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H_p-spaces, $p \leqslant 1$, and spline systems*

by

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Abstract. Using spline systems we construct unconditional bases in spaces $H_p(D)$, $0 < p \leqslant 1$. This is used to give a direct isomorphism between $H_p(D)$ and martingale H_p spaces. We also show that our systems are bases in Bergman spaces A_a^q, $q \leqslant 1$, and we characterise those spaces in terms of coefficients (this gives an explicite isomorphism of A_a^q with l_q). Applications to complemented subspaces of H_p, $p < 1$ and to properties of spline systems in L_p, $1 < p < \infty$, are also given.

This paper presents an application of orthonormal spline systems and some related systems of splines to natural H_p spaces. The main emphasis is put on $H_p(D)$, the classical Hardy space of analytic functions on the unit disc in the complex plane. Our methods, however, are mainly those from real variable H_p-theory. We use atomic decompositions of H_p-functions, as developed in [13] and [31], to prove the continuity of natural operators associated with expansions with respect to spline systems. Our basic result (Theorem 2) gives a construction of an unconditional basis in $H_p(D)$, $p \leqslant 1$. Moreover, the bases we construct can be used to give explicit isomorphisms between various H_p-spaces; most notably we show that $H_p(D)$ is isomorphic to the H_p space of dyadic martingales. This extends to $p < 1$ results of [24], [4] and [32]. Some of those results have been obtained with different proofs by Sjölin and Stromberg [30].

The above-mentioned isomorphisms provide equivalence between some chapters of martingale theory (cf. [21]), constructive function theory (cf. [7]) and H_p-spaces. This equivalence allows us to give new proofs for some results of Ciesielski's [7] and [8] on spline systems in L_p, $p > 1$ (Theorem 11 and Corollary 4).

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As an application of our results we give a characterisation of Bergman spaces A_a^q, $q\leqslant 1$ on the unit disc (Theorem 8). This characterisation is related to the one given in [12]. In particular we get that A_a^q is isomorphic, as a linear topological space, to l_q. The other application concerns complemented subspaces of $H_p(D)$. Answering the question from [22] we show that every infinite dimensional complemented subspace of $H_p(D)$, $p<1$, contains a smaller complemented subspace isomorphic to l_p.

Our notation is standard. For the general background in H_p-theory the reader may consult [13] and [31] and for elements of the theory of splines we suggest [28] and [7] and [9]. In order to make this paper more self-contained, rather lengthy preliminary sections on H_p-theory and spline systems are added.

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1. Preliminaries, spline systems. Our aim in this section is to construct some systems of splines on the unit circle T. We identify T with $[-1,1)$. If m is an integer, $m\geqslant -1$, and $\boldsymbol{V}$ is a partition of T into intervals $\boldsymbol{V}=\{I_1, I_2, \ldots, I_n\}$ then the spline of order m with respect to the partition $\boldsymbol{V}$ is any function f on T such that f is m-times continuously differentiable and for every $j=1,2,\ldots,n$, $f|I_j$ is a polynomial of degree at most $m+1$. The space of all splines of order m with respect to partition $\boldsymbol{V}$ will be denoted by $S^m(\boldsymbol{V})$.

In this paper we will consider only dyadic partitions. For $n=2^k+l$, $0\leqslant l<2^k$ we define the partition $\boldsymbol{V}_n$ of $[0,1]$ by

$$I_s=\left(\frac{s-1}{2^{k+1}}, \frac{s}{2^{k+1}}\right) \quad \text{if} \quad 1\leqslant s\leqslant 2l, \tag{1.1}$$

$$I_s=\left(\frac{s-l-1}{2^{k}}, \frac{s-l}{2^{k}}\right) \quad \text{if} \quad 2l< s\leqslant n. \tag{1.2}$$

Using this partition we define three partitions of T as follows

$$\begin{aligned} \boldsymbol{V}_n^1 &= \{I\subset T\colon I\in \boldsymbol{V}_n \text{ or } -I\in \boldsymbol{V}_n\},\\ \boldsymbol{V}_n^2 &= \{I\subset T\colon I\in \boldsymbol{V}_n \text{ or } -I\in \boldsymbol{V}_{n+1}\},\\ \boldsymbol{V}_n^3 &= \{I\subset T\colon I\in \boldsymbol{V}_{n+1} \text{ or } -I\in \boldsymbol{V}_n\}. \end{aligned}\tag{1.3}$$

We order intervals in $\boldsymbol{V}_n^1$ in such a way that I_s equals I_s of partition $\boldsymbol{V}_n$ if $s>0$ and I_s equals $-I_{|s-1|}$ of partition $\boldsymbol{V}_n$ if $s\leqslant 0$. Partitions $\boldsymbol{V}_n^2$ and $\boldsymbol{V}_n^3$ are ordered analogously. We define t_n as the point which is an endpoint of some interval from $\boldsymbol{V}_{n+1}$ but is not an endpoint of any interval from $\boldsymbol{V}_n$.

Given a partition $\boldsymbol{V}=\{I_1, I_2, \ldots, I_n\}$ of the circle T, with intervals I_j in $\boldsymbol{V}$ ordered consecutively and $n>m+2$ we define the basic splinse of order m with respect to partition $\boldsymbol{V}$, $(b_j^m)_{j=1}^n$ by the following two conditions:

(1.4) for each j the b_j^m is a non-zero spline of order m with respect to $\boldsymbol{V}$ and the support of b_j^m equals $I_j\cup I_{j+1}\cup\ldots\cup I_{j+m+1}$ (if $r>n$ then we interpret $I_r=I_{-n+r}$),

(1.5) for each j the $b_j^m\geqslant 0$ and $\sum_{j=1}^n b_j^m=1$.

Basic splines are investigated in detail in chapter 4 of [28]. In particular, it is shown there that basic splines exist. We also have that if the lengths of intervals in $\boldsymbol{V}$ are comparable (e.g. their ratios are between 1/2 and 2 as is the case for partitions $\boldsymbol{V}_n^1$, $\boldsymbol{V}_n^2$, and $\boldsymbol{V}_n^3$) then

$$\left(\int_T |b_j^m(t)|^2\right)^{1/2}\sim \frac{1}{\sqrt{n}}. \tag{1.6}$$

Moreover, in this case (cf. Th. 4.41 of [28]) the set of biorthogonal functionals λ_j on $S^m(\boldsymbol{V})$, i.e., functionals such that $\lambda_i(b_j^m)=\delta_{ij}$, $i,j=1,2,\ldots,n$, satisfies

$$|\lambda_j(f)|\leqslant c\sqrt{n}\left(\int_{\operatorname{supp} b_j^m}|f|^2\right)^{1/2} \quad \text{for every } f\in S^m(\boldsymbol{V}). \tag{1.7}$$

It is also true that every spline in $S^m(\boldsymbol{V})$ is a linear combination of basic splines.

LEMMA 1. *Let (b_j^m) denote all basic splines of order m with respect to partition $\boldsymbol{V}_n^1$ (or $\boldsymbol{V}_n^2$ or $\boldsymbol{V}_n^3$). There exists a constant K_m such that for all n and all (a_j)*

$$K_m^{-1}\left(n^{-1}\sum |a_j|^2\right)^{1/2}\leqslant\left(\int_T\Big|\sum a_j b_j^m\Big|^2\right)^{1/2}\leqslant K_m\left(n^{-1}\sum|a_j|^2\right)^{1/2}. \tag{1.8}$$

Proof. We have by (1.7) and (1.4)

$$\begin{aligned}\sum|a_j|^2 &\leqslant \sum_j\Big|\lambda_j\Big(\sum_k a_k b_k^m\Big)\Big|^2\\ &\leqslant C\sum_j n\int_{\operatorname{supp} b_j^m}\Big|\sum_k a_k b_k^m\Big|^2\leqslant Cn\int_T\Big|\sum a_j b_j^m\Big|^2.\end{aligned}$$

On the other hand,

$$\int\Big|\sum a_j b_j^m\Big|^2 = \sum_k \int_{I_k}\Big|\sum_{k-m-1}^{k} a_j b_j^m\Big|^2$$
$$\leqslant C\sum_k \int_{I_k}\sum_{k-m-1}^{k} |a_j|^2 |b_j^m|^2 \leqslant C\sum_j n^{-1}|a_j|^2.$$

The above estimates give the lemma.

Let h_n^m denote the element of $S^m(\boldsymbol{V}_n^3)$ which is orthogonal to $S^m(\boldsymbol{V}_n^1)$ and has $L_2(T)$ norm equal 1. This is well defined for $n\geqslant 1$. We put $h_0^m = \text{const}$.

LEMMA 2. *There are constants $C>0$ and q, $0<q<1$, independent of n such that*

$$|h_n^m(t)|\leqslant C\sqrt{n}\,q^{nd(t,t_n)}, \tag{1.9}$$

where $d(t,t_n)$ denotes the distance on T between t and t_n.

Proof. Let $(b_j^m)_{j=-n}^{n}$ denote basic splines with respect to the partition $\boldsymbol{V}_n^3$. Let us define the matrix $\mathfrak{A}$ by

$$a_{ij} = \langle b_i^m, b_j^m\rangle$$

($\langle\cdot,\cdot\rangle$ denotes the natural scalar product in $L_2(T)$). Clearly, we have $a_{ij}=0$ if $\operatorname{supp} b_i^m\cap\operatorname{supp} b_j^m=\emptyset$, and by (1.6) we have $|a_{ij}|\leqslant Cn^{-1}$. We infer from Lemma 1 that the matrix $\mathfrak{A}$ defines an isomorphism of the space l_2^n and $\|n\cdot\mathfrak{A}\|$ and $\|(n\mathfrak{A})^{-1}\|$ are bounded independently of n. The result of Domsta [15] or the periodic version of Lemma 2 of [14] gives that the entries b_{ij} of the matrix $\mathfrak{A}^{-1}$ satisfy

$$|b_{ij}|\leqslant Knq^{\varphi(i,j)} \tag{1.10}$$

for some constants $K>0$ and q, $0<q<1$. Symbol $\varphi(i,j)$ denotes the number of intervals from $\boldsymbol{V}_n^3$ between I_i and I_j. If we write

$$h_n^m = \sum_j a_j b_j^m \tag{1.11}$$

we have

$$\langle h_n^m, b_i^m\rangle = \sum_j a_j a_{ji} \quad\text{ so }\quad a_i = \sum_j b_{ji}\langle h_n^m, b_j^m\rangle.$$

Let us observe that if $t_n\notin\operatorname{supp} b_j^m$ then $b_j^m\in S^m(\boldsymbol{V}_n^1)$, so $\langle h_n^m, b_j^m\rangle = 0$. This observation, (1.6) and (1.10) gives

$$|a_i|\leqslant C\sqrt{n}\,q^{\varphi(i,r)}, \tag{1.12}$$

where r is defined by $t_n\in I_r\in\boldsymbol{V}_n^1$.

From (1.11) and (1.12) we get

$$|h_n^m(t)| = \Big|\sum_j a_j b_j^m(t)\Big|\leqslant \sum_{j:\,t\in\operatorname{supp} b_j^m}|a_j|\leqslant C\sqrt{n}\,q^{nd(t_n,t)}.$$

Remark 1. The above proof works only for $n>m+2$. For smaller n we clearly have the desired estimate for some constants.

COROLLARY 1. *If $h_n^{m,k}(t)$, $0\leqslant k\leqslant m+1$, denotes the k-th derivative of h_n^m, then*

$$|h_n^{m,k}(t)|\leqslant Cn^{k+1/2}q^{nd(t,t_n)} \tag{1.13}$$

or some C and q, $0<q<1$.

Proof. We use (1.11) and (1.12) and the expression for the derivative of the basic spline given in Theorem 4.16 of [28].

Let us now introduce two operators acting on functions on T.

$$\boldsymbol{D}f(t) = f'(t),$$
$$\boldsymbol{H}f(t) = \int_t^1 f(s)\,ds - \int_{-1}^{1}\int_t^1 f(s)\,ds\,dt.$$

The basic relation between those two operators is

$$\int_T \boldsymbol{D}f(t)\boldsymbol{H}h(t)\,dt = -\int_T f(t)h(t)\,dt.$$

Our next goal is to prove the analog of Corollary 1 for $\boldsymbol{H}^k h_n^m$.

LEMMA 3. *For $0\leqslant k\leqslant m+1$ we have*

$$|\boldsymbol{H}^k h_n^m(t)|\leqslant Cn^{-k+1/2}q^{nd(t,t_n)}$$

for some $C>0$ and q, $0<q<1$.

Proof. Let us start with the following claim:

$$\boldsymbol{H}^k h_n^m \text{ is orthogonal to } S^{m-k}(\boldsymbol{V}_n^1). \tag{1.14}$$

Let us take $\varphi\in S^{m-k}(\boldsymbol{V}_n^1)$. There exists $\Phi\in S^m(\boldsymbol{V}_n^1)$ such that $\boldsymbol{D}^k\Phi = \varphi - \int_T\varphi$. We have

$$\int_T \boldsymbol{H}^k h_n^m(t)\varphi(t)\,dt = \int_T \boldsymbol{H}^k h_n^m(t)\Big(\varphi(t)-\int_T\varphi\Big)\,dt + \int_T\varphi\cdot\int_T \boldsymbol{H}^k h_n^m$$
$$= \int_T \boldsymbol{H}^k h_n^m(t)\boldsymbol{D}^k\Phi(t)\,dt = (-1)^k\int_T h_n^m(t)\Phi(t)\,dt = 0.$$

Since by (1.14) $H^k h_n^m$ is orthogonal to all basic splines in $S^{m-k}(V_n^1)$, (1.5) implies that $H^k h_n^m$ has a zero in the union of every $m-k+1$ consecutive intervals from V_n^1. Let x_0 be such a zero for $H^k h_n^m$ "almost opposite" to t_n. Then

$$H^k h_n^m(t) = \int_t^{x_0} H^{k-1} h_n^m(s)\,ds.$$

Using this particular representation we can show by induction the desired estimate.

Using functions $h_n^m(t)$ we define the system of even splines on T by

$$(1.15)\qquad g_n^m(t) = \Big(\int |h_n^m(t)+h_n^m(-t)|^2 dt\Big)^{-1/2}\big(h_n^m(t)+h_n^m(-t)\big).$$

If $0 \leqslant |k| \leqslant m+1$ we define $g_n^{m,k}(t)$ as follows:

$$(1.16)\qquad g_n^{m,k}(t) = \begin{cases} D^k g_n^m(t) & \text{if } n>0 \text{ and } 0 \leqslant k \leqslant m+1,\\ (-1)^k H^{|k|} g_n^m(t) & \text{if } n>0 \text{ and } -m-1 \leqslant k \leqslant 0,\end{cases}$$

$$g_0^{m,k}(t) = \frac{1}{\sqrt{2}} \quad \text{if} \quad k \text{ is even.}$$

So $g_n^{m,k}(t)$ is indexed by $n = 0, 1, 2, \ldots$ if k is even and by $n = 1, 2, 3, \ldots$ if k is odd. Clearly, for every $m \geqslant -1$ and $0 \leqslant |k| \leqslant m+1$, $(g_n^{m,k}, g_n^{m,-k})$ is a biorthogonal system.

Let us put $\delta(t, t_n) = \min\big(d(t, t_n), d(t, -t_n)\big)$. The following omnibus theorem summarises properties of $(g_n^{m,k})$ for future reference.

THEOREM 1. *Let $m \geqslant -1$, $0 \leqslant |k| \leqslant m+1$. Then*

(a) *The system $(g_n^m)_{n=0}^\infty$ is an orthonormal system of even functions and it is complete in even functions in $L_2(T)$.*

(b) *If k is even then $(g_n^{m,k})_{n=0}^\infty$ is a system of even functions complete in even functions in $L_2(T)$. If k is odd then $(g_n^{m,k})_{n=1}^\infty$ is a system of odd functions complete in odd functions in $L_2(T)$.*

(c) *$g_n^{m,k}$ is orthogonal to $S^{m+k}(V_n^1)$.*

(d) *$g_n^{m,k}(t) = h_n^{m,k}(t) + (-1)^k h_n^{m,k}(-t)$ where $h_n^{m,k}(t) \in S^{m+k}(V_n^3)$, $h_n^{m,k}$ is orthogonal to $S^{m+k}(V_n^1)$ and for some constants C and q, $0<q<1$ we have*

$$|h_n^{m,k}(t)| \leqslant C n^{1/2+k} q^{nd(t,t_n)}.$$

(e) *For some constants C and q, $0<q<1$,*

$$|g_n^{m,k}(t)| \leqslant C n^{1/2+k} q^{n\delta(t,t_n)}.$$

(f)
$$\Big(\int_T \Big|\sum_n a_n h_n^{m,k}(t)\Big|^2 dt\Big)^{1/2} \sim \Big(\sum_n |a_n|^2 n^{2k}\Big)^{1/2}$$

and

$$\Big(\int_T \Big|\sum_n a_n g_n^{m,k}(t)\Big|^2 dt\Big)^{1/2} \sim \Big(\sum_n |a_n|^2 n^{2k}\Big)^{1/2}.$$

Proof. Everything except (f) follows immediately from previous considerations. The very important condition (f) is a theorem of Ropela [27]. Actually Ropela proved his theorem for systems on the interval, but his proof works in our case, too. The alternative proof can be found in [10].

Remark 2. Clearly, we can analogously construct a complete orthonormal system of splines $(G_n^m)_{n=0}^\infty$ on the circle T. We can also define functions $G_n^{m,k}$ by $G_0^{m,k} = 1/\sqrt{2}$, $G_n^{m,k} = D^k G_n^m$ if $k \geqslant 0$ and $G_n^{m,k} = H^{-k} G_n^m$ if $k \leqslant 0$. For $n = 2^k + l$, $0 \leqslant l < 2^k$, let $x_n = -1 + 2l/2^k$. (Remember we identify T with $[-1, 1)$.) Then for $G_n^{m,k}$ we have the following analog of Theorem 1:

THEOREM 1'. (a) *The system $(G_n^{m,k})_{n=0}^\infty$, $m \geqslant -1$, $0 \leqslant |k| \leqslant m+1$, is a complete system in $L_2(T)$ and*

$$\Big(\int_T \Big|\sum_{n=0}^\infty a_n G_n^{m,k}\Big|^2\Big)^{1/2} \sim \Big(\sum_{n=0}^\infty |a_n|^2 n^{2k}\Big)^{1/2}.$$

(b) *For some constants $C>0$ and q, $0<q<1$,*

$$|G_n^{m,k}(t)| \leqslant C n^{1/2+k} q^{nd(t,x_n)}.$$

Remark 3. As far as I know the above material was never presented exactly as above. Nevertheless it is clearly known to the specialists in spline theory. Our construction is a minor variation of the one indicated in [9].

2. Preliminaries, various H_p spaces, $p \leqslant 1$ and their relations. In this section we give precise definitions of various H_p spaces we will be interested in and we summarise their basic properties. There are various closely related H_p spaces on the circle T or on the interval $[0, 1]$. They all fall in the general framework discussed in [12].

For given $p \leqslant 1$ by s we will always mean the integer $[1/p - 1]$.

We start with the definition of p-atom (more precisely, $(p, 2)$-atom in the terminology of [31]).

DEFINITION. *A p-atom, $p \leqslant 1$, on T is either the constant function* 1 *or a function $a(t)$ such that* $\operatorname{supp} a$ *is contained in some interval $I \subsetneq T$*

and

$$\left(\int_T |a(t)|^2 dt\right)^{1/2} \leqslant |I|^{1/2-1/p}, \tag{2.1}$$

$$\int_T a(t) t^k dt = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, s. \tag{2.2}$$

In order to make the precise sense of (2.2) we identify the circle T with the interval $[-1, 1)$ in such a way that $-1 \notin I$, so I becomes a subinterval of $[-1, 1)$.

A p-atom on $[0, 1]$ is either a polynomial p of degree $\leqslant s$ and $\int_0^1 |p(t)|^2 dt \leqslant 1$ or a real function $a(t)$ satisfying (2.1) and (2.2).

Now we define $H_p(T)$ (or $H_p[0, 1]$) as the space of distributions f on T (or on $[0, 1]$) such that $f = \sum a_i a_i$ with a_i p-atoms and $\sum |a_i|^p < \infty$. We set

$$\|f\|_p = \inf\left\{\left(\sum |a_i|^p\right)^{1/p}: f = \sum a_i a_i\right\}.$$

The following proposition shows the relation between $H_p(T)$ and $H_p[0, 1]$.

Proposition 1. *Let $f \in H_p[0, 1]$, $1 \geqslant p > 1/2$, and let*

$$F(t) = \begin{cases} f(t) & \text{for} \quad t \in [0, 1), \\ f(-t) & \text{for} \quad t \in [-1, 0). \end{cases}$$

Then $F(t) \in H_p(T)$. Conversely if $F(t)$ is an even function in $H_p(T)$ then $F|[0, 1] \in H_p[0, 1]$.

The standard proof is left to the reader.

This proposition in particular allows us to apply the results for $H_1[0, 1]$ to $H_1(T)$ or $H_1(D)$ as was done in [32], Remark 1. However, it is false for $p \leqslant 1/2$. This fact forced us to develop the system of even splines on T, $(g_n^m)_{n=0}^{\infty}$.

A *p-molecule* on T centered at 0 (remember $T = [-1, 1)$) is a function $M(t)$ such that

$$\int_{-1}^{1} M(t) t^k dt = 0, \quad k = 0, 1, 2, \ldots, s \tag{2.3}$$

and

$$\zeta(M) = \left(\int |M(t)|^2 dt\right)^{a/2b} \left(\int |M(t)|^2 |t|^{2b}\right)^{1/2-a/2b} \leqslant 1 \tag{2.4}$$

where $a = 1 - 1/p + \varepsilon$, $b = 1/2 + \varepsilon$ for some fixed $\varepsilon > 1/p - 1$.

A molecule centered at $t_0 \in T$ is a suitable translation of a molecule centered at 0. The fundamental fact is that each p-molecule is in $H_p(T)$ and its norm $\|M\|_p$ is uniformly bounded. All this is well known, cf. [31].

Remark 4. Usually the above-mentioned facts are stated for the real line (or even for R^n) instead of T. The periodic case (the circle) is fully analogous to R, so all the proofs can be repeated with obvious modifications. The other way to obtain these facts for the circle is to use transference. This says simply that if $a: R \to T$ is given by $\varphi(t) = e^{it}$ then the induced map $T_\varphi: H_p(R) \to H_p(T)$, $T_\varphi(f)(t) = \sum_{k=-\infty}^{\infty} f(2k\pi + t)$ maps $H_p(R)$ onto $H_p(T)$.

The definition of a molecule stated above is not very convenient for our purposes, since the orthogonality relations (2.3) involve functions which are not continuous on the circle. The following proposition remedies this situation.

Proposition 2. *Let $M(t)$ be a function on T. Let us fix $p \leqslant 1$ and let $m+1 \geqslant s$. Assume that*

$$M \text{ is orthogonal to } S^m(V_n^1) \text{ for some } n > 8(m+1) \tag{2.5}$$

and

(2.6) *for some $t_0 \in T$ we have*

$$\left(\int_T |M(t)|^2 dt\right)^{a/2b} \left(\int_T |M(t)|^2 d(t, t_0)^{2b} dt\right)^{1/2-a/2b} \leqslant 1,$$

where a and b are as in (2.4).

Then $\|M\|_p \leqslant C$ for some absolute constant C.

Proof. By rotation we may assume that $t_0 = 0$ and we may identify T with $[-1, 1)$ in this way. Our goal is to find $b \in L_2(T)$ with $\|b\|_{L_2} \leqslant$ const such that $M - b$ will be a molecule. From our assumptions on n, m and s we see that there exist splines $\varphi_0, \varphi_1, \ldots, \varphi_s \in S^m(V_n^1)$ such that

$$\varphi_i|(-1/2, 1/2) = t^i, \quad i = 0, 1, 2, \ldots, s. \tag{2.7}$$

From (2.5) we infer that

$$\int_{-1}^{1} M(t) t^i dt = \int_{-1}^{1} M(t)\left(\varphi_i(t) + t^i\right) dt.$$

This implies that there exists a function $b(t)$ such that

$$\begin{gathered} b(t)|(-1/2, 1/2) = 0, \quad \|b\|_{L_2} \leqslant C\|M|(-1, -1/2) \cup (1/2, 1)\|_{L_2} \leqslant C \\ \text{and} \quad \int_{-1}^{1} \left(M(t) + b(t)\right) t^i dt = 0 \quad \text{for} \quad i = 0, 1, \ldots, s. \end{gathered} \tag{2.8}$$

This gives (2.3). In order to check (2.4) we estimate

$$\int_{-1}^{1} |M+b|^2 = \int_{-1/2}^{1/2} |M|^2 + \int_{-1}^{-1/2} + \int_{1/2}^{1} |M+b|^2 \leqslant C \int_{-1}^{1} |M|^2$$

and

$$\begin{aligned}\int_{-1}^{1} |(M+b)(t)|^2 t^{2b}\,dt &\leqslant \int_{-1/2}^{1/2} |M(t)|^2 t^{2b}\,dt + \Big(\int_{-1}^{-1/2} + \int_{1/2}^{1}\Big) |M(t)+b(t)|^2\, |t|^{2b}\,dt \\ &\leqslant \int_{-1/2}^{1/2} |M(t)|^2 t^{2b} + C\Big(\int_{1}^{-1/2} + \int_{1/2}^{1}\Big) |M(t)|^2 |t|^{2b}\,dt \\ &\leqslant C \int_{-1}^{1} |M(t)|^2 t^{2b}\,dt.\end{aligned}$$

This gives the proof of the proposition.

The most classical H_p space is the space $H_p(D)$. It is the space of all analytic functions on the unit disc in the complex plane such that

$$\|f\|_p = \sup_{r<1} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p\, d\theta\Big)^{1/p} < \infty.$$

The good general reference about those spaces is [16]. There is a close connection between $H_p(D)$ and $H_p(T)$, cf. [11], [13]. It is given as follows: It is well known that each $f \in H_p(D)$ has a radial limit in the sense of distributions: $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$. Since f is analytic, its real part determines the whole function up to a purely imaginary constant. We have

(2.9) $\qquad \|\mathrm{Re} f(e^{i\theta})\|_{H_p(T)} \sim \|f\|_{H_p(D)}$ for all $f(z) \in H_p(D)$ such that $f(0)$ is real.

We will also consider *martingale H_p spaces*. We will limit our attention to dyadic martingales only. The exposition of the theory of those spaces can be found in [3], [19], [21].

By a *dyadic interval on* $[0,1]$ we mean an interval of the form $[k/2^n, (k+1)/2^n]$. For function f defined on $[0,1]$ we define its *dyadic maximal function* by

$$f^*(x) = \sup \Big\{ \frac{1}{|I|} \Big| \int_I f \Big| : I \text{ is a dyadic interval and } x \in I \Big\}.$$

We say that $f \in H_p(\delta)$, the *dyadic H_p* space if $\|f\|_p = \big(\int |f^*(t)|^p\,dt\big)^{1/p} < \infty$.

It is known (cf. [3], [19]) that a function $f \in H_p(\delta)$ can be represented as a series of Haar functions $f = \sum a_n \chi_n$ and

$$\|f\|_p \sim \Big(\int \Big(\sum |a_n|^2 |\chi_n|^2\Big)^{p/2}\Big)^{1/p}. \tag{2.10}$$

The symbol $(\chi_n)_{n=0}^{\infty}$ will always denote the orthonormal Haar system. Actually the above facts are true for $0 < p < \infty$. Dyadic p-atom, $0 < p \leqslant 1$, is a function $a(t)$ such that $\operatorname{supp} a \subset I$, I dyadic interval, $\int a(t)\,dt = 0$ and $|a(t)| \leqslant |I|^{-1/p}$.

An easy and natural modification of an argument given in [13], p. 611, for $p = 1$ gives

PROPOSITION 3. *Every function $f \in H_p(\delta)$, $0 < p \leqslant 1$, has a decomposition $f = \sum \alpha_i a_i$, a_i are dyadic p-atoms and $\sum |\alpha_i|^p < \infty$. Conversely every dyadic p-atom is in $H_p(\delta)$. Moreover,*

$$\|f\|_p \sim \inf \Big\{ \Big(\sum |\alpha_i|^p\Big)^{1/p} : f = \sum \alpha_i a_i,\ a_i \text{—dyadic } p\text{-atoms} \Big\}.$$

The above proposition provides the "atomic definition" of $H_p(\delta)$.

If we consider the dyadic partition of T identified with $[-1, 1)$ we get the space of dyadic martingales on T, denoted by $H_p(T, \delta)$. All properties of this space are clearly identical with properties of $H_p(\delta)$.

To conclude the preliminaries let us make one comment on notation. $\|f\|_p$ will always mean the norm in the H_p-space which should be clear from the context. The norm in $L_p(T)$, $0 < p \leqslant \infty$, will be denoted by $\|\cdot\|_{L_p(T)}$ with one exception; $\|f\|_2$ will always mean $\big(\int |f|^2\big)^{1/2}$.

3. Unconditional bases in H_p-spaces, $0 < p \leqslant 1$. This section contains our main results. We show that the spline systems on T constructed in Section 1 are unconditional in suitable $H_p(T)$. This allows us to construct unconditional bases in $H_p(D)$ and to show that $H_p(D)$ and $H_p(\delta)$ are naturally isomorphic as linear topological spaces.

Our basic result is

THEOREM 2. *For $m \geqslant 0$ and $|k| \leqslant m+1$ the system $(g_n^{m,k})$ is an unconditional basis in its span in $H_p(T)$ for $1 \geqslant p > 1/(m+k+2)$. If k is even this span is the subspace of all even functions and if k is odd the span consists of all odd functions.*

We start the proof with two lemmas.

LEMMA 4. *Let a be a proper p-atom*, $\operatorname{supp} a \subset I$. *Let m, k, p be as in Theorem 2. Then*

$$\Big|\int_T a(t)\, g_n^{m,-k}(t)\,dt\Big| \leqslant C\, |I|^{s+2-1/p} n^{3/2-k+s} q^{n\delta(I,t_n)} \tag{3.1}$$

and

$$\sup_a \Big| \int_T a(t)\, g_n^{m,-k}(t)\, dt \Big| \leqslant C n^{1/p-k-1/2}. \tag{3.2}$$

Proof. Let us identify T with $[-1, 1)$ in such a way that I will become a subinterval in $[-1, 1)$ (cf. the definition of p-atom). From (2.2) we infer that there exists $A(t)$ such that $\operatorname{supp} A(t) \subset I$ and $\boldsymbol{D}^{s+1} A = a$. An easy estimate yields $\|A\|_2 \leqslant C|I|^{s+1}\|a\|_2$.

Using this we have

$$\Big|\int_T a(t)\, g_n^{m,-k}(t)\, dt\Big| = \Big|\int_T \boldsymbol{D}^{s+1} A(t)\, g_n^{m,-k}(t)\, dt\Big| = \Big|\int_T A(t)\, g_n^{m,s+1-k}(t)\, dt\Big|$$
$$\leqslant \|A\|_2 \cdot \Big(\int_I |g_n^{m,s+1-k}(t)|^2\Big)^{1/2} \leqslant C|I|^{s+2-1/p} n^{s-k+3/2} q^{n\delta(I,t_n)}.$$

This gives (3.1).

In order to prove (3.2) we consider two cases:

(a) $|I| < 1/n$. Since $s+2-1/p > 0$, (3.1) gives

$$\Big|\int_T a(t)\, g_n^{m,-k}(t)\, dt\Big| \leqslant C\, n^{1/p-k-1/2}. \tag{3.3}$$

(b) $|I| \geqslant 1/n$. In this case the Hölder inequality gives

$$\Big|\int_T a(t)\, g_n^{m,-k}(t)\, dt\Big| \leqslant \|a\|_2 \cdot \|g_n^{m,-k}\|_2 \leqslant C|I|^{1/2-1/p} n^{-k} \leqslant C n^{1/p-1/2-k}. \tag{3.4}$$

If we put (3.3) and (3.4) together we get (3.2).

LEMMA 5. *Let $m \geqslant 0$ and $|k| \leqslant m+1$ and $1 \geqslant p > 1/(m+k+2)$. There exist constants C_1 and C_2 depending only on m, k, p such that*

$$C_2 n^{1/2+k-1/p} \leqslant \|g_n^{m,k}\|_p \leqslant C_1 n^{1/2+k-1/p}. \tag{3.5}$$

Moreover, the right-hand side inequality holds also for $p = 1/(m+k+2)$.

Proof. To show the right-hand side inequality it is enough to show it for $h_n^{m,k}$. Using Theorem 1(d) we easily find $\|h_n^{m,k}\|_2 \leqslant Cn^k$ and

$$\int |h_n^{m,k}(t)\, d(t, t_n)^b|^2\, dt \leqslant Cn^{1+2k} \int_0^\infty q^{2nt} t^{2b}\, dt$$
$$\leqslant Cn^{2k-2b}.$$

These estimates and Proposition 2 give the desired inequality.

The left-hand side inequality follows from (3.2) by duality. We have

$$1 = \int_T g_n^{m,k}(t)\, g_n^{m,-k}(t)\, dt \leqslant \|g_n^{m,k}\|_p \cdot C n^{1/p-k-1/2}.$$

This proves Lemma 5.

Proof of Theorem 2. Let $a(t)$ be an arbitrary p-atom. Let us consider the series

$$\sum_n \pm \int a(t)\, g_n^{m,-k}(t)\, dt \cdot g_n^{m,k}. \tag{3.6}$$

We have to show that this series represents the function whose H_p norm is bounded independently of the atom $a(t)$ and of the choice of signs $+$ or $-$. Let I be an interval such that $\operatorname{supp} a \subset I$ and (2.1) holds. Let us fix an integer r such that $2^{-r-1} < |I| \leqslant 2^{-r}$. Let us write (3.6) in the form

$$\Big(\sum_{n \leqslant 2^r} + \sum_{\substack{n > 2^r \\ t_n \in I \text{ or } -t_n \in I}} + \sum_{\substack{n > 2^r \\ t_n \notin I \text{ and } -t_n \notin I}}\Big) \pm \int a(t)\, g_n^{m,-k}(t)\, dt \cdot g_n^{m,k} \tag{3.7}$$
$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

Using (3.1) and (3.5) we obtain

$$\Big\|\sum\nolimits_1\Big\|_p^p \leqslant \sum_{n \leqslant 2^r} \Big|\int a(t)\, g_n^{m,-k}(t)\, dt\Big|^p \cdot \|g_n^{m,k}\|_p^p \tag{3.8}$$
$$\leqslant C \sum_{n \leqslant 2^r} |I|^{(s+2-1/p)p} n^{(s+2-1/p)p} q^{n\delta(I,t_n)}$$
$$\leqslant C \sum_{j=0}^{r} 2^{-r(s+2-1/p)p} 2^{j(s+2-1/p)p} \leqslant \text{const}$$

since $s+2-1/p > 0$.

Using the Hölder inequality, Theorem 1(c), and (3.5) we have

$$\Big\|\sum\nolimits_3\Big\|_p^p \leqslant \sum_{\substack{n \geqslant 2^r \\ t_n \notin I \text{ and } -t_n \notin I}} \Big|\int a(t)\, g_n^{m,-k}(t)\, dt\Big|^p \|g_n^{m,k}\|_p^p \tag{3.9}$$
$$\leqslant \sum_{\substack{n \geqslant 2^r \\ t_n \notin I \text{ and } -t_n \notin I}} \|a\|_2^p \Big(\int_I |g_n^{m,-k}(t)|^2\, dt\Big)^{p/2} \|g_n^{m,k}\|_p^p$$
$$\leqslant C \sum_{\substack{n \geqslant 2^r \\ t_n \notin I \text{ and } -t_n \notin I}} |I|^{(1/2-1/p)p} n^{-kp} q^{np\delta(I,t_n)}\, n^{(k+1/2-1/p)p}$$
$$\leqslant C \sum_{n=r}^{\infty} 2^{-r(1/2-1/p)p}\, 2^{-nkp}\, 2^{n(k+1/2-1/p)p}$$
$$= C\, 2^{-r(p/2-1)} \sum_{n=r}^{\infty} 2^{n(p/2-1)} \leqslant \text{const}.$$

In order to estimate $\|\Sigma_2\|_p$ it is enough to show that

$$\sum\nolimits_2^1 = \sum_{\substack{n>2^r \\ t_n\in I \text{ or } -t_n\in I}} \pm \int a(t) g_n^{m,-k}(t)\,dt\, h_n^{m,k}$$

has uniformly bounded norm in $H_p(T)$. By Theorem 1 (f) we have

$$(3.10)\qquad \Big\|\sum\nolimits_2^1\Big\|_2 \leqslant C\|a\|_2 \leqslant C\, 2^{-r(1/2-1/p)}.$$

Let a_0 denote the center of I and let $W = 2I$. We write

$$(3.11)\qquad \Big\|\sum\nolimits_2^1 (t)\, d(t, a_0)^b\Big\|_2^2 \leqslant |W|^{2b} \int_W \Big|\sum\nolimits_2^1 (t)\Big|^2 dt + \int_{T\setminus W} \Big|\sum\nolimits_2^1 (t)\, d(t, a_0)^b\Big|^2 dt.$$

The first summand does not exceed, by (3.10),

$$(3.12)\qquad |W|^{2b} \int_T \Big|\sum\nolimits_2^1 (t)\Big|^2 dt \leqslant C\, 2^{-r(1+2b-2/p)}.$$

Using Theorem 1(d), (f) and the Hölder inequality we can estimate the second summand of (3.11) as follows:

$$(3.13)\qquad \int_{T\setminus W} \sum_{\substack{n>2^r \\ t_n\in I \text{ or } -t_n\in I}} \Big|\int a(s) g_n^{m,-k}(s)\,ds\, n^k\Big|^2 \cdot \sum_{\substack{n>2^r \\ t_n\in I \text{ or } -t_n\in I}} \Big|\frac{h_n^{m,k}(t)}{n^k}\, d(t, a_0)^b\Big|^2 dt$$

$$\leqslant C\|a\|_2^2 \cdot \int_{T\setminus W} \sum_{\substack{n>2^r \\ t_n\in I \text{ or } -t_n\in I}} d(t, a_0)^{2b}\, n q^{2nd(t, t_n)}\, dt$$

$$\leqslant C\|a\|_2^2 \int_{T\setminus W} \sum_{n=r}^{\infty} d(t, a_0)^{2b}\, 2^n \cdot 2^{n-r} q^{2^n d(t, a_0)}$$

$$\leqslant C\|a\|_2^2 \sum_{n=r}^{\infty} 2^{2n-r} \int_{2^{-r}}^{\infty} q^{2^n t} t^{2b}\, dt$$

$$\leqslant C\|a\|_2^2 \sum_{n=r}^{\infty} 2^{2n-r}\, 2^{-2bn-n} \int_{2^{n-r}}^{\infty} q^u u^{2b}\, du$$

$$\leqslant C\|a\|_2^2\, 2^{-r} 2^{r(1-2b)} \leqslant C\, 2^{r(2/p-1-2b)}.$$

If we put together (3.12) and (3.13) we get

$$(3.14)\qquad \Big\|\sum\nolimits_2^1 (t)\, d(t, a_0)^b\Big\|_2^2 \leqslant C\, 2^{r(2/p-1-2b)}.$$

Estimates (3.10) and (3.14) together with Proposition 2 show that

$$(3.15)\qquad \Big\|\sum\nolimits_2^1\Big\|_p \leqslant \text{const.}$$

The proof of the theorem follows from (3.8), (3.9) and (3.15).

Remark 5. The situation for $p \leqslant 1/(m+k+2)$ is as follows. If $p < 1/(m+k+2)$ then $g_n^{m,k}$ is not a basic sequence since the biorthogonal functionals $g_n^{m,-k}$ are not continuous on $H_p(T)$. This follows from the result of Duren–Romberg–Shields [17] (cf. our Proposition 7).

If $p = 1/(m+k+2)$ the system $g_n^{m,k}$ is a basic sequence. The case $m = -1$, $k = 0$, $p = 1$ was considered by Billard [2]. The proof in the general case will be given elsewhere.

It is known that $(g_n^{-1,0})_{n=0}^{\infty}$ is not an unconditional basis in $H_1(T)$ (cf. [23]) and $(g_n^{0,0})_{n=0}^{\infty}$ is not unconditional in $H_{1/2}(T)$. This fact follows by duality from example of Ciesielski's [6], p. 316. It seems likely that $(g_n^{m,k})$ is never unconditional in H_p for $p = 1/(m+k+2)$.

For $g \in L_2(T)$ let $\tilde{g}$ denote the trigonometric conjugate of g. Since the trigonometric conjugation operator $\tilde{}$ extends to an isomorphism of $H_p(T)$ (mod constants) and $\tilde{}$ maps even distributions into odd and vice versa, we obtain

COROLLARY 2. *For $m \geqslant 0$, $|k| \leqslant m+1$ the system*

$$(g_n^{m,k})_{n=0}^{\infty} \cup (\tilde{g}_n^{m,k})_{n=1}^{\infty} \quad \textit{if} \quad k \textit{ is even}$$

and the system

$$\{1\} \cup (g_n^{m,k})_{n=1}^{\infty} \cup (\tilde{g}_n^{m,k})_{n=1}^{\infty} \quad \textit{if} \quad k \textit{ is odd}$$

is an unconditional basis in $H_p(T)$ for $p > 1/(m+k+2)$.

Remark 6. A much more natural basis for $H_p(T)$ is given by the systems $(G_n^{m,k})_{n=0}^{\infty}$ (cf. Remark 2). The same proof as the proof of Theorem 2 gives

THEOREM 2′. *The system $(G_n^{m,k})_{n=0}^{\infty}$ is an unconditional basis in $H_p(T)$ for $p > 1/(m+k+2)$.*

Now we are in a position to produce an unconditional basis in $H_p(D)$. For every real function $f \in L_2(T)$ we define an analytic function on D (its Cauchy integral) $Cf(z)$ by

$$Cf(z) = Cf(re^{i\theta}) = f(re^{i\theta}) + i\tilde{f}(re^{i\theta})$$

where f and $\tilde{f}$ are extended to D via the Poisson formula. If we apply this to the system $g_n^{m,k}$ we obtain a complex system $(Cg_n^{m,k})_{n=0}^{\infty}$ ($Cg_0^{m,k}$ always means a constant).

From Corollary 2 and (2.9) we infer

THEOREM 3. *For $m \geqslant 0$, $|k| \leqslant m+1$, the system $(Cg_n^{m,k})_{n=0}^{\infty}$ is an unconditional basis in a complex space $H_p(D)$ for $1 \geqslant p > 1/(m+k+2)$.*

The standard application of the Khintchine inequality and Theorem 3 yields the following square function type characterisation of $H_p(D)$.

THEOREM 4. *Let $1 \geqslant p > 1/(m+k+2)$, $m \geqslant 0$, $|k| \leqslant m+1$. The function $f(z) = \sum_{n=0}^{\infty} a_n Cg_n^{m,k}(z)$ belongs to $H_p(D)$ if and only if*

$$\Big(\int_{-\pi}^{\pi}\Big(\sum_{n=0}^{\infty}|a_n|^2|Cg_n^{m,k}(e^{i\theta})|^2\Big)^{p/2}d\theta\Big)^{1/p} < \infty.$$

The next theorem establishes a linear topological isomorphism between $H_p(D)$ and $H_p(\delta)$.

THEOREM 5. *Let $T^{m,k}\colon H_p(D)\to H_p(\delta)$ be defined by*

$$T^{m,k}(Cg_n^{m,k}) = n^k\chi_n.$$

The operator $T^{m,k}$ establishes an isomorphism between $H_p(D)$ and $H_p(\delta)$ for $1 \geqslant p > 1/(m+k+2)$.

Proof. We start with the proof that $T^{m,k}$ is bounded. It is enough to check that $\|T^{m,k}(Ca)\|_p$ is uniformly bounded for all atoms a. This reduces to the estimate

$$\Big\|\sum_n \int a(t)g_n^{m,-k}(t)\,dt\,n^k\chi_n\Big\|_p \leqslant \text{const}.$$

As in (3.7) we split it into three sums $\sum_1$, $\sum_2$ and $\sum_3$. Since

$$\|g_n^{m,k}\|_p \sim n^k\|\chi_n\|_p, \tag{3.16}$$

estimate (3.8) shows that $\|\sum_1\|_p \leqslant \text{const}$ and estimate (3.9) shows that $\|\sum_3\|_p \leqslant \text{const}$.

The properties of dyadic intervals give that

$$\Big|\operatorname{supp}\sum\nolimits_2\Big| \leqslant C\,2^{-r} \tag{3.17}$$

(we mean $\operatorname{supp} a \subset I$ with $2^{-r-1} < |I| \leqslant 2^{-r}$).

Using (2.10), the Hölder inequality, estimate (3.17) and Theorem 1 (f) together with (3.16) we obtain

$$\begin{aligned}\Big\|\sum\nolimits_2\Big\|_p^p &\leqslant C\int_0^1\Big(\sum_{\substack{n>2^r\\ t_n\in I \text{ or } -t_n\in I}}\Big|\int a(t)g_n^{m,-k}(t)\,dt\Big|^2(n^k\chi_n(s))^2\Big)^{p/2}ds\\ &\leqslant C\Big|\operatorname{supp}\sum\nolimits_2\Big|^{(2-p)/p}\Big\|\sum\int a(t)g_n^{m,-k}(t)\,dt\,n^k\chi_n\Big\|_2^p\\ &\leqslant C\,2^{-r(2-p)/p}\|a\|_2^p \leqslant \text{const}.\end{aligned}$$

So $T^{m,k}$ is continuous.

In order to show that $(T^{m,k})^{-1}$ is continuous we use Proposition 3. Let us take a real dyadic atom $a(t)$, supported on a dyadic interval I. We have

$$a(s) = \sum_{n:\operatorname{supp}\chi_n\subset I}\int a(t)\chi_n(t)\,dt\,\chi_n(s).$$

So

$$(T^{m,k})^{-1}(a) = \sum_{n:\operatorname{supp}\chi_n\subset I}\Big(\int a(t)\chi_n(t)\,dt\Big)n^{-k}Cg_n^{m,k}.$$

By (2.9)

$$\|(T^{m,k})^{-1}(a)\|_{H_p(D)} \sim \Big\|\sum_{n:\operatorname{supp}\chi_n\subset I}\Big(\int a(t)\chi_n(t)\,dt\Big)\,n^{-k}g_n^{m,k}\Big\|_p.$$

This last norm is estimated exactly as $\|\sum_2\|_p$ is estimated in the proof of Theorem 2. This completes the proof.

The fact that $H_1(\delta)$ and $H_1(D)$ are isomorphic was discovered by Maurey in [24] but his proof was not constructive. The constructive proof was given by Carleson [4] and the author in [32]. The case $k=0$ of Theorem 5 with $H_p[0,1]$ instead of $H_p(D)$ follows from [30] (cf. also our Theorem 12).

We also have

THEOREM 5′. *Let $S^{m,k}\colon H_p(T)\to H_p(\delta)$ be defined by*

$$S^{m,k}(G_n^{m,k}) = n^k\chi_n.$$

The operator $S^{m,k}$ establishes an isomorphism between $H_p(T)$ and $H_p(\delta)$ for $1 \geqslant p > 1/(m+k+2)$.

The proof is the same as the proof of Theorem 5.

4. Bergman spaces and complemented subspaces of H_p, $0 < p < 1$. This section is more functional analytic in spirit than the rest of the paper. We show how the existence of unconditional bases in $H_p(D)$ formally gives unconditional bases in some Bergman spaces. Those bases provide natural isomorphisms of Bergman spaces and l_p-spaces. Later we apply this to the proof that every complemented subspace of H_p, $0 < p < 1$, contains l_p complemented.

Let us start with some definitions. A p-norm on a linear space X is a function $\|\cdot\|$: $X \to R^+$ such that

$$\|x\| > 0 \quad \text{for} \quad x \neq 0,$$
$$\|ax\| = |a| \cdot \|x\|,$$
$$\|x+y\|^p \leqslant \|x\|^p + \|y\|^p.$$

A *p-Banach space* is a linear space X equipped with the p-norm $\|\cdot\|$ and complete with respect to the metric $\|x-y\|^p$. A p-Banach space X has the property that for every bounded sequence $x_n \in X$, $\|x_n\| \leqslant 1$ and for every sequence of scalars (a_n) with $\sum |a_n|^p \leqslant 1$, $\sum\limits_n a_n x_n \in X$ and $\|\sum\limits_n a_n x_n\| \leqslant 1$.

If X is a p-Banach space and Y is a q-Banach space and $\boldsymbol{T}$: $X \to Y$ is linear then $\|\boldsymbol{T}\| = \sup\{\|\boldsymbol{T}x\|: \|x\| \leqslant 1\}$. $\boldsymbol{T}$ is continuous if and only if $\|\boldsymbol{T}\| < \infty$. Let now X be a p-Banach space, $p < 1$ and let $p < q \leqslant 1$ be given. The q-envelope of X, denoted $q - X$, is the completion of X with respect to the q-quasinorm

$$\|x\|_q = \inf\left\{\left(\sum |a_n|^q\right)^{1/q} \text{where } x = \sum a_n x_n,\ x_n \in X,\ \|x_n\| \leqslant 1\right\}.$$

The proof of the following standard proposition is omitted:

PROPOSITION 4. *Let X be a p-Banach space and Y be a q-Banach space, $p < q$, and let $\boldsymbol{T}$: $X \to Y$ be a continuous linear operator. Then there exists a unique extension $\hat{\boldsymbol{T}}$: $q - X \to Y$ and $\|\hat{\boldsymbol{T}}\| \leqslant \|T\|$.*

Now we introduce the *weighted Bergman spaces* on the unit disc D. The space A_a^p, $0 < p \leqslant \infty$, $-1 < a < \infty$ consists of all functions analytic in D such that

$$\|f\|_{p,a} = \left(\int\limits_D |f(z)|^p (1-|z|)^a dz\right)^{1/p} < \infty.$$

The following Theorem 6 is essentially known. The case $q = 1$ was proved in [29] and [17]. We give the sketch of the proof for the sake of completeness.

THEOREM 6. *Let $0 < p < q \leqslant 1$. Then $q - H_p(D) = A^q_{q/p-2}$ and the norms are equivalent.*

We start with the classical lemma due to Hardy–Littlewood [20] (cf. [16], 5.11).

LEMMA 6. *If $0 < p < q \leqslant \infty$, $\lambda \geqslant p$ and $a = 1/p - 1/q$ then for $f \in H_p(D)$ we have*

$$\int\limits_0^1 (1-r)^{\lambda a - 1}\left(\int\limits_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta\right)^{\lambda/2} dr \leqslant C\|f\|_p^\lambda.$$

In particular, ($\lambda = q$) the identity is a continuous map from $H_p(D)$ into $A^q_{q/p-2}$ for $0 < p < q \leqslant 1$.

Proof of Theorem 6. In view of Lemma 6 it is enough to repre-ent every $f(z) \in A^q_{q/p-2}$, $\|f\|_{q,q/p-2} = 1$ as a sum $f(z) = \sum a_n f_n(z)$ with $a_n|^q \leqslant$ const and $\|f_n\|_p \leqslant$ const. This is done by approximating the egral representation formula. It can be done by hand (cf. [29], Th. 2) we can use Theorem 2 of [12], which in our special case gives

$$f(z) = \sum \lambda_i \frac{(1-|\zeta_i|^2)^{1/p}}{(1-z\zeta_i)^{2/p}}, \tag{4.1}$$

where $\zeta_i \in D$ and $\sum |\lambda_i|^q \leqslant$ const. The $H_p(D)$ norms of functions appearing in (4.1) are uniformly bounded (cf. [16], p. 65).

This completes the proof.

Now let (f_n) be an arbitrary unconditional basis for $H_p(D)$ and let $p < q \leqslant 1$. Using Proposition 4, Theorem 6 and Lemma 6 we see that there exists a constant C such that for every sequence of scalars (a_n) and every sequence of signs (ε_n) we have

$$\left\|\sum \varepsilon_n a_n f_n\right\|_{q,q/p-2} \leqslant c \left\|\sum a_n f_n\right\|_{q,q/p-2}.$$

In other words f_n is an unconditional basis in $A^q_{q/p-2}$. So using Theorem 3 we obtain

THEOREM 7. *The system $(Cg_n^{m,k})_{n=0}^\infty$, $m \geqslant 0$, $|k| \leqslant m+1$ is an unconditional basis in A^q_a, $q \leqslant 1$ for*

$$q > 1/(m+k+2) \quad \text{and} \quad -1 < a < q(m+k+2)-2.$$

Our goal now is to characterise A_a^q in terms of coefficients with respect to the system $(Cg_n^{m,k})_{n=0}^\infty$. We start with

PROPOSITION 5. *Let $f = \sum\limits_{n=0}^{\infty} a_n Cg_n^{m,k} \in H_p$, $p > 1/(m+k+2)$, and let $q > p$. Then for some constant $C = C(m, k, p, q)$,*

$$\left(\sum_{n=0} |a_n|^q n^{(1/2+k-1/p)q}\right)^{1/q} \leqslant C\|f\|_p.$$

Proof. As usual, it is enough to show that for every p-atom on T we have

$$\Big(\sum_{n=0}^{\infty}\Big|\int_T a(t)g_n^{m,-k}(t)\,dt\Big|^q n^{(1/2+k-1/p)q}\Big)^{1/q} \leqslant \text{const.}$$

We proceed analogously as in the proof of Theorem 2. Let $\operatorname{supp} a \subset I$, $\|a\|_2 \leqslant |I|^{1/2-1/p}$ and $2^{-r-1} < |I| \leqslant 2^{-r}$. We write

$$\sum_n \Big|\int a(t)g_n^{m,-k}(t)\,dt\Big|^q n^{(1/2+k-1/p)q}$$
$$= \Big(\sum_{n=0}^{2^r} + \sum_{\substack{n>2^r\\ t_n\in I \text{ or } -t_n\in I}} + \sum_{\substack{n>2^r\\ t_n\notin I \text{ and } -t_n\notin I}}\Big)\Big|\int a(t)g_n^{m,-k}(t)\,dt\Big|^q n^{(1/2+k-1/p)q}$$
$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

Estimates (3.8) and (3.9) give that $\sum_1$ and $\sum_3$ are finite if q is replaced by p. Since we have $p < q$, we have the desired inequality for $\sum_1$ and $\sum_3$.

To estimate $\sum_2$ we use Theorem 1 (f) and the Hölder inequality to obtain

$$\sum\nolimits_2 \leqslant \Big(\sum\Big|\int a(t)g_n^{m,-k}(t)\,dt\Big|^2 n^{2k}\Big)^{q/2}\cdot\Big(\sum_{\substack{n>2^r\\ t_n\in I \text{ or } -t_n\in I}} n^{\left(\frac{1}{2}-\frac{1}{p}\right)\frac{2q}{2-q}}\Big)^{\frac{2-q}{2}}$$
$$\leqslant C\|a\|_2^q\Big(\sum_{n=r}^{\infty} 2^{n-r}2^{n\left(\frac{1}{2}-\frac{1}{p}\right)\frac{2q}{2-q}}\Big)^{\frac{2-q}{2}}$$
$$= C\|a\|_2^q 2^{-r\left(\frac{2-q}{2}\right)}\Big(\sum_{n=r}^{\infty} 2^{n\left(1+\frac{2q}{2-q}\frac{p-2}{2p}\right)}\Big)^{\frac{2-q}{2}} \leqslant \text{const.}$$

The last inequality uses (2.1) and the fact that $q > p$.

Lemma 5 and (2.9) give $\|Cg_n^{m,k}\|_p \sim n^{1/2+k-1/p}$. This observation and Proposition 5 imply that $q-H_p(D)$ for $1 \geqslant q > p$ consists of all sums $\sum_{n=0}^{\infty} a_n Cg_n^{m,k}$ such that

$$\sum_{n=0}^{\infty} |a_n|^q n^{(1/2+k-1/p)q} < \infty,$$

where p, m and k are related as in Theorem 2. This fact and Theorem 6 yield the following characterisation of A_α^q.

THEOREM 8. *Let* $1/(m+k+2) < q \leqslant 1$, $-1 < \alpha < q(m+k+2)-2$. *The function* $f(z)$ *is in* A_α^q *if and only if*

$$f(z) = \sum_{n=0}^{\infty} a_n Cg_n^{m,k}(z)$$

and

$$\sum_{n=0}^{\infty} |a_n|^q n^{(1/2+k-(\alpha+2)/q)q} < \infty.$$

This theorem can be compared (in the special case of the unit disc) with Theorem 2 of [12]. Our decomposition has the advantage over decomposition from [12] in being unique. On the other hand the functions $Cg_n^{m,k}$ we use are much less natural than the functions associated with the Bergman kernel as used in [12] (cf. (4.1)).

In the language of functional analysis Theorem 8 means that the basis $n^{(\alpha+2)/q-k-1/2}Cg_n^{m,k}(z)$ in A_α^q, with α, q, m, k as in Theorem 8, is equivalent to the unit vector basis in l_q. In particular we get

COROLLARY 3. *The space* A_α^q, $0 < q \leqslant 1$, $-1 < \alpha < \infty$, *is isomorphic to* l_q.

This corollary is known (cf. [22], Th. 2.4 and 3.3) but as far as we know Theorem 8 gives the first explicit construction of a basis in A_α^q.

Remark 7. Since the dual of $H_p(D)$, $p < 1$, is clearly the same as the dual of $A_{1/p-2}^1$ (cf. [29]) Theorem 8 allows us to give the formal description of this dual.

PROPOSITION 6. *The dual of* $H_p(D)$, $p < 1$, *can be identified with all infinite series* $\sum_{n=0}^{\infty} a_n Cg_n^{m,-k}$, $p > 1/(m+k+2)$ *such that* $\sup_n |a_n|\, n^{-(1/2+k-1/p)} < \infty$.

On the other hand, the dual of $H_p(D)$, $p < 1$, has been described in [17] (cf. also [29] and [12]).

A function $f \in C(T)$ belongs to Λ_α, $0 < \alpha \leqslant 1$, if

$$|f(t_1)-f(t_2)| \leqslant Cd(t_1, t_2)^\alpha, \qquad t_1, t_2 \in T$$

and is said to belong to Λ_* if

$$|f(t+h)-2f(t)+f(t-h)| \leqslant C|h|$$

for all h and $t \in T$.

The result of Duren–Romberg–Shields [17] is as follows

PROPOSITION 7. *The dual of* $H_p(D)$, $p < 1$, *can be identified with the space of all functions* f *continuous in* $\bar{D}$ *and analytic in* D *such that if*

$1/(n+1) < p < 1/n$ $(n = 1, 2, 3, \ldots)$ *then*

$$D^{n-1} f(e^{i\theta}) \in \Lambda_{1/p-n};$$

if $p = 1/(n+1)$ $(n = 1, 2, 3, \ldots)$ *then*

$$D^{n-1} f(e^{i\theta}) \in \Lambda_*.$$

If we put together Propositions 6 and 7 and take into account that the pairing is the same in both Propositions we get a constructive characterisation of smooth analytic functions in terms of coefficients of expansions with respect to systems $Cg_n^{m,k}$. We also see that the dual of $H_p(D)$, $p < 1$, i.e., the space of smooth analytic functions is isomorphic to l_∞. The non-analytic version of those results has been obtained by Ciesielski (cf. [7]).

Now we intend to apply our results to the investigation of complemented subspaces of $H_p(D)$. Our main result in this direction is the following.

THEOREM 9. *Let X be an infinite dimensional complemented subspace of $H_p(D)$, $0 < p < 1$. Then X contains a complemented subspace isomorphic to l_p.*

This theorem answers the question asked in [22]. Let us remark that for $\infty > p \geqslant 1$ the analogous statement is not true.

LEMMA 7. *If a p-Banach space $X = X_1 + X_2$ is a direct sum of its subspaces X_1 and X_2 and if $p < q \leqslant 1$ then*

$$q - X = (q - X_1) + (q - X_2).$$

The standard proof of this lemma is omitted.

Proof of Theorem 9. Let X be an infinite dimensional complemented subspace of $H_p(D)$, i.e., $H_p(D) = X + Y$. By Lemma 7 and Theorem 6 $1 - H_p(D) = A^1_{1/p-2} = (1 - X) + (1 - Y)$. Since X is infinite dimensional, we infer that $1 - X$ is an infinite dimensional Banach space. Moreover, by the definition of 1-envelope of X $\{x \in X: \|x\|_p \leqslant 1\}$ is a non-compact subset of $1 - X$. Let us take m such that $p > 1/(m+2)$ and let us consider $(Cg_n^m)_{n=0}^\infty$, the unconditional basis in $H_p(D)$ and simultaneously in $A^1_{1/p-2}$. By a standard perturbation argument we may assume that there are functions $\varphi_r \in X$ such that

$$\varphi_r = \sum_{j=k_r+1}^{k_{r+1}} a_j C g_j^m, \tag{4.2}$$

where every $k_r = 2^{s(r)}$ and $s(r)$ is a strictly increasing sequence of integers

$$\|\varphi_r\|_p = 1, \tag{4.3}$$

$$\|\varphi_r\|_{1,1/p-2} \geqslant C > 0 \quad \text{for} \quad r = 1, 2, \ldots \tag{4.4}$$

The proof of Theorem 9 reduces to the following two propositions:

PROPOSITION 8. *Let (φ_r) satisfies (4.2), (4.3) and (4.4). Then a certain subsequence of (φ_r) is equivalent in $H_p(D)$ to a unit vector basis in l_p.*

PROPOSITION 9. *Let $\psi_r = \sum_{j=k_r+1}^{k_{r+1}} \beta_j C g_j^m$ be such that*

$$\int_T \varphi_r(t)\psi_r(t)\,dt = 1 \quad \text{for} \quad r = 1, 2, \ldots \tag{4.5}$$

and

$$\|\psi_r\|_{H_p(D)^*} \leqslant C \quad \text{for} \quad r = 1, 2, \ldots \tag{4.6}$$

Then for every p-atom $a(t)$

$$\sum_r \Big| \int_T a(t)\psi_r(t)\,dt \Big|^p \leqslant \text{const}. \tag{4.7}$$

Assuming those propositions hold we conclude the proof of Theorem 9 as follows. Let φ_r denote the subsequence given by Proposition 8. From (4.4) we infer that there exist ψ_r's as required by Proposition 9. We define the projection P: $H_p \to \operatorname{span}\{\varphi_r\}$ as

$$P(f) = \sum_r \int_T f(t)\psi_r(t)\,dt \cdot \varphi_r.$$

In order to show that P is continuous it is enough to show that for every $f \in H_p(D)$

$$\sum \Big| \operatorname{Re} \int_T f(t)\psi_r(t)\,dt \Big|^p < C\|f\|_p.$$

But for f such that $f(0) = 0$ we have

$$\operatorname{Re} \int_T f(t)\psi_r(t)\,dt = 2 \int_T \operatorname{Re} f(t) \cdot \operatorname{Re} \psi_r(t)\,dt$$

so P is continuous by (2.9) and (4.7).

Proof of Proposition 8. We will use some facts about uniform integrability of functions. The facts we need are summarised in the following

LEMMA 8. *Let $1 \leqslant a < 2$, $C \geqslant 1$ and let μ be a probability measure. Let $f_1, f_2, \ldots$ in $L_a(\mu)$ be such that $\int |f_n|^a d\mu = 1$, $n = 1, 2, \ldots$, and for every sequence of scalars $c_1, c_2, \ldots$ we have*

$$C \Big(\int \Big| \sum_n c_n f_n \Big|^a d\mu \Big)^{1/a} \geqslant \Big(\sum_n |c_n|^a \Big)^{1/a}. \tag{4.8}$$

Then there exists a constant γ and a sequence of disjoint sets $A_1, A_2, \ldots$ such that for some subsequence (f_{n_k}) we have

$$\Big(\int_{A_k} |f_{n_k}|^\alpha d\mu\Big)^{1/\alpha} \geqslant \gamma. \tag{4.9}$$

The proof of this lemma can be found in [18]; the proof of condition (d) in the proof of Theorem 3.1.

Let now φ denote any of the φ_r's. Let us factor $\varphi(z) = g(z)\cdot h(z)$ in such a way that $g \in H_\alpha(D)$, $1 \leqslant \alpha < 2$ and $h \in H_\beta(D)$, $0 < \beta < 2$, $1/\alpha + 1/\beta = 1/p$ and on the unit circle T we have $|g| = |\varphi|^{p/\alpha}$ and $|h| = |\varphi|^{p/\beta}$.

We have by the Hölder inequality

$$C \leqslant \|\varphi\|_{1,1/p-2} = \frac{1}{2\pi}\int_0^1 \int_{-\pi}^{\pi} |\varphi(re^{i\theta})|(1-r)^{1/p-2}\, dr\, d\theta \tag{4.10}$$

$$\leqslant \Big(\frac{1}{2\pi}\int_0^1 \int_{\pi}^{\pi} |g(re^{i\theta})|^2 (1-r)^{2/\alpha-2}\, dr\, d\theta\Big)^{1/2} \times$$

$$\times \Big(\frac{1}{2\pi}\int_0^1 \int_{-\pi}^{\pi} |h(re^{i\theta})|^2 (1-r)^{2/\beta-2} dr\, d\theta\Big)^{1/2}.$$

Lemma 6 applied for $q = 2$, $p = \beta$, $\lambda = 2$ gives

$$\Big(\frac{1}{2\pi}\int_0^1 \int_{-\pi}^{\pi} |h(re^{i\theta})|^2 (1-r)^{2/\beta-2}\, dr\, d\theta\Big)^{1/2} \leqslant C\|h\|_\beta = C\|\varphi\|_p^{p/\beta} = C. \tag{4.11}$$

From (4.10) and (4.11) we get

$$\Big(\frac{1}{2\pi}\int_0^1 \int_{-\pi}^{\pi} |g(re^{i\theta})|^2 (1-r)^{2/\alpha-2}\, dr\, d\theta\Big)^{1/2} \geqslant C. \tag{4.12}$$

If we write $g(z) = \sum_{n=0}^{\infty} b_n z^n$ the direct evaluation of the integral in (4.12) gives

$$\Big(\sum_{n=0}^{\infty} |b_n|^2 n^{1-2/\alpha}\Big)^{1/2} \geqslant C. \tag{4.13}$$

Application of Hölder's inequality and the Hausdorff–Young theorem in (4.13) gives $(1/\alpha + 1/\alpha' = 1)$

$$C \leqslant \Big(\sum_{n=0}^{\infty} |b_n|^\alpha n^{\alpha-2}\Big)^{1/2\alpha} \Big(\sum_{n=0}^{\infty} |b_n|^{\alpha'}\Big)^{1/2\alpha'} \tag{4.14}$$

$$\leqslant \Big(\sum_{n=0}^{\infty} |b_n|^\alpha n^{\alpha-2}\Big)^{1/2\alpha} \Big\|\sum_{n=0}^{\infty} b_n z^n\Big\|_\alpha^{1/2} = \Big(\sum_{n=0}^{\infty} |b_n|^\alpha n^{\alpha-2}\Big)^{1/2\alpha}.$$

Let us now consider the operator $T\colon H_\alpha(D) \to l_\alpha$ given by

$$T\Big(\sum_{n=0}^{\infty} a_n z^n\Big) = (a_n n^{(\alpha-2)/\alpha})_{n=0}^{\infty}.$$

By [16], Theorem 6.2, it is a continuous linear operator. If $\varphi_n = g_n \cdot h_n$ we have by (4.14) $\|Tg_n\| \geqslant C$. Since $\varphi_n(z)$ converges pointwise to zero in D, the same is true about $g_n(z)$. This implies that $T(g_n)$ converges to zero coordinatewise in l_α. Using the standard gliding hump argument (cf. [26]) we infer that some subsequence of $T(g_n)$ is equivalent to the unit vector basis in l_α. This means that the subsequence of g_n satisfies the assumptions of Lemma 8. So for further subsequence we have a sequence of disjoint sets $A_k \subset T$ such that

$$\int_{A_k} |g_{n_k}|^\alpha = \int_{A_k} |\varphi_{n_k}|^p \geqslant \gamma^\alpha. \tag{4.15}$$

We have

$$\int_T \Big|\sum_k a_k \varphi_{n_k}\Big|^p \leqslant \sum_k \int_T |a_k \varphi_{n_k}|^p \leqslant \sum |a_k|^p. \tag{4.16}$$

To prove the other inequality we use the fact that φ_{n_k} is an unconditional basic sequence, so by the Khintchine inequality and (4.15) we have

$$\int_T \Big|\sum_k a_k \varphi_{n_k}(t)\Big|^p dt \geqslant C \int_0^1 \int_T \Big|\sum_k a_k r_k(s) \varphi_{n_k}(t)\Big|^p dt\, ds \tag{4.17}$$

$$\geqslant C \int_T \Big(\sum_k |a_k|^2 |\varphi_{n_k}(t)|^2\Big)^{p/2} dt$$

$$\geqslant C \sum_k \int_{A_k} |a_k|^p |\varphi_{n_k}(t)|^p \geqslant C\gamma^\alpha \sum_k |a_k|^p.$$

Inequalities (4.16) and (4.17) complete the proof of Proposition 8.

Proof of Proposition 9. It is enough to show (4.7) with $\psi_r(t)$ replaced by $\operatorname{Re}\psi_r(t)$. Let us denote

$$\operatorname{Re}\psi_r(t) = \eta_r(t) = \sum_{k_r+1}^{k_{r+1}} \beta_n g_n^m.$$

Let us take a p-atom $a(t)$, $\operatorname{supp} a(t) \subset I$, $\|a\|_2 \leqslant |I|^{1/2-1/p}$. Let us fix R such that $k_R^{-1} > |I| > k_{R+1}^{-1}$.

For $r < R$ we have by Proposition 6 and (3.1)

$$\Big|\int a(t)\eta_r(t)\,dt\Big| \leqslant C \sum_{k_r+1}^{k_{r+1}} n^{(1/2-1/p)} \Big|\int_T a(t) g_n^m(t)\,dt\Big|$$
$$\leqslant C\,|I|^{s+2-1/p} \sum_{s(r)}^{s(r+1)} 2^{n(1/2-1/p)} 2^{n(3/2+s)}$$
$$\leqslant C\, k_R^{-s-2+1/p} k_{r+1}^{2+s-1/p}$$

so

$$\sum_{r<R} \Big|\int a(t)\eta_r(t)\,dt\Big|^p \leqslant C k_R^{-s-2+1/p} \sum_{r<R} k_{r+1}^{2+s-1/p} \leqslant \text{const}. \tag{4.18}$$

To consider the case $r > R$ let us observe that Theorem 1 (c) implies

$$\sum_{n=2^k}^{2^{k+1}} |g_n^m(t)| \leqslant C\cdot 2^{k/2}. \tag{4.19}$$

From (4.19) and Proposition 6 we infer

$$\Big|\int a(t)\eta_r(t)\,dt\Big| \leqslant \|a\|_2\cdot \Big(\int_I \Big|\sum_{k_r+1}^{k_{r+1}} \beta_n g_n^m(t)\Big|^2 dt\Big)^{1/2}$$
$$\leqslant C\,\|a\|_2 \Big(\int_I \Big|\sum_{s(r)}^{s(r+1)} 2^{n(1/2-1/p)} 2^{n/2}\Big|^2 dt\Big)^{1/2}$$
$$\leqslant C\,\|a\|_2 |I|^{1/2} \sum_{s(r)}^{s(r+1)} 2^{n(1-1/p)} \leqslant C\,|I|^{1-1/p} 2^{s(r)(1-1/p)}$$

so

$$\sum_{r>R} \Big|\int a(t)\eta_r(t)\,dt\Big|^p \leqslant C\, k_{R+1}^{-(p-1)} 2^{s(R+1)(p-1)} \leqslant \text{const}. \tag{4.20}$$

We have

$$\Big|\int a(t)\eta_R(t)\,dt\Big| \leqslant \|a\|_p\cdot \|\psi_r\|_{H_p(D)^*} \leqslant \text{const}. \tag{4.21}$$

If we put together (4.18), (4.20) and (4.21) we get (4.7).

5. Unconditionality in L_p, $1 < p < \infty$. In this section we will concentrate our attention on T and $[0,1]$ and $p \geqslant 1$. Our main tool will be dyadic H_1 and interpolation theorems. On T we will work with systems $(G_n^{m,k})_{n=0}^{\infty}$ and on $[0,1]$ we will work with Ciesielski's systems $(f_n^{m,k})_{n\geqslant|k|-m}$. For the definition and detailed investigation of those systems we refer to [7]. Let us only remark that $(f_n^{m,0}) = (f_n^m)$ is a system of orthonormal splines on $[0,1]$, i.e., for $-m \leqslant n \leqslant 0$ (f_n^m) is the orthonormalisation of the monomials $1, t, \ldots, t^m$ and next we have an orthonormal system of splines of order m corresponding to a natural dyadic partition of $[0,1]$. $f_n^{m,k}$ denotes the kth derivative of f_n^m if $k \geqslant 0$ and $(-k)$th antiderivative if $k \leqslant 0$.

The difference between $(G_n^{m,k})$ considered on $[-1,1)$ and Ciesielski's systems lies in different behaviour at the endpoints. This brings in certain asymmetry.

The interpolation theorem we will be using is the following special case of Theorem D of [13].

PROPOSITION 10. *Let T be a continuous linear operator from H_1 into L_1 (in particular, from H_1 into H_1; any H_1-space we are considering works) and from L_2 into L_2. Then T is a continuous linear operator from L_p into L_p for $1 < p < 2$.*

A look at Theorem 2′ gives that $(G_n^{m,k})_{n=0}^{\infty}$ is an unconditional basis in $H_1(T)$ for $m \geqslant 0$ and $k \neq -m-1$. Since $(G_n^{m,k})_{n=0}^{\infty}$ is also an unconditional basis in $L_2(T)$, Proposition 10 gives that the system $(G_n^{m,k})_{n=0}^{\infty}$, $m \geqslant 0$, $|k| \leqslant m+1$, $m \neq -m-1$, is an unconditional basis in $L_p(T)$, $1 < p \leqslant 2$. In this range of m and k we can interpolate operators $S^{m,k}$ (cf. Theorem 5′ and Theorem 1′(a)). The results are summarised in

PROPOSITION 11. *The system $(G_n^{m,k})_{n=0}^{\infty}$, $m \geqslant 0$, $|k| \leqslant m+1$, $k \neq -m-1$ is an unconditional basis in L_p, $1 < p \leqslant 2$, equivalent to $n^k\chi_n$.*

By duality we get

PROPOSITION 12. *The system $(G_n^{m,k})_{n=0}^{\infty}$, $m \geqslant 0$, $|k| \leqslant m+1$, $k \neq m+1$ is an unconditional basis in L_p, $2 \leqslant p < \infty$, equivalent to $n^k\chi_n$.*

The next theorem allows us to consider also the exceptional cases $k = -m-1$ and $k = m+1$.

THEOREM 10. *For every sequence $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$, $\varepsilon_n = \pm 1$ we define an operator $T_\varepsilon^{m,k}$: $H_1(T, \delta) \to L_1(T)$ by*

$$T_\varepsilon^{m,k}\Big(\sum_{n=0}^{\infty} a_n G_n^{m,k}\Big) = \sum_{n=0}^{\infty} a_n \varepsilon_n G_n^{m,k}.$$

For $m \geqslant -1$ and $|k| \leqslant m+1$ we have $\sup_\varepsilon \|T_\varepsilon^{m,k}\| < \infty$.

Proof. Once more the proof is patterned after the proof of Theorem 2. Let a denote the dyadic 1-atom and let I be a dyadic interval such that $\operatorname{supp} a \subset I$, $\|a\|_2 \leqslant |I|^{-1/2}$. Let us denote $|I| = 2^{-N}$. Clearly, we have to show

$$\sup_s \sup_a \|T_s^{m,k} a\|_{L_1(T)} \leqslant \text{const}.$$

Let us write

$$T_s^{m,k}(a) = \sum_{n\leqslant 2^N} + \sum_{\substack{n>2^N\\ t_n\in 2I}} + \sum_{\substack{n>2^N\\ t_n\notin 2I}} \int a(t) G_n^{m,-k}(t)\,dt\, G_n^{m,k}$$
$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

To estimate $\sum_1$ we observe that because I is dyadic $G_n^{m,-k}|I$ is a polynomial, so if A is such that $\operatorname{supp} A \subset I$ and $A' = a$ we have

$$\left|\int a(t) G_n^{m,-k}(t)\,dt\right| = \left|\int A(t) G_n^{m,-k+1}(t)\,dt\right|$$
$$\leqslant \|A\|_2 \cdot \Big(\int_I |G_n^{m,-k+1}(t)|^2 dt\Big)^{1/2} \leqslant C\, 2^{-N/2} n^{-k+1} q^{nd(I,t_n)}.$$

Using this and the estimate $\|G_n^{m,k}\|_{L_1(T)} \leqslant Cn^{k-1/2}$ we get that $\|\sum_1\|_{L_1(T)} \leqslant$ const.

The sum $\sum_3$ is estimated exactly like (3.9). To estimate $\|\sum_2\|_{L_1}$ we may observe that dyadic 1-atom is 1-atom, so like in the proof of Theorem 2 we can show that $\sum_2$ is a 1-molecule, so its norm in $L_1(T)$ is uniformly bounded. We can also give a direct estimate as follows:

$$\int_{T\setminus 4I} \Big| \sum_{\substack{n\geqslant 2^N\\ t_n\in 2I}} \int a(t) G_n^{m,-k}(t)\,dt\, G_n^{m,k}(s)\Big|\,ds$$
$$\leqslant \int_{T\setminus 4I} \Big(\sum_{\substack{n\geqslant 2^N\\ t_n\in 2I}} \Big|\int a(t) G_n^{m,-k}(t)\,dt\Big|^2 n^{2k}\Big)^{1/2} \Big(\sum_{\substack{n\geqslant 2^N\\ t_n\in 2I}} \frac{|G_m^{m,k}(s)|^2}{n^{2k}}\Big)^{1/2} ds$$
$$\leqslant C\|a\|_2 \int_{T\setminus 4I} \Big(\sum_{\substack{n\geqslant 2^N\\ t_n\in 2I}} nq^{2nd(t_n,s)}\Big)^{1/2} ds$$
$$\leqslant C\|a\|_2 \int_{T\setminus 4I} \Big(\sum_{n=N}^{\infty} 2^n q^{2\cdot 2^n d(s,I)}\Big)^{1/2} ds$$
$$\leqslant C\|a\|_2 \sum_{n=N}^{\infty} 2^{n/2} \int_{T\setminus 4I} q^{2^n d(s,I)}\,ds$$
$$\leqslant C\|a\|_2 \sum_{n=N}^{\infty} 2^{n/2} \int_{2^{-N}}^{\infty} q^{2^n s}\,ds$$
$$\leqslant C\, 2^{N/2} \sum_{n=N}^{\infty} 2^{-n/2} \int_{2^{n-N}}^{\infty} q^u\,du \leqslant \text{const}.$$

Also

$$\int_{4I} \Big| \sum_{\substack{n\geqslant 2^N\\ t_n\in 2I}} \int a(t) G_n^{m,-k}(t)dt\, G_n^{m,k}(s)\Big|\,ds \leqslant \|a\|_2 \cdot 2\,|I|^{1/2} \leqslant 2.$$

This completes the proof of the theorem.

We can summarise our consideration as follows:

THEOREM 11. *The system $G_n^{m,k}$, $m \geqslant -1$, $|k| \geqslant m+1$ is an unconditional basis in $L_p(T)$, $1 < p < \infty$. If $|k| \leqslant m$ then this basis is equivalent to $n^k \chi_n$.*

The above theorem is a periodic analog of results of Ciesielski [8]. Unfortunately our method does not give the case $|k| = m+1$ in the equivalence result.

Remark 8. Despite Theorem 10 the system $(G_n^{m,k})_{n=0}^\infty$ need not be a basis for $H_1(T, \delta)$. The trouble is that the norm of $G_n^{m,k}$ in $H_1(T, \delta)$ can be substantially bigger than $n^{k-1/2}$.

Now we will briefly describe the situation for Ciesielski's systems. The argument fully analogous to the proofs of Theorem 2 and Theorem 5 gives

THEOREM 12. *The system $(f_n^{m,k})_{n\geqslant |k|-m}$, $m \geqslant -1$, $-m-1 \leqslant k \leqslant 0$, is an unconditional basis in $H_p[0,1]$ for $p > 1/(m+k+2)$. Moreover, the basis $(f_n^{m,k})_{n\geqslant |k|-m}$ in $H_p[0,1]$ is equivalent to the basis $n^k \chi_n$ in $H_p(\delta)$ for $p > 1/(m+k+2)$.*

For $k = 0$ this theorem was established in a different way in [30].

The trouble with the derivative (i.e., the case $k > 0$) is that its norm in $H_p[0,1]$ may be bigger than it should be. To indicate this we will prove

PROPOSITION 13. *Let f_n denote $f_n^{0,0}$. The norm of f'_{2^n} in $H_p[0,1]$ is greater than or equal to $C \cdot n \cdot 2^{n/2}$.*

Proof. The system f_n is the classical Franklin system investigated in detail in [5] and [6]. Let us define

$$\varphi(t) = \begin{cases} n & \text{if} \quad 0 \leqslant t \leqslant 2^{-n}, \\ -\log_2 t & \text{if} \quad 2^{-n} \leqslant t \leqslant 1. \end{cases}$$

It is an easy and well known exercise that $\|\varphi\|_{\mathrm{BMO}} \leqslant \mathrm{const}$ (cf. [25], Ex. 2.4). On the other hand,

$$\int_0^1 f'_{2^n}(t)\varphi(t)\,dt = f_{2^n}(t)\varphi_n\ (t)\Big|_0^1 - \int_0^1 f_{2^n}(t)\varphi'(t)\,dt.$$

Using Lemma 3 of [5] and exponential inequalities for Franklin functions we infer that the first summand is of the order of magnitude $n\cdot 2^{n/2}$. The second summand is estimated as

$$\Big|\int_0^1 f_{2^n}(t)\varphi'(t)\,dt\Big| \leqslant \int_0^1 |f_{2^n}(t)|\,dt\cdot \sup|\varphi'(t)| \leqslant C2^{n/2}.$$

Since $H_1[0,1]^* = \mathrm{BMO}$, those inequalities prove the proposition.

Nevertheless we can repeat the proof of Theorem 10 to get

THEOREM 13. *The operator* $T_\varepsilon^{m,k}$, $\varepsilon = (\varepsilon_n)$, $\varepsilon_n = \pm 1$, $m \geqslant -1$, $|k| \leqslant m+1$, *defined by*

$$T_\varepsilon^{m,k}(f_n^{m,k}) = \varepsilon_n n^k \chi_n$$

is a continuous map from $H_1(\delta)$ *into* L_1, *and* $\sup_\varepsilon \|T_\varepsilon^{m,k}\| < \infty$.

This theorem and interpolation give

COROLLARY 4. *The system* $(f_n^{m,k})_{n \geqslant |k|-m}$, $m \geqslant -1$, $|k| \leqslant m+1$, *is an unconditional basis in* $L_p[0,1]$, $1 < p < \infty$.

Remark 9. Theorems 10 and 13 can be extended to $p < 1$ also. The extension is obvious. It was not given because we do not see interesting applications.

Remark 10. Like in Remark 7 and Propositions 6 and 7 there is a connection between smooth functions on $[0,1]$ and $H_p[0,1]^*$. We can also get the characterisations of smooth functions in terms of coefficients with respect to the systems $(f_n^{m,k})$. Those results have been obtained in [7].

The q-envelopes of $H_p[0,1]$, $p < q \leqslant 1$, have been described by Aleksandrov [1].

Addendum (October 1982). After this paper has been completed in November 1981 some additional results connected with splines in H_p-spaces were obtained. P. Sjölin and J.-O. Stromberg (*Spline systems as bases in Hardy spaces*, Dept. of Math. Stockholms Univ. Report No. 1, 1982) have shown that the Ciesielski system $(f_n^{m,0})$ is not an unconditional basis in $H_p[0,1]$ for $p = (m+2)^{-1}$ (cf. Remark 5). J.-O. Stromberg has constructed a very nice systems of splines on R^n for which the results of [30] hold (*A modified Franklin system and higher order spline systems on R^n as unconditional basis of Hardy spaces*, Dept. of Math. Stockholms Univ. Report No. 21, 1981). This construction of J.-O. Stromberg have been modified by the author in order to show that Hardy spaces of analytic functions on the complex ball are isomorphic for $1 \leqslant p < \infty$ to Hardy spaces on the disc (Annals of Mathematics 118(1983), 21–34). Almost everywhere convergence of spline expansions of H_p functions was investigated by P. Sjölin (Dept. of Math. Stockholms Univ. Report No. 25, 1982).

References

[1] A. B. Aleksandrov, *Essays on non locally convex Hardy classes*, Springer-Verlag, Lecture Notes in Math. vol. 864, 1–89.
[2] P. Billard, *Bases dans H et bases de sous espaces de dimension finie dans A*, Proc. Conf. Oberwolfach (August 14–22, 1971) ISNM vol. 20, Birkhauser, Bassel and Stuttgart 1972.
[3] D. Burkholder, R. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. 124 (1970), 249–304.
[4] L. Carleson, *An explicit unconditional basis in H_1*, Bull. Sci. Math. 104 (1980), 405–416.
[5] Z. Ciesielski, *Properties of the orthonormal Franklin system*, Studia Math. 23 (1963), 141–157.
[6] — *Properties of the orthonormal Franklin system*, II, ibid. 27 (1966), 289–323.
[7] — *Constr uctive function theory and spline systems*, ibid. 53 (1975), 277–302.
[8] — *Equivalence, unconditionality and convergence a.e. of the spline bases in L_p spaces*, in: *Approximation theory*, Banach Center Publications, vol. 4, Warszawa 1979, 55–68.
[9] — *Bases and approximation by splines*, Proc. Intl. Congress of Mathematicians, Vancouver 1974, 47–51.
[10] Z. Ciesielski, T. Figiel, *Spline bases in classical function spaces on compact C^n manifolds*, I, II, Studia Math. 76 (1983), 1–58, 95–136.
[11] R. R. Coifman, *A real characterisation of H_p*, ibid. 51 (1974), 269–274.
[12] R. R. Coifman, R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L_p*, Asterisque 77 (1980), 1–65.
[13] R. R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.
[14] S. Demko, *Inverses of band matrices and local convergence of spline projections*, SIAM J. Numer. Anal. 14 (1977), 616–619.
[15] J. Domsta, *A theorem on B-splines II, the periodic case*, Bull. Acad. Polon. Sci. 24 (1976), 1077–1084.
[16] P. L. Duren, *Theory of H_p-spaces*, Academic Press, 1970.
[17] P. L. Duren, B. W. Romberg, A. L. Shields, *Linear functionals on H_p spaces with $0 < p < 1$*, J. Reine Angew. Math. 238 (1969), 32–60.
[18] P. Enflo, H. P. Rosenthal, *Some results concerning $L^p(\mu)$ spaces*, J. Functional Analysis 14 (1973), 325–348.

[19] A. Garsia, *Martingale inequalities: seminar notes on recent progress*, W. A. Benjamin, Reading, Mass. 1973.
[20] G. H. Hardy, J. E. Littlewood, *Some properties of fractional integrals* II, Math. Z. 34 (1932), 403–439.
[21] C. Herz, *H_p-spaces of martingales, $0 < p \leqslant 1$*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete 28 (1974), 189–209.
[22] N. J. Kalton, D. A. Trautman, *Remarks on subspaces of H_p when $0 < p < 1$*, Michigan Math. J. 29 (1982), 163–171.
[23] S. Kwapień, A. Pełczyński, *Some linear topological properties of the Hardy spaces H_p*, Compositio Math. 33 (1976), 261–288.
[24] B. Maurey, *Isomorphismes entre espaces H_1*, Acta Math. 145 (1980), 79–120.
[25] U. Neri, *Some properties of functions with bounded mean oscillation*, Studia Math. 61 (1977), 63–75.
[26] A. Pełczyński, *Projections in certain Banach spaces*, ibid. 19 (1960), 209–228.
[27] S. Ropela, *Spline bases in Besov spaces*, Bull. Acad. Polon. Sci. 24 (1976), 319–325.
[28] L. Schumaker, *Spline Functions: Basic Theory*, J. Wiley & Sons, New York 1981.
[29] J. H. Shapiro, *Mackey topologies, reproducing kernels, and diagonal maps on the Hardy and Bergman spaces*, Duke Math. J. 43 (1976), 197–202.
[30] P. Sjölin, J.-O. Stromberg, *Basis properties of Hardy spaces*, Ark. Mat. 21.1 (1983), 111–125
[31] M. H. Taibleson, G. Weiss, *The molecular characterisation of cetrain Hardy spaces*, Asterisque 77 (1980), 67–150.
[32] P. Wojtaszczyk, *The Franklin system is an unconditional basis in H_1*, Ark. Mat. Ark. Mat. 20.2 (1982), 293–300.

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