The Dunford–Pettis property for the ball-algebras, the polydisc-algebras and the Sobolev spaces

by

J. BOURGAIN (Brussels)

Abstract. Let X be one of the spaces considered in the title. It is proved that then X^* has Dunford–Pettis property.

0. Introduction. Let us recall the definition. A Banach space X is said to have *Dunford–Pettis property* (DPP) provided $\lim \langle x_n, x_n^* \rangle = 0$ whenever (x_n) is a weakly null sequence in X and (x_n^*) a weakly null sequence in X^*. The best known examples of spaces having this property are the $C(K)$ and $L^1(\mu)$-spaces [5]. It was investigated for spaces of analytic functions by several people, was established for the disc algebra A in [3] and for H^∞ in [1]. The reader will find further details on DPP and related properties in the survey work [4].

Our interest goes here to spaces of several variable functions, i.e. the ball-algebras $A(B_d)$, the polydisc algebras $A(D^d)$ and spaces of smooth functions $C^{(k)}(\Pi^d)$. Precise definitions will be stated later on. We will prove that the dual of each of these spaces (and hence the space itself) has DPP. This will be a consequence of

THEOREM 1. *Let X be one of the spaces $A(B_d)$, $A(D^d)$ or $C^{(k)}(\Pi^d)$. Then any bounded sequence (x_n^*) in X^* either tends uniformly to zero on weakly compact subsets of X^{**} or does not tend uniformly to zero on a c_0-sequence in X.*

This result answers questions considered by A. Grothendieck in [5], by A. Pełczyński in [11] (Section 8), [12] (exp. 2) and by J. Diestel in [4].

The proof of Th. 1, which is rather soft, is based on sequence arguments and at this point it is not clear how they can be localized in order to show DPP of the bidual spaces.

1. A construction of c_0-sequences. In this section we give the key lemma to derive Th. 1, which is a simple procedure to obtain c_0-sequences. Let G be a compact topological space, E a finite-dimensional Hilbert space with norm $|\ | = |\ |_E$ and $C_E(G)$ the space of continuous E-valued func-

tions on G (a vector-valued setting is only required to deal with the Sobolev spaces). We consider a subspace X of $C_E(G)$ and denote by $i : X \to \to C_E(G)$ the injection. In the sequel "dist" means always norm-distance.

Notice that for $\varphi \in C(G)$, multiplication by φ defines an operator φ on $C_E(G)$. We will use the same notation for the adjoints.

PROPOSITION 2. *Assume (x_n^*) to be a bounded sequence in X^* and $\delta > 0$ such that the following property holds:*

(∗) *For each $\varphi \in C(G)$ and each $\varepsilon > 0$, there is a weakly null sequence (x_n^{**}) in the unit ball of X^{**} such that*

(i) $\varlimsup_n |\langle x_n^*, x_n^{**} \rangle| > \delta$,

(ii) $\operatorname{dist}(\varphi \cdot i^{**}(x_n^{**}), X^{**}) < \varepsilon$ *for each n.*

Then there is a sequence (x_k) in X, equivalent to the usual c_0-basis such that $\sup_n |\langle x_n^*, x_k \rangle| > \delta/2$ for all k.

Proof. Consider for each n a norm-preserving extension $\mu_n \in M_E(G)$ of x_n^*. Let (ε_k) be a sequence of positive numbers such that $\sum \varepsilon_k < \delta/10$.

By induction on k, we will construct a sequence of integers (n_k), a decreasing sequence (N_k) of infinite subsets of N, a sequence (y_k) in X and a sequence (ψ_k) in $C_{[0,1]}(G)$.

Step 1. Since (x_m^{**}) is weakly null, there is a finite convex combination $(\lambda_m)_{m \in D}$ such that

$$\left\| \sum_D \sigma_m \lambda_m x_m^{**} \right\| < \tfrac{1}{2} (\dim E)^{-1} \varepsilon_1^2 = \varepsilon_1'$$

for all choices of signs $\sigma_m = \pm 1$.

So, by local reflexivity, there are elements $(x_m)_{m \in D}$ in the unit ball of X such that

(a) $\langle x_m, x_m^* \rangle = \langle x_m^{**}, x_m^* \rangle$ for $m \in D$,

(b) $\left\| \sum_D \sigma_m \lambda_m x_m \right\| < \varepsilon_1'$ for all $\sigma_m = \pm 1$.

Now (b) implies that $\left\| \sum \lambda_m |x_m|_E \right\|_\infty < \varepsilon_1^2$ and hence $\left\| \sum \lambda_m \tau_m \right\|_\infty < \varepsilon_1$ if for each $m \in D$ one chooses $\tau_m \in C_{[0,1]}(G)$ such that

(c) $\tau_m = 1$ on $[|x_m| > 2\varepsilon_1]$ and $\tau_m = 0$ on $[|x_m| \leqslant \varepsilon_1]$.

It follows that for each n,

$$\sum \lambda_m \| \tau_m \mu_n \| < \varepsilon_1,$$

which allows us to fix some $n_1 \in D$ and an infinite set $N_1 \subset N$ so that

$$\| \psi_1 \mu_n \| < \varepsilon_1 \quad \text{for} \quad n \in N_1,$$

where $\psi_1 = \tau_{n_1}$. Define $y_1 = x_{n_1}$. Thus $|\langle y_1, x_{n_1}^* \rangle| > \delta$ and $\psi_1 = 1$ if $|y_1| > 2\varepsilon_1$.

Inductive step. Assume the construction done up to step k. Define $\varphi = (1 - \psi_1) \ldots (1 - \psi_k)$, $\varepsilon = \varepsilon_{k+1}$ and use (∗) to obtain an appropriate sequence (x_n^{**}) in X^{**}. Thus there are elements $z_n^{**} \in X^{**}$ such that $\| \varphi \cdot x_n^{**} - z_n^{**} \| < \varepsilon$ for each n. Consider again a finite subset D of N_k such that

$$\left\| \sum_D \sigma_m \lambda_m x_m^{**} \right\| < \tfrac{1}{2} (\dim E)^{-1} \varepsilon_{k+1}^2 = \varepsilon_{k+1}' \qquad (\sigma_m = \pm 1)$$

for some convex combination $(\lambda_m)_{m \in D}$.

By a local reflexivity argument, one can then obtain $(x_m)_{m \in D}$ and $(z_m)_{m \in D}$ in X satisfying

(a) $\|x_m\| \leqslant 1$,

(b) $\langle x_m, x_m^* \rangle = \langle x_m^{**}, x_m^* \rangle$,

(c) $\left\| \sum_D \sigma_m \lambda_m x_m \right\| < \varepsilon_{k+1}'$ $(\sigma_m = \pm 1)$,

(d) $\| \varphi \cdot x_m - z_m \| < \varepsilon_{k+1}$.

Again $\left\| \sum_D \lambda_m |x_m| \right\|_\infty < \varepsilon_{k+1}^2$ and hence $\left\| \sum_D \lambda_m \tau_m \right\|_\infty < \varepsilon_{k+1}$ if for each $m \in D$ we fix some $\tau_m \in C_{[0,1]}(G)$ so that

(e) $\tau_m = 1$ on $[|x_m| > 2\varepsilon_{k+1}]$ and $\tau_m = 0$ on $[|x_m| \leqslant \varepsilon_{k+1}]$.

For each $n \in N_k$

$$\sum_D \lambda_m \| \tau_m \cdot \mu_n \| < \varepsilon_{k+1},$$

which allows us to fix some $n_{k+1} \in D$ for which there exists an infinite subset N_{k+1} of N_k with

$$\| \psi_{k+1} \mu_n \| < \varepsilon_{k+1} \quad \text{for} \quad n \in N_{k+1},$$

where $\psi_{k+1} = \tau_{n_{k+1}}$. Define $y_{k+1} = X_{n_{k+1}}$. Then $|\langle y_{k+1}, x_{n_{k+1}}^* \rangle| > \delta$, $\psi_{k+1} = 1$ on $|y_{k+1}| > 2\varepsilon_{n_{k+1}}$ and $\operatorname{dist}((1 - \psi_1) \ldots (1 - \psi_k) y_{k+1}, X) < \varepsilon_{k+1}$. This completes the construction.

Consider a sequence (x_k) in X taking:

$x_1 = y_1$,

x_{k+1} such that $\|(1 - \psi_1) \ldots (1 - \psi_k) y_{k+1} - x_{k+1}\| < \varepsilon_{k+1}$.

Then

$$\left\| \sum |x_k| \right\|_\infty \leqslant \left\| |y_1| + \sum_{k>1} (1 - \psi_1) \ldots (1 - \psi_{k-1}) |y_k| \right\|_\infty + \sum_{k>1} \varepsilon_k.$$

For fixed $t \in G$, suppose $|y_k(t)| > 2\varepsilon_k$ for some k and let k_l be the smallest

integer with this property. By construction

$$\Big[|y_1| + \sum_{k>1} (1-\psi_1)\dots(1-\psi_{k-1})\,|y_k|\Big](t) \leqslant 2\sum_{k>k_t} \varepsilon_k + \|y_{k_t}\|_\infty.$$

This shows that

$$\Big\|\sum |x_k|\Big\|_\infty \leqslant 1 + 3\sum \varepsilon_k < 2.$$

Also, by construction, since $n_k \in N_j$ for $j < k$,

$$|\langle x_{n_k}^*, x_k \rangle| \geqslant |\langle \mu_{n_k}, (1-\psi_1)\dots(1-\psi_{k-1})\,y_k \rangle| - \varepsilon_k\|\mu_{n_k}\|$$

$$\geqslant |\langle x_{n_k}^*, y_k \rangle| - \sum_{j<k}\|\psi_j \mu_{n_k}\| - \varepsilon_k$$

$$> \delta - \sum \varepsilon_j$$

and this clearly proves Prop. 2.

2. The ball-algebras. Let $d \geqslant 2$ be some integer. Denote by $\langle .,. \rangle$ the usual scalar product on $\mathbf{C}^d$; the unit ball $B_d = \{\zeta \in \mathbf{C}^d; \langle \zeta, \zeta \rangle \leqslant 1\}$ and the unit sphere $S_d = \{\zeta \in \mathbf{C}^d; \langle \zeta, \zeta \rangle = 1\}$. On S_d we have the natural rotation-invariant probability measure σ. Define $A = A(B_d)$ as the space of functions which are continuous on B_d and analytic in the interior of B_d. Equipped with sup-norm, $A(B_d)$ embeds in $C(S_d)$ restricting the functions to the sphere. For $f \in A$, the Cauchy formula is given by

$$f(z) = \int_S \frac{f(\zeta)\,d\sigma(\zeta)}{(1-\langle z, \zeta \rangle)^n} \quad \text{for} \quad |z| < 1.$$

The reader will find the basic theory in [14]. We will use the following elementary fact

LEMMA 3. *Let φ be a polynomial in $z_1, \dots, z_d, \bar{z}_1, \dots, \bar{z}_d$. For $|z| < 1$, consider the function*

$$\Phi_z(\zeta) = \frac{\varphi(z) - \varphi(\zeta)}{(1-\langle z, \zeta \rangle)^d}.$$

Then $\{\Phi_z; |z| < 1\}$ is relatively compact in $L^1(S)$.

Proof. We show that $\int\limits_{\|\zeta - z\| < \varepsilon} |\Phi_z(\zeta)|\,\sigma(d\zeta) \to 0$ uniformly in z for $\varepsilon \to 0$.
Clearly

$$|\varphi(z) - \varphi(\zeta)| \leqslant C\|z - \zeta\| = \sqrt{2}\,C(1 - \mathrm{Re}\,\langle z, \zeta \rangle)^{1/2} \leqslant \sqrt{2}\,C\,|1 - \langle z, \zeta \rangle|^{1/2}$$

and

$$\int\limits_{\|\zeta - z\| < \varepsilon} |1 - \langle z, \zeta \rangle|^{-d+1/2}\,d\sigma(\zeta) = O(\varepsilon).$$

LEMMA 4. *If (x_n^{**}) is weakly null in A^{**} and $\varphi \in C(S)$, then*

$$\lim_{n\to\infty} \mathrm{dist}\,(\varphi \cdot x_n^{**}, A^{**}) = 0.$$

Proof. By density, φ can be assumed as in Lemma 3. From Lemma 3, it follows that given $\varepsilon > 0$,

$$\sup_{|z|<1} |\langle \bar{\Phi}_z, i^{**}(x_n^{**}) \rangle| < \varepsilon$$

for n large enough.

Fix such n. Again by Lemma 3, there is a net (x_α) in A such that

(a) $\|x_\alpha\| \leqslant \|x_n^{**}\|$,

(b) $x_n^{**} = \lim_\alpha x_\alpha\,\sigma(A^{**}, A^*)$,

(c) $\sup_{|z|<1} |\langle \bar{\Phi}_z, x_\alpha \rangle| < \varepsilon$ for each α.

Since $\varphi \cdot x_n^{**} = \lim_\alpha \varphi \cdot x_\alpha\,\sigma(C^{**}, C^*)$, it will suffice to prove $\mathrm{dist}\,(\varphi \cdot x_\alpha, A) \leqslant \varepsilon$ for each α. Define

$$x_\alpha'(z) = \int_S \frac{\varphi(\zeta)x_\alpha(\zeta)}{(1-\langle z, \zeta \rangle)^d}\,d\sigma(\zeta) \quad \text{for} \quad |z| < 1.$$

Since $x_\alpha \in A$, we find for $z \in S$ (a.e.)

$$|\varphi(z)x_\alpha(z) - x_\alpha'(z)| = \overline{\lim_{r\to1}}\Big|\int_S \frac{\varphi(r\cdot z) - \varphi(\zeta)}{(1-\langle rz, \zeta \rangle)^d}\,x_\alpha(\zeta)\,d\sigma(\zeta)\Big| \leqslant \overline{\lim_{r\to1}} |\langle x_\alpha, \bar{\Phi}_{r\cdot z} \rangle|$$

and thus, by (c), $\|\varphi \cdot x_\alpha - x_\alpha'\|_\infty < \varepsilon$. Thus also $x_\alpha' \in H^\infty(B_d)$ and therefore $\mathrm{dist}\,(\varphi \cdot x_\alpha, A) \leqslant \varepsilon$, proving the lemma.

Proof of Th. 1 for $X = A(B_d)$. Assume (x_n^*) to be a bounded sequence in X^* and (x_n^{**}) a weakly null sequence in X^{**} such that $\overline{\lim_n} |\langle x_n^*, x_n^{**} \rangle| > 0$. Lemma 4 shows that $(*)$ is fulfilled and it remains to apply Prop. 2.

3. The polydisc-algebras. $D = \{z \in \mathbf{C}; |z| < 1\}$ is the open unit disc and D^d the subset of $\mathbf{C}^d$ obtained by d-fold product. $A(D^d)$ is the space of functions which are continuous on $\bar{D}^d$ and analytic on D^d. This space equipped with supremum norm identifies again with a closed subspace of $C(\Pi^d)$, Π = circle. Our reference here is [13].

For $\Lambda \subset \mathbf{Z}^d$, let C_Λ be the subspace of the functions $f \in C(\Pi^d)$ with spectrum $\mathrm{Spec}(f)$ contained in Λ. With this notation $A(D^d) = C_{\mathbf{Z}_+ \times \ldots \times \mathbf{Z}_+}$.

If J is a non-empty subset of $\{1, 2, \ldots, d\}$ and $M = (m_j)_{j \in J}$ a multi-index, the projection $P_{J,M}$ on $C(\Pi^d)$ is defined by

$$P_{J,M}(f)(\Theta) = \left\{ \int f e^{-\sum_J m_j \theta_j} \Pi_J d\theta^j \right\} \cdot e^{\sum_J m_j \theta_j}.$$

The range of the restriction of $P_{J,M}$ to $A(D^d)$ can obviously be identified with $A(D^{d-|J|})$.

Fix a sequence of functions $(K_r)_{r=1,2,\ldots}$ on Π such that $\|K_r\|_1 \leqslant C$ and $\mathrm{Spec}(K_r)$ is finite,

$$\hat{K}_r(m) = 1 \quad \text{for} \quad |m| \leqslant r.$$

We consider the measure

$$\varrho_r = \bigotimes_{j=1}^{d} [\delta^{(j)} - K_r^{(j)}],$$

where δ stands for Dirac measure and (j) means jth variable. Let P_r denote the ϱ_r-convolution operator. Let us summarize some straightforward facts.

LEMMA 5. (i) $\|P_r\| \leqslant C(d)$.

(ii) $I - P_r$ can be written as linear combination of $P_{J,M}$ projections.

(iii) If $(m_1, \ldots, m_d) \in \mathrm{Spec}\, \varrho_r$, then $|m_j| > r$ for $1 \leqslant j \leqslant d$.

To prove Th. 1 for $X = A(D^d)$, we proceed by induction on d. The case $d = 1$ is known. Assume the result is valid for $A(D^{d-1})$ $(d > 1)$. Let (x_n^*) be a bounded sequence in X^* and (x_n^{**}) a weakly null sequence in X^{**}. From a preceding observation, if for some $P_{J,M}$

$$\overline{\lim_n} |\langle P_{J,M}^*(x_n^*), x_n^{**} \rangle| > 0,$$

the induction hypothesis allows us to conclude. Thus, using Lemma 5, we can assume for some $\delta > 0$

$$\overline{\lim_n} |\langle x_n^*, P_r^{**}(x_n^{**}) \rangle| \geqslant \delta$$

for each r.

Notice that by construction if $x \in A$ and φ is a trigonometric polynomial on Π^d with $\mathrm{Spec}(\varphi) \subset \{-r, -r+1, \ldots, r\}_d^d$, then $\varphi \cdot P_r(x) \in A$. It is therefore clear that

$$\lim_{r \to \infty} \mathrm{dist}\left(\varphi \cdot P_r^{**}(x^{**}), X^{**}\right) = 0$$

uniformly on the unit ball of X^{**}, for given $\varphi \in C(\Pi^d)$. Thus (x_n^*) satisfies $(*)$ of Prop. 2.

4. Spaces of smooth functions. Let U be a d-dimensional compact manifold. Denote by $C^{(k)}(U)$ the space of complex-valued functions on U which are continuous with all derivatives of order $\leqslant k$. From a result of [10], the space $C^{(k)}(U)$ is linearly isomorphic to $C^{(k)}(\Pi^d)$. We represent latter space as a translation invariant subspace of $C(\Pi^d, E)$ for an appropriate finite-dimensional Hilbert space E, identifying $f \in C^{(k)}(\Pi^d)$ with the element

$$(D^J f) |J| \leqslant k, \quad \text{where} \quad D^J = \frac{\partial^{|J|}}{\partial_1^{j_1} \ldots \partial_d^{j_d}},$$

$J = (j_1, \ldots, j_d)$ and $|J| = j_1 + \ldots + j_d$. (Details can be found in [8].)

We will use the following fact:

LEMMA 6. *There is a bounded sequence (P_r) of convolution operators on Π^d with finite spectrum such that*

$$\|D^J(f - P_r f)\|_\infty \leqslant 1/r \|f\|_{C^{(k)}} \quad \text{for} \quad |J| < k.$$

Proof. Consider for instance the Jackson kernel K_r on Π (see [15]) for which $\int_\Pi K_r(\theta)\theta\, d\theta \sim 1/r$. Let P_r be the convolution operator by the d-fold kernel $K_r^{(d)} = K_r \otimes \ldots \otimes K_r$. For $f \in C^{(1)}(\Pi^d)$, we find

$$|f - P_r f| \leqslant \int_{\Pi^d} |f_\Psi - f| K_r^{(d)}(\Psi)\, d\Psi \leqslant C(d) \|\nabla f\|_\infty \int_{\Pi^d} (|\psi_1| + \ldots + |\psi_d|) \hat{K}_r^{(d)}(\Psi)$$

and hence

$$\|f - P_r f\|_\infty \leqslant C'(d)\, 1/r \|\nabla f\|_\infty$$

which clearly proves the lemma.

Let X be the representation of $C^{(k)}(\Pi^d)$ in $C(\Pi^d, E)$. To obtain Th. 1 from Prop. 2, we show

LEMMA 7. *Assume (x_n^{**}) to be a weakly null sequence in X^{**} and φ a trigonometric polynomial on Π^d. Then $\lim_{n \to \infty} \mathrm{dist}(\varphi \cdot x_n^{**}, X^{**}) = 0$.*

Proof. Since the P_r-operators are of finite rank, $\lim_{n \to \infty} \|P_r^{**}(x_n^{**})\| = 0$ and hence $\|P_r^{**}(x_n^{**})\| < \tau$ for n sufficiently large. For such n, consider a sequence (x_α) in X satisfying

(a) $\|x_\alpha\| \leqslant \|x_n^{**}\|$,

(b) $x_n^{**} = \lim_\alpha x_\alpha \sigma(X^{**}, X^*)$,

(c) $\|P_r(x_\alpha)\| < \tau$ for each α.

Assume x_a to be the representation of $f_a \in C^{(k)}(\Pi^d)$. Then

$$\operatorname{dist}(\varphi \cdot x_a, X) \leqslant \operatorname{const} \max_{|J| \leqslant k} \|\varphi \cdot D^J f_a - D^J(\varphi f_a)\| \leqslant \operatorname{const} \|\varphi\|_{C^{(k)}} \|f_a\|_{C^{(k-1)}}.$$

From (c) and Lemma 6

$$\|f_a\|_{C^{(k-1)}} \leqslant \|P_r x_a\| + \|f_a - P_r f_a\|_{C^{(k-1)}}$$
$$< \tau + \operatorname{const} 1/r \|f_a\|_{C^{(k)}} < \tau + \operatorname{const} 1/r \|x_n^{**}\|.$$

Since $\varphi \cdot x_n^{**} = \lim_a \varphi \cdot x_a \sigma(C_E^{**}, C_E^*)$, it follows that

$$\operatorname{dist}(\varphi \cdot x_n^{**}, X^{**}) \leqslant \operatorname{const} \|\varphi\|_{C^{(k)}}(\tau + 1/r).$$

This proves the lemma.

5. Further results and remarks.

1. As a formal consequence of Th. 1, we can state

COROLLARY 8. *Any l^1-sequence in the dual of one of the spaces $A(B_d)$, $A(D^d)$ or $C^{(k)}(\Pi^d)$ has a subsequence with ω^*-complemented closed linear span.*

2. Closely related to the results presented here is the problem whether or not the duals of these spaces are weakly complete. It can be shown (by adaptation of G. M. Henkin's method [6], [7]) that the dual of $C^{(k)}(\Pi^d)$ has the form $S \oplus L$, where S is some separable space and L is an $L^1(\mu)$-space (details will appear elsewhere). The reader will find a description of $A(D^2)^*$ in [2].

References

[1] J. Bourgain, *New Banach space properties of the disc algebra and H^∞*, to appear in Acta Math.
[2] —, *Non-isomorphism of the polydisc algebras in dimension two and three*, preprint.
[3] J. Chaumat, *Une généralisation d'un théorème de Dunford–Pettis*, Université de Paris XI, Orsay 1974.
[4] J. Diestel, *A survey of results related to the Dunford–Pettis property*, Contemporary Math. 2 (1980), 15–60.
[5] A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. 5 (1953), 129–173.
[6] G. M. Henkin, *Non isomorphism of some spaces of functions of different numbers of variables*, Funkt. Analiz. i Priložen. 4 (1967), 57–68.
[7] —, *Banach spaces of analytic functions on the ball and on the bicilinder are not isomorphic*, ibid. 2, 4 (1968), 82–91.
[8] S. Kwapień, A. Pełczyński, *Absolutely summing operators and translation invariant spaces of functions on compact abelian groups*, Math. Nachr. 94 (1980), 303–340.
[9] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces*, Springer Lecture Notes in Math. 338, 1973.

[10] B. S. Mitjagin, *The homotopy structure of the linear group of a Banach space*, Uspehi Mat. Nauk 25, 5 (1970), 63–106.
[11] A. Pełczyński, *Banach spaces of analytic functions and absolutely summing operators*, CBMS Regional Conference Series in Math. AMS 30 (1976).
[12] —, Publ. Math. de l'Univ. Pierre et Marie Curie No 29, Séminaire d'Initiation à l'Analyse 1978/79.
[13] W. Rudin, *Function Theory in Polydiscs*, Math. Lecture Notes Series, Benjamin, 1969.
[14] — *Function Theory in the Unit Ball of C^n*, Grundlehren der math. Wissenschaften 241, Springer, 1980.
[15] A. Zygmund, *Trigonometric Series*, Cambridge University Press, 1959.

DEPARTMENT OF MATHEMATICS
VRIJE UNIVERSITEIT BRUSSEL
PLEINLAAN 2-F7, 1050 BRUSSELS