Remote points in spaces with π-weight ω_1

by

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Abstract. Every nonpseudocompact ccc space with π-weight ω_1 has remote points. If there is an ω_1-scale then the ccc assumption may be dropped. Results for ccc spaces with larger π-weight are also obtained.

0. Introduction. All spaces are completely regular and X^* denotes $\beta X - X$. A point $p \in \beta X - X$ is called a *remote point* of X if $p \in \mathrm{cl}_{\beta X} A$ for each nowhere dense subset A of X. Remote points and similar notions, such as remote filters, nice remote filters and remote linked systems, have proven very useful in the investigation of Čech–Stone compactifications (cf. [vD], [vM1], [vM2]). Much more recently the existence of remote linked systems, surprisingly, has been used in forcing proofs using large cardinals [TW].

In 1962, Fine and Gillman [FG] introduced remote points and proved that, under CH, the reals had remote points. The authors of [KvMM] improved this result to include all nonpseudocompact spaces, X with $|C^*(X)| \leqslant \omega_1$. Van Douwen [vD], and, independently, Chae and Smith [CS], showed that if X is a nonpseudocompact space with countable π-weight, then X has remote points (and nice remote filters). However, there are nonpseudocompact spaces with π-weight ω_2 which do not have remote points (by [vDvM] and [KvMM]). This leaves open the question for π-weight ω_1 spaces.

A partial answer was provided by van Mill [vM1]: a nonpseudocompact space which is a product of ω_1 spaces, each with countable π-weight, has remote points (and nice remote filters, which was needed for his applications). Recall that a space is ccc if each family of pairwise disjoint open sets is countable. In this paper we show that a nonpseudocompact ccc space with π-weight ω_1 has remote points and that it is independent with not CH that they all have nice remote filters. We also show that it is consistent with not CH that all nonpseudocompact spaces with π-weight ω_1 have remote points. In [D3] we show it is consistent that there is a nonpseudocompact space with π-weight ω_1 having no remote points. The reason for the nonpseudocompactness assumption in the above results is that a pseudocompact space with π-weight less than the first measurable cardinal does not have remote points [Te].

1. Preliminaries. Let X be a space. A set $\mathcal{F}\subset\mathcal{P}(X)$ is called *remote* if for each nowhere dense set $D\subset X$ there is an $F\in\mathcal{F}$ with $F\cap D=0$. A *remote filter* will refer to a filter of closed sets which is remote. If X is the topological sum of $X_n (n<\omega)$, denoted $\sum X_n$, then a closed filter F is called *nice* provided that $|\{n<\omega: F\cap X_n=\emptyset\}|<\omega$ for all $F\in\mathcal{F}$, and $\bigcap\mathcal{F}=\emptyset$. A π-base B for X is a set of non-empty open subsets of X with the property that each open subset of X contains a member of B. The *π-weight* of X, $\pi w(X)$, is the minimum cardinality of a π-base for X. A family of sets is *cellular* if its members are pairwise disjoint.

We now establish our general framework for constructing remote points and filters. This approach has been used repeatedly ([vD][D1,2]).

1.1. Lemma. *Let $\{X_n: n<\omega\}$ be a locally finite family of regular closed subsets of X and let, for $n<\omega$, B_n be a π-base for X_n. Let Γ be the set of all maximal cellular families of sets from $\bigcup B_n$ and let $\mathcal{F}$ be a closed filter on X.*

(1) *If F has the property that for each $\sigma\in\Gamma$ there is an $F\in\mathcal{F}$ with $F\subset\bigcup\{\bar{a}: a\in\sigma\}$, then $\mathcal{F}$ is remote.*

(2) *If, for each $\sigma\in\Gamma$, there is an $F\in\mathcal{F}$ so that for each $n<\omega$ there is a finite $\sigma_n\subset\sigma\cap B_n$ with $F\subset\bigcup\{\bar{a}: a\in\bigcup_n\sigma_n\}$ then F is remote and extends to a remote point of X.*

Proof. (1) If $D\subset X$ is nowhere dense, we can choose a subset

$$\sigma\subset\{a\in\bigcup B_n:\ \bar{a}\cap D=0\}$$

so that σ is a maximal cellular family. Since, for each $n\in\omega$, $D\cap X_n$ is nowhere dense in X_n and B_n is an π-base, $\sigma\in\Gamma$. If $F\in\mathcal{F}$ and $F\subset\bigcup\{\bar{a}: a\in\sigma\}$ then $F\cap D=0$. Hence $\mathcal{F}$ is remote.

(2) In this case, for a nowhere dense set $D\subset X$, let $\sigma\subset\{a\in\bigcup B_n$: a is completely separated from $D\}$ be chosen as in (1). Now let $F\in\mathcal{F}$ and, for $n<\omega$ let σ_n be a finite subset of $\sigma\cap B_n$ so that $F\cap X_n\subset\bigcup\{\bar{a}: a\in\sigma_n\}$. It is easily seen that $\bigcup\{\bar{a}: a\in\bigcup_n\sigma_n\}$ is completely separated from D, and therefore $\mathrm{cl}_{\beta X}F\cap\mathrm{cl}_{\beta X}D=\emptyset$. Now any $p\in\bigcap\{\mathrm{cl}_{\beta X}F: F\in\mathcal{F}\}$ is a remote point.

Our procedure then, will be to take X_n, B_n and Γ as in 1.1 and try to select, for $\sigma\in\Gamma$, finite sets $\sigma_n\subset\sigma\cap B_n$ for $n\in\omega$ so that $\{\bigcup_n\bigcup\{\bar{a}: a\in\sigma_n\}: \sigma\in\Gamma\}$ forms a filter base. If we do so then evidently 1.1(2) is satisfied.

For sets A and B the set of functions from A to B is denoted by ${}^AB$ and for a cardinal λ, $[A]^\lambda=\{B\subset A: |B|=\lambda\}$. For a poset $(P,<)$, a subset $D\subset P$ is *cofinal* in P if for each $p\in P$ there is a $q\in D$ with $p\leqslant q$. Let u be a filter on ω and let $f,g\in{}^\omega\omega$; define $f<_u g$ to mean $\{n: f(n)<g(n)\}\in u$. Of course, if u is the cofinite filter, cof, then we shall use $<^*$ instead of $<_u$. In general, for a filter u $({}^\omega\omega,<^u)$ does not contain a cofinal chain but if it does then let λ_u denote the least cardinal of a cofinal chain. If $({}^\omega\omega,<_*)$ has a cofinal chain then there is said to be a *λ-scale* where $\lambda=\lambda_{\mathrm{cof}}$. If u is an ultrafilter then $({}^\omega\omega,<_u)$ does have a cofinal chain. We shall let $\varkappa_0$ denote the smallest cardinal such that $\lambda_u<\varkappa_0$ for all $u\in\omega^*$. It is well known that $\lambda_u\geqslant\omega_1$ for each $u\in\omega^*$. We shall also let $\eta_0=\min\{\lambda_u: u\in\omega^*\}$.

The more familiar cardinals, b and d can be defined as $b=\min\{|B|: B\subset{}^\omega\omega$ and for each $g\in{}^\omega\omega$ there is an $f\in{}^\omega\omega$ with $f\not<_*g\}$ and $d=\min\{|D|: D$ is cofinal in $({}^\omega\omega,<_*)$ (although D need not be a chain$\}$. If $d=b$ then there is a λ-scale with $\lambda=d$ and conversely. Note also that if there is a λ-scale then $\lambda\geqslant\omega_1$.

2. The main results. The method we use in this paper is a modification of that used in an earlier paper [D1]. The idea was to index all maximal cellular families from a π-base and define a sequence of finite subsets of them such as in 1.1(2) but it was the "responsibility" of the γth cellular family to meet that already chosen for the αth cellular family for $\alpha<\gamma$. It is not difficult to set it up so that this is possible for a ccc space so long as there are no unbounded subsets of $({}^\omega\omega,<_*)$ of cardinality $|\gamma|$. The problem now is that even with small π-weight there are always at least c maximal cellular families. The way we get around this is to index a π-base in a special way. The following definition is slightly more general than we require for the π-weight ω_1 case but it will be useful in the next section.

2.1. Definition. For a cardinal $\varkappa$, let us say that a π-base B for a topological sum $\sum X_n$ is *well-indexed by $\varkappa$* if $B=\{a(n,\alpha,m): n,m<\omega,\alpha<\varkappa\}$ where, for each (n,α,m), $a(n,\alpha,m)\subset X_n$ and for each maximal cellular family $\sigma\subset B$ there is an $\alpha<\varkappa$ with $\sigma\subset\{a(n,\alpha,m): n,m<\omega\}$.

The above definition says that B is like a π-base for maximal cellular families. It is probably true that, for a space $X=\sum X_n$, the minimum $\varkappa$ such that X has a π-base well indexed by $\varkappa$ is a function of the cardinal $\pi w(X)$ rather then the space X itself. In the next section we look at this more closely, for now all we require is the following fact whose proof is trivial.

2.2. Fact. *If $X=\sum X_n$ is a ccc space with $\pi w(x)\leqslant\omega_1$ then X has a π-base B well indexed by ω_1.*

Proof. For each n, let B_n be a π-base for X_n and let $B=\bigcup B_n$. Let $B=\{b_\gamma: \gamma\in\omega_1\}$ and for each $(n,\alpha)\in\omega\times\varkappa$ let $\{a(n,\alpha,m): m\in\omega\}$ be an indexing of $\{b_\gamma: \gamma\in\alpha\}\cap B_n$. Now if σ is a maximal cellular family, σ is countable. Therefore there is an $\alpha\in\omega_1$ with $\sigma\subset\{b_\gamma: \gamma\in\alpha\}$ and so $\sigma\subset\{a(n,\alpha,m): n,m<\omega\}$.

We now prove our main results. We shall first present the result about nice remote filters because the proof is easier.

2.3. Theorem. *Assume that $b=d$. A topological sum $X=\sum X_n$, which is ccc and has π-weight ω_1, has nice remote filters.*

Proof. Let $\lambda=b=d$ and choose $\{r_\xi: \xi\in\lambda\}\subset{}^\omega\omega$ so that $\{r_\xi: \xi\in\lambda\}$ is well-ordered by $<_*$ and for each $f\in{}^\omega\omega$ there is a $\xi\in\lambda$ with $f<_*r_\xi$. By 2.2, let $B=\{a(n,\alpha,m): \alpha\in\omega_1, n,m<\omega\}$ be a well-indexed π-base for X. Define Γ to be all maximal cellular families from B.

Let $\sigma\in\Gamma$ and choose $\alpha=\alpha(\sigma)\in\omega_1$ so that $\sigma\subset\{a(n,\alpha,m)\colon n,m<\omega\}$. Our procedure will be to define a function $g_\sigma\in{}^\omega\omega$ so that $\bigcup\{a(n,\alpha,m)\colon a(n,\alpha,m)\in\sigma$ and $m<g_\sigma(n)\}$ will be put in the nice remote filter (see 1.1(2)).

FACT 1. *Let $\xi<\lambda$ and let $\delta=\min(\xi,\omega_1)$. There is an ordinal $S_\sigma(\xi)>\xi$ satisfying the following: for any $j<\omega$ and any sequence $\beta_i\colon i<j\subset\delta$ there is a cofinite subset, A, of ω such that, for any $n\in A$ and sequence $\{m_i\colon i<j\}$ of integers less than $r_\xi(n)$, $\bigcap_{i<j}a(n,\beta_i,m_i)\neq 0$ implies there is an $m<r_{S_\sigma(\xi)}(n)$ with $a(n,\alpha,m)\cap\bigcap_{i<j}a(n,\beta_i,m_i)\neq 0$.*

Let us defer the proof of Fact 1 until we see its use. The following loosely stated consequence of Fact 1 may be more readable. The function $r_{S_\sigma(\xi)}$ is large enough so that if we have ordinals $\beta_i<\delta$ $(i<j)$ and integers $m(n,i)<r_\xi(n)$ with $\bigcap a(n,\beta_i,m(n,i))\neq 0$ for cofinitely many n then for cofinitely many n there is an $m<r_{S_\sigma(\xi)}(n)$ so that $a(n,\alpha,m)$ meets this intersection.

Since $\{r_\xi\colon \xi\in\lambda\}$ is cofinal in $({}^\omega\omega,<_*)$ we can choose $\xi_0(\sigma)$ with $\alpha<\xi_0(\sigma)<\lambda$ so that for cofinitely many $n<\omega$ there is an $m<r_{\xi_0(\sigma)}(n)$ with $a(n,\alpha,m)\in\sigma$. We shall recursively define a sequence $\xi_j(\sigma)$ for $j<\omega$ (but for convenience we will just call them ξ_j). For each $j<\omega$, simply let $\xi_{j+1}=S_\sigma(\xi_j)$. Now we define $g_\sigma(n)=\max(r_{\xi_j}(n)\colon j\leqslant n)$ for $n<\omega$.

Let us show $\{\bigcup\{a(n,\alpha,m)\colon \alpha=\alpha(\sigma),\ a(n,\alpha,m)\in\sigma$ and $m<g_\sigma(n)\}\colon \sigma\in\Gamma\}$ forms a nice filter base. Let $\Gamma_1\subset\Gamma$ with $|\Gamma_1|=k<\omega$. Recursively choose, for $i<k$, $\sigma_i\in\Gamma_i$ so that $\xi_i(\sigma_i)$ is a minimum for $\{\xi_i(\sigma)\colon \sigma\in\Gamma\setminus\{\sigma_j\colon j<i\}\}$. Let $\beta_i=\alpha(\sigma_i)$ $(i<k)$ and note that for $i<j<k$, $\beta_i<\xi_0(\sigma_i)\leqslant\xi_i(\sigma_i)\leqslant\xi_{j-1}(\sigma_j)<\xi_j(\sigma_j)$.

Now, choose $N_0\geqslant k$, so that for $n\geqslant N_0$ there is an $m(n,0)<r_{\xi_0(\sigma_0)}(n)$ with $a(n,\beta_0,m(n,0))\in\sigma_0$. Suppose that $j+1<k$ and that we have chosen N_i $(i\leqslant j)$ and $m(n,i)<r_{\xi_i(\sigma_i)}(n)$ for $n\geqslant N_i$ so that for $n\geqslant N_j$,

$$\bigcap_{i\leqslant j}a(n,\beta_i,m(n,i))=0\quad\text{and}\quad a(n,\beta_i,m(n,i))\in\sigma_i\ (i\leqslant j)\,.$$

We can choose an integer $N'_{j+1}\geqslant N_j$ so that, for $n\geqslant N'_{j+1}$, $r_{\xi_j(\sigma_{j+1})}(n)\geqslant r_{\xi_i(\sigma_i)}(n)$ $(i\leqslant j)$ since $\xi_i(\sigma_{j+1})\geqslant\xi_i(\sigma_i)$ $(i\leqslant j)$. Therefore, for $n>N'_{j+1}$ and $i\leqslant j$, $m(n,i)<r_{\xi_j(\sigma_{j+1})}$. By Fact 1 and the definition of $\xi_{j+1}(\sigma_{j+1})$ there is an $N_{j+1}\geqslant N'_{j+1}$ so that for each $n\geqslant N_{j+1}$ there is an $m(n,j+1)<r_{\xi_{j+1}(\sigma_{j+1})}(n)$ with $\bigcap_{i\leqslant j+1}a(n,\beta_i,m(n,i))\neq 0$ and $a(n,\beta_i,m(n,i))\in\sigma_i$ $(i\leqslant j+1)$. It follows that for $n\geqslant N_{k-1}$,

$$\bigcap_{\sigma\in\Gamma_1}\bigcup\{a(n,\alpha(\sigma),m)\colon a(n,\alpha,m)\in\sigma\text{ and }m<g_\sigma(n)\}\supset\bigcap_{i<k}a(n,\beta_i,m(n,i))\neq 0$$

since, for $i<k$, $m(n,i)<r_{\xi_i(\sigma_i)}(n)\leqslant g_{\sigma_i}(n)$. Therefore we have a nice remote filter base.

It remains to prove Fact 1. Let $\xi<\lambda$ and $\delta=\min(\xi,\omega_1)$. For any sequence $(\beta_i\colon i<j)\subset\delta$ and $n\in\omega$, there are only finitely many open sets of the form $\bigcap\{a(n,\beta_i,m_i)\colon i<j\}$ with $m_i<r_\xi(n)$ $(i<j)$. Since σ is a maximal cellular family, there is an integer k_n so that, for each of the non-empty sets there is an $m<k_n$ so that $a(n,\alpha,m)$ is in σ and meets the set. It follows that there is a ξ' so that for cofinitely many n $r_{\xi'}(n)\geqslant k_n$. Now we simply let $S_\sigma(\xi)<\lambda$ be larger than ξ' for each of the $|\xi|\omega$-many such sequences.

2.4. THEOREM. *Each nonpseudocompact ccc space with π-weight ω_1, has remote points.*

Proof. By Lemma 1.1 it suffices to show the result for $X=\sum X_n$. Let u be any ultrafilter on ω and choose $\{r_\xi\colon \xi\in\lambda_u\}\subset{}^\omega\omega$ to be well-ordered and cofinal in $({}^\omega\omega,<_u)$. The proof then proceeds exactly as in 2.3 except that every set of integers chosen to be cofinite is now chosen so as to belong to u and $<_*$ is replaced by $<_u$. The remote filter that we construct will then trace on $\sum X_n$ along the filter u.

We now drop the ccc assumption. As mentioned in the introduction, under CH, each nonpseudocompact space with $|C^*(X)|=c$ has remote points. Whereas it is consistent that there is a space with π-weight ω_1 (and $|C^*(X)|=c$) which does not have remote points. Therefore some set-theoretic assumption is needed for the existence of remote points for spaces with π-weight ω_1. Our result shows exactly what assumption is needed. Also we not only weaken the CH assumption but extend the result itself because $\pi w(x)=\omega_1$ does not imply $|C^*(X)|\leqslant c$. The proof will again require only minor modifications to that of 2.3.

2.5. THEOREM. (1) *Every nonpseudocompact space with π-weight ω_1 has remote points iff $\eta_0=\omega_1$. (recall $\eta_0=\min\{\lambda_u\colon u\in\omega^*\}$).* (2) *Each topological sum with π-weight ω_1 has nice remote filters iff there is an ω_1-scale (i.e. $b=d=\omega_1$).*

Proof. The only if of both (1) and (2) is proven in [D3]. We shall prove (2); the modifications to this proof for (1) are exactly as those for 2.4.

Let $X=\sum X_n$ and assume that $b=d=\omega_1$. Fix ω_1-scale $\{r_\xi\colon \in\omega_1\}\subset{}^\omega\omega$ and let, for $n\in\omega$, $\{b(n,\alpha)\colon \alpha<\omega_1\}$ be a π-base for X_n. We cannot hope to get a well-indexed π-base because X need not be ccc. However it will suffice to let $\{a(n,\alpha,m)\colon m\in\omega\}$ be a listing of $\{\bigcap\{b(n,\beta)\colon \beta\in F\}\colon F$ is a finite subset of $\alpha+1\}\setminus\{\emptyset\}$, for each $n\in\omega$ and $\alpha\in\omega_1$. Again let Γ be the collection of all maximal cellular families from the π-base.

Let $\sigma\in\Gamma$. For each $n<\omega$, there is a $\gamma_n<\omega_1$ such that $b(n,\gamma_n)\in\sigma$; let $\alpha_0(\sigma)<\omega_1$ with $\gamma_n<\alpha_0$ for $n<\omega$. Choose $\xi_0(\sigma)\geqslant\alpha_0(\sigma)$ so that for cofinitely many $n\in\omega$ there is an $m<r_{\xi_0}(n)$ with $a(n,\alpha_0,m)\in\sigma$.

We shall need a slight modification of the Fact 1 in 2.3. The proof is virtually the same and is omitted.

FACT 2. *Let $\xi<\alpha<\omega_1$ and suppose that for each $a\in\{a(n,\xi,m)\colon n,m<\omega\}$ there is an $a'\in\sigma\cap\{a(n,\alpha,m)\colon n,m<\omega\}$ with $a\cap a'\neq\emptyset$. There is an $S_\sigma(\xi)<\omega_1$ with $\alpha<S_\sigma(\xi)$ so that for any $j<\omega$ and any sequence $(\beta_i\colon i<j)\subset\delta$ there is a co-finite subset A of ω such that for any $n\in A$ and sequence $\{m_i\colon i<j\}$ of integers less than $r_\xi(n)$, $\bigcap_{i<j}a(n,\beta_i,m_i)\neq 0$ implies there is an $m<r_{S_\sigma(\xi)}(n)$ with $a(n,\alpha,m)\cap\bigcap_{i<j}a(n,\beta_i,m_i)\neq 0$.*

The only difference between the above fact and that in 2.3 is that α is large to ensure that the αth level of the π-base contains enough of σ. Note that each level contains all finite intersections of elements from earlier levels.

FACT 3. *For each $\xi \in \omega_1$, there is an $\alpha > \xi$ so that for each $a \in \{a(n, \xi, m)\colon n, m < \omega\}$ there is an $a' \in \sigma \cap \{a(n, \alpha, m)\colon n, m < \omega\}$ with $a \cap a' \neq 0$.*

Indeed, for each $n, m < \omega$, choose $a_{n,m} \in \sigma$ so that $a_{n,m} \cap a(n, \xi, m) \neq 0$. Now let $\alpha > \xi$ be large enough so that $\{a(n, \alpha, m)\colon n, m < \omega\} \supset \{a_{n,m}\colon n, m < \omega\}$.

We recursively choose ξ_j and α_j for $j \in \omega$ as follows: $\xi_{j+1} = S_\sigma(\xi_j)$ from Fact 2 (where $\alpha = \alpha_j$) and then α_{j+1} is chosen from Fact 3. Now, of course, our remote filter is $\{\bigcup_n \bigcup \{a(n, \alpha_j, m)\colon a(n, \alpha_j, m) \in \sigma,\ j \leqslant n \text{ and } m < r_{\xi_j}(n)\}\colon \sigma \in \Gamma\}$. To check that this is a nice filter, let $\Gamma_1 \subset \Gamma$ with $|\Gamma_1| = k < \omega$. Again choose, for $j < k$, $\sigma_j \in \Gamma_1$ so that $\xi_j(\sigma_j)$ is a minimum for $\{\xi_j(\sigma)\colon \sigma \in \Gamma_1 \setminus \{\sigma_0, \ldots, \sigma_{j-1}\}\}$. Now let $\beta_j = \alpha_j(\sigma_j)$ $(j < k)$. Observe that for $i < j < k$ $\beta_i = \alpha_i(\sigma_i) < \xi_i(\sigma_i) \leqslant \xi_{j-1}(\sigma_j)$. We proceed from here exactly as in 2.3.

3. Larger π-weight. In this section we investigate to what extent the results in section 2 will extend to spaces with larger π-weight. It is known that 2.5 does not generalize but with ccc spaces it is possible, in both 2.3 and 2.4 [D1]. Our generalizations at present depend on which cardinals $\varkappa$ have the property that a ccc space of π-weight $\varkappa$ have a π-base well-indexed by $\varkappa$.

3.1. THEOREM. *Let X be a nonpseudocompact ccc space which has a π-base well-indexed by $\varkappa$.*

(1) *If $b = d$, $\varkappa \leqslant b$ and if X is a topological sum then X has nice remote filters.*

(2) *If $\varkappa < \varkappa_0$ then X has remote points.*

Proof. To prove (1) we need only substitute $\varkappa$ for ω_1 in Fact 1. For (2), we must choose $u \in \omega^*$ so that $\lambda_u \geqslant \varkappa$ and substitute $\varkappa$ for ω_1 in Fact 1.

It would be very nice if it were true that if X was a ccc space with π-weight less than $\varkappa_0$ then X has a π-base well-indexed by some $\lambda \leqslant \varkappa_0$. This, of course, would mean that any such nonpseudocompact X has remote points. I do not know if it is true but it is true for certain π-weights and if there are no measurable cardinals then it is true for all.

3.2. DEFINITION. For a cardinal $\varkappa$, let $C(\varkappa)$ be the minimum cardinality of a family $\mathscr{A} \subset [\varkappa]^\omega$ with the property that if $B \in [\varkappa]^\omega$ then there is an $A \in \mathscr{A}$ with $B \subseteq A$.

The following result is very easy to prove and is probably folklore.

3.3. LEMMA. *For $n \in \omega$, $C(\omega_n) = \omega_n$.*

Our reason for making Definition 3.2 is the next result.

3.4. THEOREM. *Let X be a ccc nonpseudocompact space with $\pi w(X) = \varkappa$. If $C(\varkappa) < \varkappa_0$ then X has remote points.*

Proof. It suffices to assume that $X = \sum X_n$. We simply show that X has a π-base well-indexed by $C(\varkappa)$ and apply 3.1. Indeed let $\{A_\xi\colon \xi < C(\varkappa)\} \subset [\varkappa]^\omega$ satisfy the property in 3.2. Also, for $n \in \omega$, let $\{b(n, \alpha)\colon \alpha \in \varkappa\}$ be a π-base for X_n. Then, for $\xi < C(\varkappa)$ and $n < \omega$, let $\{a(n, \xi, m)\colon m \in \omega\}$ be a listing of $\{b(n, \alpha)\colon \alpha \in A_\xi\}$. Now if σ is a maximal cellular family from the π-base, let $B = \{\alpha\colon$ for some $n, b(n, \alpha) \in \sigma\}$. By the choice of $\{A_\xi\colon \xi < C(\varkappa)\}$, there is a $\xi < C(\varkappa)$ so that $B \subset A_\xi$. Clearly $\sigma \subset \{a(n, \xi, m)\colon n, m \in \omega\}$.

3.5. COROLLARY. *For each $n \in \omega$ for which $\omega_n < \varkappa_0$, any nonpseudocompact ccc space of π-weight ω_n has remote points.*

3.6. THEOREM. *If there are no measurable cardinals then each nonpseudocompact ccc space with π-weight less than $\varkappa_0$ has remote points.*

Proof. The assumption we really require is Jensen's Covering Axiom: Whenever A is an uncountable set of ordinals then there is a $B \in L$ (Godel's constructible universe) so that $A \subset B$ and $|A| = |B|$. Jensen has shown that if there are no measurable cardinals then the above condition holds [KaMa]. It is very easy to show that in this case, $C(\varkappa) = \varkappa$ for $\varkappa$ regular and $C(\varkappa) \leqslant \varkappa^+$ for all $\varkappa$. Also, since λ_u is regular for all $u \in \omega^*$, $\varkappa_0$ cannot be the successor of a singular. Therefore if X is a space with $\pi w(X) < \varkappa_0$ then $C(\pi w(X)) < \varkappa_0$. The result now follows from 3.4.

We have one final little result on the function $C(.)$ which seems interesting.

3.7. PROPOSITION. *If $M \subset [\varkappa]^\omega$, for a cardinal $\varkappa$, where for each $B \in [\varkappa]^\omega$ there is an $M \in \mathscr{M}$ with $|B \cap M| = \omega$ then $C(\varkappa) \leqslant \mathscr{M}$.*

Proof. Let $\{a_\alpha\colon \alpha \in \varkappa\}$ be a listing of the finite subsets of $\varkappa$. For each $M \in \mathscr{M}$, let $A_M = \bigcup \{a_\alpha\colon \alpha \in M\}$. Let $B \in [\varkappa]^\omega$ be arbitrary. Choose recursively $\alpha_n \in \varkappa$ so that $a_{\alpha_n} \subset a_{\alpha_{n+1}}$ and $B = \bigcup \{a_{\alpha_n}\colon n < \omega\}$. Now there is an $M \in \mathscr{M}$ with $|\{\alpha_n\colon n \in \omega\} \cap M| = \omega$ and therefore $B \subset \bigcup \{a_{\alpha_n}\colon \alpha_n \in M\} \subset A_M$. Hence $C(\varkappa) \leqslant |M|$.

4. A space that is not nice.. In [D4] we constructed a topological sum of ccc spaces with π-weight ω_2 for which it is consistent that it has no nice remote filters. In this section we modify the space (and the model) so that it now has π-weight ω_1 and still no nice remote filters. We will use some basic foreing techniques and refer the reader to Kunen's book [K] for background and undefined notions. We will also be quite sketchy in our presentation and refer the reader to [D4] for more detail.

Let M be our model of set theory and let $P = \{p\colon p$ is a function, $\mathrm{dom}(p) \subset \omega_2 \times \omega$, $\mathrm{range}(p) \subset \omega\}$. Let G be a generic filter of P. For each $\alpha < \omega_2$ let $f_\alpha \in {}^\omega\omega$ be the function defined by $f_\alpha(n) = k$ where $p(\alpha, n) = k$ for some $p \in G$ for $n < \omega$. We work in $M[G]$ for a while.

Let $S = \{s\colon$ for some $n < \omega$, s is a function from n to $\omega\}$. For $s \in S$, let $d(s)$ be the domain of s. For $\alpha \in \omega_1$ and $s \in S$, let $U(s, \alpha) = \{t \in S\colon s \subset t$ and for each n, $d(s) \leqslant n < d(t)$, $t(n) > f_\alpha(n)\}$. The collection $\{U(s, \alpha)\colon s \in S, \alpha < \omega_1\}$ forms a subbase for a 0-dimensional topology on S. Let $Y = \beta S$ and $X = \omega \times Y$. For $n < \omega$, $s \in S$ and $\alpha \in \omega_1$, let $a(n, s, \alpha) = \{n\} \times \mathrm{cl}_{\beta S} U(s, \alpha)$; each $a(n, s, \alpha)$ is clopen in X.

FACT 4. *For $n < \omega$, $s, t \in S$ and $\alpha, \gamma < \omega_1$, $a(n, s, \alpha) \cap a(n, t, \gamma) \neq 0$ iff $s \subset t$ and $t(k) > f_\alpha(k)$ for $d(s) \leqslant k < d(t)$ or $t \subset s$ and $s(k) > f_\gamma(k)$ for $d(t) \leqslant k < d(s)$.*

We will show that, for $X_n = \{n\} \times Y$, $\sum X_n$ does not have a nice remote filter. We now work in M and assume that, where appropriate, we are using nice names of sets in $M[G]$.

Let $\mathscr{F}$ be a remote filter on X. Since Y is compact it is easy to show that the situation described in 1.1(2) is equivalent to $\mathscr{F}$ being remote. That is, for each maximal cellular family σ, there are finite subset $\sigma_n \subset \sigma$ $(n<\omega)$ with $\bigcup_n \bigcup \sigma_n \in \mathscr{F}$. Now, for $\alpha<\omega_1$, $\{a(n, s, \alpha)\colon d(s)>f_\alpha(n)\}$ is easily seen to contain a maximal cellular family σ_α. Let $\sigma_\alpha(n)$ be fixed so that $\bigcup_n \bigcup \sigma_\alpha(n) \in \mathscr{F}$ and define $h_\alpha \in {}^\omega\omega$ by $h_\alpha(n) = \max\{d(s)\colon a(n, s, \alpha) \in \sigma_\alpha(n)\}$. For each $\alpha<\omega_1$, $n<\omega$, there is a maximal antichain $A_{\alpha,n} \subset P$ so that p decides $h_\alpha(n)$ for $p \in A_{\alpha,n}$. Since P is ccc, each $A_{\alpha,n}$ is countable. Choose $\lambda<\omega_2$ so that $\lambda \times \omega \cap \operatorname{dom}(p) = 0$ for $p \in \bigcup\{A_{\alpha,n}\colon \alpha<\omega_1, n<\omega\}$. Let σ be a maximal cellular family contained in $\{a(n, s, 0)\colon d(s)>f_\lambda(n)\}$. As above there are finite sets $\sigma(n) \subset \sigma$ $(n<\omega)$ with $\bigcup_n \bigcup \sigma(n) \in \mathscr{F}$. For $n<\omega$, let $A_n \subset P$ be a maximal antichain so that for each $p \in A_n$ $(n<\omega)$ completely decides $\sigma(n)$. Let $\alpha<\omega_1$, with $\{\alpha\} \times \omega \cap \operatorname{dom}(p) = \emptyset$ for $p \in \bigcup A_n$.

Finally, suppose that $p \in P$ and $p \Vdash \{n<\omega\colon \bigcup\sigma_\alpha(n) \cap \bigcup\sigma(n) = \emptyset\}$ is finite; we shall derive a contradiction thus showing that X has no nice remote filters in $M[G]$. Let $j<\omega$ and $p_1 \supset p$ so that $p_1 \Vdash$ for $n>j$, $\bigcup \sigma_\alpha(n) \cap \bigcup \sigma(n) \neq \emptyset$. Since p_1 is finite we may choose $n>j$ so that $\omega_2 \times \{n\} \cap \operatorname{dom}(p_1) = \emptyset$. Choose $q_1 \in A_{\alpha,n}$ so that $p_1 \cup q_1 \in P$ and let $k<\omega$ be large enough so that $(\alpha, m) \in \operatorname{dom} q_1$ implies $m<k$ and $q_1 \Vdash h_\alpha(n)<k$ (i.e. we can extend $p_1 \cup q_1$ so as to make $f_\alpha(k)$ as large as we like). Choose $q_2 \in A_n$ and $m<\omega$ so that $p_1 \cup q_1 \cup q_2 \in p$ and $q_2 \Vdash f_\lambda(n)>k$ and $s(k)<m$ for each $s \in \sigma(n)$; (we may do this since $(\lambda_1, n) \in \operatorname{dom}(p_1 \cup q_1)$). Finally we may choose $p_2 \supset p_1 \cup q_1 \cup q_2$ so that $p_2 \Vdash f_\alpha(k)>m$. Therefore $p_2 \Vdash \exists n >j\, \exists k\, (h_\alpha(n)<k<f_\lambda(n)$ and $s(k)<f_\alpha(k)$ for $s \in \sigma(n))$. However, since $q_1 \subset p_2$, $p_2 \Vdash d(t) <h_\alpha(n)$ for $t \in \sigma_\alpha(n)$ and so by Fact 4, $p_2 \Vdash \exists n>j(\bigcup \sigma_\alpha(n) \cap \bigcup \sigma(n) = \emptyset)$. This is our desired contradiction.

This is in fact a very interesting situation. We have just shown that, for each remote filter $\mathscr{F}$, there is an $F \in \mathscr{F}$ with $|\{n<\omega\colon F \cap X_n = \emptyset\}| = \omega$. Therefore $\omega^* \neq \bigcap \{\mathrm{cl}_{\beta\omega}\{n<\omega\colon F \cap X_n \neq \emptyset\}\colon F \in \mathscr{F}\}$. On the other hand, since $\pi w(X) = \omega_1$, by 2.4, we have for each $u \in \omega^*$ a remote filter $\mathscr{F}_u$ on X with

$$\{\{n<\omega\colon F \cap X_n \neq \emptyset\}\colon F \in \mathscr{F}_n\} = u\,.$$

5. Questions.

(1) Do all ccc nonpseudocompact spaces with weight at most c have remote points?

(2) Do all ccc nonpseudocompact spaces have remote points?

(3) Is there a compact space X which is nowhere ccc (i.e. no non-empty open subset of X is ccc) and yet $\omega \times X$ has remote points in ZFC?

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