# On dendroids and their ramification points in the classical sense

by

**Jacek Nikiel** (Wrocław)

**Abstract.** A family $T$ of triods is said to be an *almost disjoint family* if, for any $S \in T$, there are two end-points $a$, $b$ of $S$ such that the arc from $a$ to $b$ (contained in $S$) is disjoint with any triod $S' \in T$ distinct from $S$. In this paper we show that if the set of all ramification points (in the classical sense) of a dendroid $X$ cannot be covered by countably many arcs, then $X$ contains some uncountable almost disjoint family of triods. Then we apply this result to planable dendroids, in particular, we show that the set of all ramification points of a plane dendroid is a $G_{\delta\sigma}$-set.

In [7] we showed that if $X$ is such a dendroid that the set $R(X)$ of all its ramification points (in the classical sense) cannot be covered by countably many arcs and $X$ does not contain any uncountable family of pairwise disjoint triods, then $X$ contains the Gehman dendrite. In this paper we obtain some further results on such dendroids and we then apply them to plane dendroids. In particular, we estimate the Borel class of the set of all ramification points of a plane dendroid. Such an estimation has been known only for the set of all end-points of a plane dendroid ([4]).

Let $X$ be a dendroid. If $x, y \in X$, then $[x, y]$ denotes the only arc from $x$ to $y$ and $(x, y] = [x, y] \setminus \{x\}$, $(x, y) = [x, y] \setminus \{x, y\}$. If confusion is possible, we write $[x, y]_X$ instead of $[x, y]$. A subset $B$ of $X$ is said to be a *branch of* $X$, if there is a point $x \in X$ such that $B$ is an arc-component of $X \setminus \{x\}$. In this case we say that a *branch $B$ of $X$ begins at* $x$. If $B$ is a branch of $X$, then there is a unique point at which $B$ begins. We denote this point by $x(B)$. If $Y$ is any nonempty subset of $X$ and $B$ is a branch of $X$ such that $x(B) \in Y$ and $B \cap Y = \emptyset$, then we say that $B$ *begins from* $Y$. If $x$ is a point of $X$, then we define its order (in the classical sense), $r(x)$, as a cardinality of the set of all branches of $X$ each of which begins at $x$. If $r(x) = 1$ (resp. $r(x) \geqslant 3$), then $x$ is said to be an *end-point* (resp. a *ramification point*) *of* $X$. The set of all end-points (resp. ramification points) of $X$ is denoted by $E(X)$ (resp. $R(X)$). If $B$ is a branch of $X$, we put $R(B) = B \cap R(X)$ and $E(B) = B \cap E(X)$. A subset $Y$ of $B$ is said to be an *m-arc* if $Y = (x, y]$ for some $y \in E(B)$, $x = x(B)$.

A dendroid $T$ is a triod if $R(X)$ consists of a single point $x_T$ and $r(x_T) = 3$. If $\mathcal{T}$ is any family of triods, then we say that $\mathcal{T}$ is an *almost disjoint family* if for each $T \in \mathcal{T}$ there are points $a, b \in E(T)$ such that $[a, b]_T \cap \bigcup(\mathcal{T} \setminus \{T\}) = \emptyset$.

LEMMA 1 ([1], Lemma 3, p. 18). *Let $X$ be a dendroid and $f$ a one-to-one continuous function $f$: $[0, \infty) \to X$ (here $[0, \infty)$ denotes the ray of non-negative reals). Then the closure of the set $f([0, \infty))$ is an arc.*

If $\alpha$ is an ordinal number, $\alpha<\omega_1$, then we define $P_\alpha$ as the set of all sequences $\{(n_0, n_1, \ldots, n_\gamma, \ldots)$: $n_\gamma<\omega_0, \gamma<\alpha\}$. If $\alpha<\omega_1$, $p = (n_0, n_1, \ldots, n_\gamma, \ldots) \in P_\alpha$ and $n<\omega_0$ is fixed, then we put $(p, n) = (k_0, k_1, \ldots, k_\gamma, \ldots, k_\alpha) \in P_{\alpha+1}$, where

$$k_\gamma = \begin{cases} n_\gamma & \text{for } \gamma < \alpha, \\ n & \text{for } \gamma = \alpha. \end{cases}$$

If $\alpha, \beta<\omega_1$, $\beta\leqslant\alpha$, $p = (n_0, n_1, \ldots, n_\gamma, \ldots) \in P_\alpha$, then $p|_\beta$ denotes the restriction of $p$ to $\beta$, i.e., $p|_\beta = (m_0, m_1, \ldots, m_\gamma, \ldots) \in P_\beta$, where $m_\gamma = n_\gamma$ for each $\gamma<\beta$. If $\alpha, \beta<\omega_1$, $p \in P_\alpha$, $q \in P_\beta$, then $q \prec p$, if $\beta\leqslant\alpha$ and $p|_\beta = q$.

THEOREM 1. *Let $X$ be a dendroid. If*

(1) *$R(X)$ cannot be covered by countably many arcs,*

*then $X$ contains some uncountable almost disjoint family of triods.*

Proof. Let us suppose that

(2) $X$ does not contain any uncountable almost disjoint family of triods.

Therefore

(3) $X$ does not contain any uncountable pairwise disjoint family of triods.

Moreover, we always have the following

(4) if $B$ is any branch of $X$ such that $R(B)$ is nonempty, then $B$ contains some triod.

Now, we use transfinite induction to describe the structure of $X$. Namely, we define some families of branches of $X$.

Let $I$ be a fixed arc in $X$ which is maximal (in the sense of inclusion) and let $x_0$ be a fixed end-point of $I$.

Let $\mathscr{B}$ be a family of all branches beginning from $I$ such that if $B \in \mathscr{B}$ then $R(B) \neq \emptyset$. By (3) and (4) $\mathscr{B}$ is countable, and so we may index it as follows: $\mathscr{B} = \{B_n: n<\omega_0\}$ (if $\mathscr{B}$ is finite, we put $B_n = \emptyset$ for sufficiently large $n$). We see that

(5) $$R(X) \subset I \cup \bigcup \mathscr{B}.$$

Let $\alpha$ be an ordinal number, $0<\alpha<\omega_1$, and suppose that for each $\beta<\alpha$ we have already defined subsets $B_p$ of $X$, for $p \in P_{\beta+1}$, such that if $B_p \neq \emptyset$ then

(6) $B_p$ is a branch of $X$,

(7) $B_p \cap B_{p'} = \emptyset$, for $p, p' \in P_{\beta+1}, p \neq p'$,

(8) if $\beta<\gamma<\alpha$, $p \in P_{\beta+1}$, $q \in P_{\gamma+1}$, $p \prec q$, then $B_p \supset B_q$ and
$$[x_0, x(B_p)] \cap [x(B_p), x(B_q)] = \{x(B_p)\}.$$



Case A. There is an ordinal number $\beta$ such that $\alpha = \beta+1$. Let $p \in P_{\beta+1} = P_\alpha$ be fixed. If $B_p = \emptyset$, then put $B_{(p,n)} = \emptyset$ for each $n<\omega_0$. If $B_p \neq \emptyset$, then let $I_p$ be some fixed $m$-arc of $B_p$. Let $\mathscr{B}_p$ be a family of all branches beginning from $I_p$ such that if $B \in \mathscr{B}_p$ then $R(B) \neq \emptyset$. By (3) and (4) $\mathscr{B}_p$ is countable, and so we may index it: $\mathscr{B}_p = \{B_{(p,n)}: n<\omega_0\}$ (if $\mathscr{B}_p$ is finite, we put $B_{(p,n)} = \emptyset$ for sufficiently large $n$). We see that (6), (7), (8) hold. Moreover

(9) $$R(B_p) \subset I_p \cup \bigcup \mathscr{B}_p.$$

Case B. $\alpha$ is a limit ordinal number. Since $\alpha<\omega_1$, we can find a sequence $\{\beta_n: n<\omega_0\}$ such that $\beta_0<\beta_1< \ldots <\alpha$ and $\lim \beta_n = \alpha$. Let us fix such a sequence and let $p \in P_\alpha$ be fixed. Put $p_n = p|_{\beta_n}$, and so $p_n \prec p_{n+1} \prec p$ for $n<\omega_0$. If $B_{p_k} = \emptyset$ for some $k<\omega_0$, then put $B_{(p,n)} = \emptyset$ for each $n<\omega_0$. Otherwise, put $x_n = x(B_{p_n})$. The set $S = \bigcup \{[x_n, x_{n+1}]: n<\omega_0\}$ is (by (8)) a continuous and one-to-one image of $[0, \infty)$, so (by Lemma 1) $\mathrm{cl}(S)$ is an arc. We see that $\mathrm{cl}(S) = [x_0, x_p]$, where $x_p = \lim_{n\to\infty} x_n$. Moreover, by (8), the definition of the point $x_p$ does not depend on the particular choice of a sequence $\{\beta_n: n<\omega_0\}$ converging to $\alpha$. Note that the point $x_p$ need not be an end-point of $X$.

Let $\mathscr{B}_p$ be a family of all branches of $X$ beginning at $x_p$ and such that if $B \in \mathscr{B}_p$ then $x_0 \notin B$ and $R(B) \neq \emptyset$. By (3) and (4), $\mathscr{B}_p$ is countable, and so we may index it: $\mathscr{B}_p = \{B_{(p,n)}: n<\omega_0\}$ (if $\mathscr{B}_p$ is finite, we put $B_{(p,n)} = \emptyset$ for sufficiently large $n$). We see that (6), (7) and (8) hold.

The construction is finished. Put

$$C_\alpha = \begin{cases} \{x \in R(X): x = x_p \text{ for some } p \in P_\alpha\} & \text{if } \alpha \text{ is a limit ordinal number, } 0<\alpha<\omega_1, \\ \emptyset & \text{if } \alpha \text{ is a non-limit ordinal number or } \alpha = 0, \end{cases}$$

and

$$Q_\alpha = \begin{cases} \{p \in P_\alpha: \mathscr{B}_p \neq \emptyset\} & \text{if } \alpha \text{ is a limit ordinal number, } 0<\alpha<\omega_1, \\ \emptyset & \text{if } \alpha \text{ is a non-limit ordinal number or } \alpha = 0. \end{cases}$$

(10) For each $\alpha<\omega_1$, the set $C_\alpha$ is countable.

Indeed, if $x \in C_\alpha$, then $r(x)\geqslant 3$; so let $D_1, D_2, D_3$ be 3 distinct branches of $X$ beginning at $x$. At most one of them, say $D_3$, can contain the point $x_0$. Let $x_1 \in D_1$ and $x_2 \in D_2$ be any points, so $T_x = [x_1, x] \cup [x_2, x] \cup [x_0, x]$ is a triod. If $x, x' \in C_\alpha$, $x \neq x'$, then (by (7)) $T_x \cap T_{x'} = (x, x_0] \cap (x', x_0]$, i.e., the family $\{T_x: x \in C_\alpha\}$ is almost disjoint. By (2) we obtain (10). Furthermore:

(11) for each $\alpha<\omega_1$, the set $Q_\alpha$ is countable.

If $p \in Q_\alpha$, then $\mathscr{B}_p$ is nonempty, i.e., there is a branch $E_p$ of $X$ beginning at $x$ such that $R(E_p) \neq \emptyset$ and $x_0 \notin E_p$. By (7), $E_p \cap E_{p'} = \emptyset$ for $p, p' \in Q_\alpha$, $p \neq p'$. By (4), there is a triod $T_p$ contained in $E_p$ for each $p \in Q_\alpha$. By (3) we obtain (11).

Put $I_p = \emptyset$ if $p \in P_\alpha$ and $\alpha$ is a limit ordinal number, $0<\alpha<\omega_1$.

(12) For each $0<\alpha<\omega_1$, the family $\{I_p: p \in P_\alpha, I_p \neq \emptyset\}$ is countable.

If $\alpha = \beta+1$ for some limit ordinal number $\beta$, then (by (11)) $Q_\beta$ is countable. Since each family $\mathscr{B}_p = \{B_{(p,n)}: n<\omega_0\}$ is countable and each nonempty $I_{(p,n)}$ corresponds to exactly one $B_{(p,n)} \in \mathscr{B}_p$ and this correspondence is one-to-one for $(p, n) \in P_{\beta+1} = P_\alpha$, we see that in this case (12) holds. Suppose that $\alpha = \beta+n+1$ for some limit ordinal number $\beta<\omega_1$, $0<n<\omega_0$, and we have already shown (12) for $\beta+n$. If $p \in P_{\beta+n}$ and $I_p \neq \emptyset$, then each arc $I_{(p,k)}$, $k<\omega_0$, is contained in $B_{(p,k)}$ and if $I_p = \emptyset$ then also $I_{(p,k)} = \emptyset$ for each $k<\omega_0$. Therefore

$$\{I_{(p,k)}: p \in P_{\beta+n}, k<\omega_0, I_p \neq \emptyset\} \supset \{I_q: q \in P_{\beta+n+1}, I_q \neq \emptyset\}$$

and these sets are countable.

Now we show

(13) $\quad R(X) \subset I \cup \bigcup\{I_p: p \in P_\beta, \beta<\alpha\} \cup \bigcup\{C_\beta: \beta<\alpha\} \cup \bigcup\{B_p: p \in P_\alpha\}$,

if $\alpha$ is a non-limit ordinal number; and

(14) $\quad R(X) \subset I \cup \bigcup\{I_p: p \in P_\beta, \beta<\alpha\} \cup \bigcup\{C_\beta: \beta\leqslant\alpha\} \cup \bigcup\{B_p: p \in P_{\alpha+1}\}$,

if $\alpha$ is a limit ordinal number.

We proceed by induction. For $\alpha = 0$, (14) is the same as (5). Suppose that $\alpha$ is given, $0<\alpha<\omega_1$, and both (13) and (14) hold for each $\beta<\alpha$. There are three cases to consider:

Case I. $\alpha = \beta+1$ for some non-limit ordinal number $\gamma$. Therefore (13) holds for $\gamma$ and, by (9), we see that

$$\bigcup\{R(B_p): p \in P_\gamma\} \subset \bigcup\{I_p: p \in P_\gamma\} \cup \bigcup\{B_{(p,n)}: p \in P_\gamma, n<\omega_0\},$$

so (13) holds for $\alpha$.

Case II. $\alpha = \gamma+1$ for some limit ordinal number $\gamma$. Therefore (14) holds for $\gamma$. But

$$\bigcup\{I_p: p \in P_\beta, \beta<\gamma\} = \bigcup\{I_p: p \in P_\beta, \beta\leqslant\gamma\} = \bigcup\{I_p: p \in P_\beta, \beta<\alpha\}$$

(because $I_p = \emptyset$, for $p \in P_\gamma$) and

$$\bigcup\{C_\beta: \beta\leqslant\gamma\} = \bigcup\{C_\beta: \beta<\alpha\};$$

so (13) holds for $\alpha$.

Case III. $\alpha$ is a limit ordinal number, $0<\alpha<\omega_1$. Let $\{\gamma_n: n<\omega_0\}$ be an increasing sequence of non-limit ordinal numbers which converges to $\alpha$. We know that (13) holds for each $\gamma_n$. Moreover,

(15) $\quad I \cup \bigcup\{I_p: p \in P_\beta, \beta<\gamma_n\} \cup \bigcup\{C_\beta: \beta<\gamma_n\} \subset$
$\qquad \subset I \cup \bigcup\{I_p: p \in P_\beta, \beta<\alpha\} \cup \bigcup\{C_\beta: \beta\leqslant\alpha\}$ for each $n<\omega_0$.

Suppose that (14) does not hold for $\alpha$, i.e., there is an $x \in R(X)$ such that

(16) $\quad x \notin I \cup \bigcup\{I_p: p \in P_\beta, \beta<\alpha\} \cup \bigcup\{C_\beta: \beta\leqslant\alpha\} \cup \bigcup\{B_p: p \in P_{\alpha+1}\}$.



By (13) used for each $\gamma_n$ and by (15) and (16) we see that

(17) $\quad x \in \bigcup\{B_p: p \in P_{\gamma_n}\}$ for each $n<\omega_0$,

and

(18) $\quad x \notin C_\alpha \cup \bigcup\{B_p: p \in P_{\alpha+1}\}$.

By (17), (7) and (8) there is a $q \in P_\alpha$ such that if $q_n = q|_{\gamma_n}$ then $x \in B_{q_n}$ for each $n<\omega_0$. Since (by (18)) $x \notin C_\alpha$, we see that $x \neq x_q$, so there is a unique branch $B$ of $X$ beginning at $x_q$ and such that $x \in B$. Suppose first that $[x_0, x_q] \cap [x, x_q] \neq \{x_q\}$. By the hereditary unicoherence of $X$, there is a point $y \in X$ such that $y \in [x, x_q]$ and $[x_0, x_q] \cap [x, y] = \{y\}$, so $y \neq x_q$. Since $x_q = \lim_{n\to\infty} x(B_{q_n})$, there is a $k<\omega_0$ such that $y \in [x(B_{q_k}), x(B_{q_{k+1}}))$. We have $x \in B$ and $x \in B_{q_{k+1}}$ (by (17)). But $B$ and $B_{q_{k+1}}$ are branches of $X$ which begin from the arc $[x(B_{q_k}), x(B_{q_{k+1}})]$ at distinct points $y$ and $x_{q_{k+1}}$. Therefore $B \cap B_{q_{k+1}} = \emptyset$ — a contradiction. Hence $[x_0, x_q] \cap [x, x_q] = \{x_q\}$ and $x_0 \notin B$ (because if $x_0 \in B$ then $[x_0, x_q] \cap [x, x_q] \neq \{x_q\}$). This means that

$$x \in \bigcup \mathscr{B}_q \subset \bigcup\{B_p: p \in P_{\alpha+1}\} \quad (\text{since } x \in R(B) \neq \emptyset).$$

The last face contradicts (18). In this way both (13) and (14) are proved.

(19) If $\alpha<\omega_1$ is a non-limit ordinal number, then there is a $p \in P_\alpha$ such that $B_p \neq \emptyset$.

Indeed, if $\alpha$ is a non-limit ordinal number and $B_p = \emptyset$ for each $p \in P_\alpha$, then by (13), (12) and (10) the set $R(X)$ can be covered by countably many arcs (since $\alpha<\omega_1$) — which contradicts (1).

(20) If $p \in P_\alpha$, $q \in P_\beta$, for some $0<\alpha$, $\beta<\omega_1$, $p \neq q$, then $I_p \cap I_q = \emptyset$.

Indeed, put $\gamma = \min\{\delta: p|_\delta \neq q|_\delta\}$. Suppose that $\gamma\leqslant\alpha$ and $\gamma\leqslant\beta$. It is clear that $\gamma$ cannot be a limit ordinal number because in this case $p|_\gamma = q|_\gamma$. If $\gamma$ is a non-limit ordinal number, then (by (7)) $B_{p|_\gamma} \cap B_{q|_\gamma} = \emptyset$, $I_p \subset B_p \subset B_{p|_\gamma}$, $I_q \subset B_q \subset B_{q|_\gamma}$, so we obtain (20). Therefore we may assume that $\gamma = \alpha+1$, i.e., $q|_\alpha = p$ and $\alpha<\beta$. We have $I_q \subset B_q \subset B_{q|\alpha+1}$ and $B_{q|\alpha+1}$ is a branch of $X$ beginning from $I_p$, so $B_{q|\alpha+1} \cap$ $\cap I_p = \emptyset$. (20) is proved.

For each limit ordinal number $\alpha$, $\alpha \neq \emptyset$, let $p_\alpha \in P_\alpha$ and $m_\alpha, n_\alpha<\omega_0$ be such that $B_{((p_\alpha, m_\alpha), n_\alpha)} \neq \emptyset$; thus also $B_{(p_\alpha, m_\alpha)} \neq \emptyset$ and therefore $I_{(p_\alpha, m_\alpha)} \neq \emptyset$, $I_{((p_\alpha, m_\alpha), n_\alpha)} \neq \emptyset$. We have $I_{(p_\alpha, m)} = (x_{p_\alpha}, y_1]$ and $I_{((p_\alpha, m_\alpha), n_\alpha)} = (x(B_{((p_\alpha, m_\alpha), n_\alpha)}), y_2]$ for some $y_1, y_2 \in E(X)$. Moreover,

$$(x_{p_\alpha}, y_1] \cap [x(B_{((p_\alpha, m_\alpha), n_\alpha)}), y_2] = \{x(B_{((p_\alpha, m_\alpha), n_\alpha)})\}.$$

Let $a$ be any point such that $a \in (x_p, x(B_{((p_\alpha, m_\alpha), n_\alpha)}))$. By the previous remarks, the set $T_\alpha = [a, y_1] \cup I_{((p_\alpha, m_\alpha), n_\alpha)}$ is a triod. By (20), $T_\alpha \cap T_\beta = \emptyset$ for $\alpha \neq \beta$. The set $\{\alpha<\omega_1: \alpha \neq 0$ and $\alpha$ is a limit ordinal number$\}$ is uncountable, and so the collection $\{T_\alpha: 0<\alpha<\omega_1$ and $\alpha$ is a limit ordinal number$\}$ is an uncountable family of pairwise disjoint triods contained in $X$. This contradicts (3). The proof of Theorem 1 is finished.

THEOREM 2. *Let $X$ be a dendroid and assume that*

(i) *$X$ does not contain any uncountable family of pairwise disjoint triods,*

*and*

(ii) *$X$ contains an uncountable almost disjoint family $\mathscr{T}$ of triods.*

*Then $R(X)$ cannot be covered by countably many arcs.*

Proof. Let us suppose that some family $\{[a_n, b_n]: n = 1, 2, \ldots\}$ covers $R(X)$. Let $\{r_T: T \in \mathscr{T}\}$ denote the set of ramification points of triods from the family $\mathscr{T}$, i.e., $r_T \in T$, $\{r_T\} = R(T)$, for $T \in \mathscr{T}$. Since $\mathscr{T}$ is an almost disjoint family, $r_T \neq r_{T'}$, for $T, T' \in \mathscr{T}$, $\mathscr{T} \neq T'$. Therefore there is a positive integer $k$ such that the arc $[a_k, b_k]$ contains uncountably many points $r_T$, i.e., the family

$$\mathscr{T}' = \{T \in \mathscr{T}: r_T \in [a_k, b_k]\}$$

is uncountable. Put

$$\mathscr{T}'' = \{T \in \mathscr{T}': T \cap [a_k, b_k] = \{r_T\}\}.$$

By the hereditary unicoherence of $X$, if $T, T' \in \mathscr{T}''$, $T \neq T'$, then $T \cap T' = \emptyset$, i.e., $\mathscr{T}''$ is the family of pairwise disjoint triods. Put $\mathscr{S} = \mathscr{T}' \setminus \mathscr{T}''$; so by (i) $\mathscr{S}$ is an uncountable almost disjoint family of triods. Moreover, if $T \in \mathscr{S}$, then $T \cap [a_k, b_k] \neq \{r_T\}$. By the hereditary unicoherence of $X$, $T \cap [a_k, b_k]$ is some nondegenerate arc, for $T \in \mathscr{S}$. Let $s_T$ be any point such that $s_T \neq r_T$ and $[r_T, s_T] \subset T \cap [a_k, b_k]$ for each $T \in \mathscr{S}$. Let "$<$" be a unique natural ordering of the arc $[a_k, b_k]$ such that $a_k < b_k$. Put $\mathscr{S}_1 = \{T \in \mathscr{S}: r_T < s_T\}$ and $\mathscr{S}_2 = \{T \in \mathscr{S}: s_T < r_T\}$, so $\mathscr{S}_1 \cup \mathscr{S}_2 = \mathscr{S}$. Therefore either $\mathscr{S}_1$ or $\mathscr{S}_2$ is uncountable, say, $\mathscr{S}_1$. The family $\{[r_T, s_T]: T \in \mathscr{S}_1\}$ is an uncountable family of nondegenerate subarcs of $[a_k, b_k]$. Hence $[r_T, s_T] \cap [r_{T'}, s_{T'}] \neq \emptyset$ for some $T, T' \in \mathscr{S}_1$, $T \neq T'$. This means that $r_T < s_T$, $r_{T'} < s_{T'}$, $r_{T'} < s_T$, and so $r_T < r_{T'} < s_T$. Therefore $r_{T'} \in [r_T, s_T]$, and so $r_{T'} \in T \cap T'$. But the family $\mathscr{T}$ is almost disjoint, in particular, $r_{T_1} \notin T_2$ for any distinct triods $T_1, T_2 \in \mathscr{T}$ — a contradiction.

LEMMA 2. *There is no uncountable almost disjoint family of triods contained in the plane $E^2$.*

Proof. Lemma 2 is a particular case of [3], Corollary 1, p. 275.

COROLLARY. *Let $X$ be a planable dendroid. Then*

(a) *$R(X)$ can be covered by countably many arcs;*

(b) *$R(X)$ is a $G_{\delta\sigma}$-set;*

(c) *the set of all points of $X$ each of which is of order $\geqslant 5$ is countable.*

Proof. (a) is an immediate consequence of Theorem 1 and Lemma 2.

(b) follows from (a) and [6], Theorem 1.

(c) follows from (a) and Lemma 2.



EXAMPLES. 1. In [7] the dendroid $X$ is constructed so that $R(X)$ cannot be covered by countably many arcs and $X$ does not contain any uncountable family of pairwise disjoint triods.

2. The points of the 3-dimensional Euclidean space $E^3$ are denoted by ordered three-tuples of real numbers $\langle x_1, x_2, x_3\rangle$. If $x, y \in E^3$, then $\overline{xy}$ is the straight-line segment from $x$ to $y$.

Let $C$ be the Cantor set contained in the real line, $0 \leqslant x \leqslant 1$ for $x \in C$. Put

$$X = \overline{\langle 0,0,0\rangle\langle 1,0,0\rangle} \cup \bigcup \{\overline{\langle x,0,0\rangle\langle x,1,0\rangle} \cup \overline{\langle x,0,0\rangle\langle x,0,1\rangle} \cup \overline{\langle x,0,0\rangle\langle x,0,-1\rangle}: x \in C\}.$$

We see that $X$ is a dendroid, that it contains an uncountable family of pairwise disjoint triods, but that it is a comb, i.e., $R(X) \subset I$ for some arc $I \subset X$. This example shows that assumption (i) of Theorem 2 is essential.

3. Put $X = \overline{\langle 0,0\rangle\langle 1,0\rangle} \cup \bigcup \{\overline{\langle p/q, 1/q\rangle}: p$ and $q$ are positive integers, $p < q$ and $p, q$ are relatively prime$\} \subset E^2$. Therefore $X$ is a planable comb and $R(X)$ is an $F_\sigma$-set but not a $G_\delta$-set.

4. Let $B$ be the Cantor comb, i.e., $B = I \times \{0\} \cup C \times I \subset E^2$ (where $I$ is the closed unit interval). Let $D$ (resp. $E$) denote the set of all left (resp. right) end-points of intervals deleted from $I$ during the construction of $C$. For each $t \in I \setminus C$ let $c_t$ be the greatest point of $C$ which is less than $t$ (since $C$ is compact such a point exists) and if $t \in E$, then let $c_t \in D$ be the second end-point of the deleted interval to which $t$ belongs. We consider a decomposition $\mathscr{D}$ of the plane $E^2$ into points and segments $\overline{\langle t, 0\rangle\langle c_t, t - c_t\rangle}$ for $t \in (I \setminus C) \cup E$. This decomposition is easily seen to be an upper semi-continuous and monotone decomposition no element of which separates the plane. By the Moore theorem ([2], § 61, IV, Theorem 8, p. 533) the quotient space $E^2/\mathscr{D}$ is homeomorphic to the plane (the straightforward construction of the homeomorphism $E^2/\mathscr{D} \to E^2$ is not difficult). Let $\pi$ denote the quotient map $\pi: E^2 \to E^2/\mathscr{D}$ and put $f = \pi|_B: B \to \pi(B) = Y$. One can observe that $f$ is monotone relative to the point $\langle 0, 0\rangle \in B$ (i.e., for each subcontinuum $Q$ of $Y$ such that $f(\langle 0,0\rangle) \in Q$ the inverse image $f^{-1}(Q)$ is connected), so by [5], Corollary 2.8, p. 721, $Y$ is a smooth dendroid (with respect to the point $f(\langle 0,0\rangle)$). The set $R(Y)$ is homeomorphic to $C \setminus D$. Therefore $R(Y)$ is a $G_\delta$-set and by the well-known Baire theorem $R(Y)$ is not an $F_\sigma$-subset of $Y$.

$Y$ is a planable smooth comb such that $R(Y)$ is a $G_\delta$-set but not an $F_\sigma$-set.

5. $X$ is the dendroid of Example 3 and $Y$ that of Example 4. We can embed $X$ and $Y$ into $E^2$ in such a way that $X \cap Y$ is a single point $x$ which is the initial point of both $X$ and $Y$. Hence $Z = X \cup Y$ is a smooth planable dendroid such that $R(Z) = R(X) \cup R(Y) \cup \{x\}$ is not either an $F_\sigma$-set or a $G_\delta$-set.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
pl. Grunwaldzki 2/4, 50-384 Wrocław
Poland

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# Fixed point free equivariant homotopy classes

by

**Dariusz Wilczyński** (Poznań)

**Abstract.** Let $G$ be a compact Lie group. For an equivariant self-map $f$ of a compact smooth $G$-manifold $M$, an equivariant homotopy invariant $L(f)$ is defined, and it is shown that, under given conditions on $M$, this invariant detects the equivariant homotopy classes of fixed point free maps. In this context, the question of the existence of a non-singular equivariant vector field extending a vector field given on the boundary is also discussed.

**0. Introduction.** The Lefschetz fixed point theorem states that, if $f\colon X \to X$ is homotopic to a fixed point free map, then the Lefschetz number of $f$ vanishes. This theorem is valid for a wide class of spaces $X$ and maps $f$, e.g., for all compact ANRs and continuous maps. The question arises whether the vanishing of the Lefschetz number is a sufficient condition for $f$ to be fixed point free up to a homotopy. The answer is negative in general, even for polyhedra. If $X$ is a Wecken space the answer is positive. The simplest and most important example of a Wecken space is a compact simply connected manifold, with or without boundary, of dimension at least three. A full, detailed exposition of the Lefschetz fixed point theorem, its converse and related topics can be found in [3].

In this paper the question of the existence of a fixed point free map homotopic to a given self-map of a compact smooth manifold is considered in a $G$-equivariant category, $G$ being any compact Lie group. In Section 2 with each equivariant map $f$ we associate a family of integers, denoted by $L(f)$, which depends only on the equivariant homotopy class of $f$ and has properties analogous to that of the usual Lefschetz number. In particular, the Lefschetz theorem 2.6 is valid. This invariant detects the equivariant homotopy classes of fixed point free maps. Of course, some additional hypotheses, as in the non-equivariant case, are needed. In fact, we prove the following theorem:

THEOREM A. *Let $M$ be a compact smooth $G$-manifold such that all connected components of $M^H$ are simply connected and of dimension at least three for any isotropy subgroup $H$ with a finite Weyl group in $G$. Then an equivariant map $f\colon M \to M$ is fixed point free up to an equivariant homotopy if and only if $L(f) = 0$.*

If $f$ is the identity the same is true without any restrictions on the fundamental group and dimension.