G)$. In view of Theorem 2 of [10], L' is a covering $(k+1)$-group of index $q = \text{g.c.d.}\{q_t\}_{t \in T}$ of the $(n+1)$-group $\delta(G)$.

Finally, one can show that

$$g_{(\cdot)}(\lambda_{t_i}(a_i), \underbrace{\lambda_{t_i}(c_{t_i}), \ldots, \lambda_{t_i}(c_{t_i})}_{k}) = \lambda^*_{t_i}(a_i, \varepsilon_i)$$

and

$$g_{(\cdot)}(\lambda_{t_i}(b_i), \underbrace{\lambda_{t_i}(c_{t_i}), \ldots, \lambda_{t_i}(c_{t_i})}_{\mu_i k}) = \lambda^*_{t_i}(b_i, \mu_i).$$

Hence $a\,\theta\, b$ if and only if $l_i \equiv m_i (\mathrm{mod}\, k)$ and $\lambda^*_{t_i}(a_i, \varepsilon_i) = \lambda^*_{t_i}(b_i, \mu_i)$ for each $i = 1, \ldots, r$.

3. Free n-groups. Free n-groups generated by one element are exactly infinite cyclic n-groups; thus every free n-group is a free product of infinite cyclic n-groups.

Hence Theorems 1 and 2 enable us to give a construction of free n-groups. The construction we shall present here will be founded on the description of the free product given in Theorem 2.

Let a nonempty set X be given. Call a word each expression of the form $w = \overset{l_1}{x_1} \ldots \overset{l_r}{x_r}$ where $x_i \in X$, $x_i \neq x_{i+1}$, $l_i \neq 0$ for each $i = 1, \ldots, r$ and $l_1 + \ldots + l_r \equiv 1 (\mathrm{mod}\, n)$. In the set G of words define an $(n+1)$-ary operation f which is accomplished by forming the word obtained from $n+1$ words by juxtaposing them and performing the necessary operations, i.e., if in the just formed "long word" there appear neighbouring expressions $\overset{l_i}{x}$ and $\overset{l_j}{x}$, then $\overset{l_i}{x}\overset{l_j}{x}$ should be replaced by $\overset{l_i+l_j}{x}$. If after proceeding in this way we obtain $\overset{0}{x}$, we just cancel it. We iterate this procedure until the reduced "long word" becomes a word.

THEOREM 5. *The $(n+1)$-groupoid (G, f) is an $(n+1)$-group. Moreover, it is a free $(n+1)$-group freely generated by X.*

Proof. The set $A = nZ+1$ with addition forms an $(n+1)$-group. It is an infinite cyclic $(n+1)$-group generated by 1. The free covering group of this $(n+1)$-group is an additive group of integers (cf. [1], [12]).

Take any nonempty set X. Let $G_x = \{\overset{u}{x}\colon u \in nZ+1\}$ for $x \in X$. In G_x define an $(n+1)$-ary operation f by $f(\overset{u_1}{x}, \ldots, \overset{u_{n+1}}{x}) = \overset{u_1+\ldots+u_{n+1}}{x}$.

The $(n+1)$-groupoid (G_x, f) is an infinite cyclic $(n+1)$-group generated by $\overset{1}{x}$. Thus (up to an isomorphism) $\Phi_n(G_x) = \{\overset{l}{x}\colon l \in Z\}$, the group operation f^* is defined by the formula $f^*(\overset{l_1}{x}, \overset{l_2}{x}) = \overset{l_1+l_2}{x}$ and $\overset{0}{x}$ is the neutral element. The epimorphism $\zeta_x\colon \Phi_n(G_x) \to \mathfrak{C}_{n,2}$ is then defined by $\zeta_x(\overset{l}{x}) \equiv l-1 (\mathrm{mod}\, n)$. Let $G = \coprod_{x \in X} G_x$. Then G is the free $(n+1)$-group freely generated by X. Nonempty elements of the set $\coprod_{x \in X} \Phi_n(G_x)$ are of the form $\overset{l_1}{x_1} \ldots \overset{l_r}{x_r}$ where $x_i \neq x_{i+1}$ and $l_i \neq 0$ for $i = 1, \ldots, r$. The epimorphism $\zeta\colon \Phi_n(G) \to \mathfrak{C}_{n,2}$ is defined by

$$\zeta(\overset{l_1}{x_1} \ldots \overset{l_r}{x_r}) = \varphi_{(r-1)}(\zeta_{x_1}(\overset{l_1}{x_1}), \ldots, \zeta_{x_r}(\overset{l_r}{x_r})) \equiv (l_1-1) + \ldots + (l_r-1) + (r-1) (\mathrm{mod}\, n)$$

and $\zeta(\varnothing) = n-1$. Thus, by Lemma 1, an element $\overset{l_1}{x_1} \ldots \overset{l_r}{x_r}$ belongs to G if and only if $\varphi_{(r-1)}(l_1-1, \ldots, l_r-1) = 0$, i.e., $l_1 + \ldots + l_r \equiv 1 (\mathrm{mod}\, n)$. This completes the proof of Theorem 5.

Sub-n-groups of free n-groups were investigated by Artamonov in [1]. The construction of free abelian n-groups is due to Sioson (cf. [14]).

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Received 2 March 1982;
in revised form 8 June 1982

On metrizability of continuous images of compact ordered spaces

by

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Abstract. We prove here the following generalization of Treybig's Product Theorem: if a Hausdorff space X is a continuous image of a compact ordered space, then for every open map from X into a Hausdorff space Y the set of all points of Y having infinite preimages is metrizable.

1. Introduction. It is shown in [1] that if a Hausdorff space X is a continuous image of a compact ordered space then each Hausdorff space which can be obtained as an open infinite-to-one continuous image of X is metrizable. This result generalizes the Theorem of Treybig [6].

If an open continuous map from X onto a Hausdorff space Y has at least one finite fibre, then the space Y need not be metrizable. However, as shown in Section 3, the set of all points of Y having infinite preimages is metrizable. To prove this, we will need some technical lemmas, given in Section 2, and concerning the behaviour of long decreasing sequences of closed subsets of continuous images of compact ordered spaces.

By a (*compact*) *ordered space* we mean a linearly ordered set which is a (compact) space when equipped with the usual open-interval topology. If some convex sets are added to the topology of an ordered space, the resulting space is called a *GO-space*.

Let K be an ordered space and let A be a subset of K. A set $C \subset A$ is called a *convex component* of A if it is the maximal subset of A with respect to the property of being convex. A sequence $\{A_n: n = 1, 2, \ldots\}$ of subsets of X is said to be *increasing* (*decreasing*) if any element of A_n is less (greater) than any element of A_m for $n < m$.

All ordinals below are regarded as ordered spaces. A subset of a regular uncountable cardinal $\varkappa$ is called *stationary* if it meets all closed unbounded subsets of $\varkappa$.

Let S be a subset of a cardinal $\varkappa$. A map $f\colon S \to \varkappa$ will be called *regressive* if $f(\alpha) < \alpha$ for $\alpha \in S - \{0\}$. The following theorem will be used below.

Pressing Down Lemma (G. Fodor, see [5], Theorem 8, p. 347). *Let S be a stationary subset of a regular uncountable cardinal $\varkappa$. If $f\colon S \to \varkappa$ is regressive, then there exists an ordinal $\alpha < \varkappa$ such that $f^{-1}(\{\alpha\})$ is stationary.*