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On the canonical extensions for distributions in the space $B_{p,\mu}$

by

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Abstract. We consider the problem of extension of $u \in \mathscr{D}'(R_{n+1}^+)$, $R_{n+1}^+ = \{(x, t) \in R_{n+1};\ t > 0\}$, over R_{n+1} so as to vanish for $t < 0$. The canonical extension $(u \,|\, R_{n+1}^+)_\sim$ exists for every $u \in B_{p,\mu}(R_{n+1})$ with $1 \leqslant p < \infty$ if and only if $(1+\tau^2)^{-1/2}\mu^{-1}(0, \tau) \in L^{p'}$ for $p > 1$ and $\inf(1+\tau^2)^{1/2}\mu(0, \tau) > 0$ for $p = 1$, where $1/p+1/p' = 1$. Another equivalent conditions are given in connection with the notion of multiplication of distributions.

Let R_N be an N-dimensional Euclidean space and let Ξ_N be its dual. A positive-valued continuous function $\mu(\xi)$ defined on Ξ_N is called a *temperate weight function* if there exist positive constants C and k such that $\mu(\xi+\eta) \leqslant C(1+|\xi|^k)\mu(\eta)$ for every $\xi, \eta \in \Xi_N$. By $B_{p,\mu}(R_N)$, $1 \leqslant p \leqslant \infty$, we shall understand the space of $u \in \mathscr{S}'(R_N)$ such that its Fourier transform $\hat{u}$ is a locally summable function and

$$\|u\|_{p,\mu}^p = (2\pi)^{-N} \int_{\Xi^N} |\hat{u}(\xi)|^p \mu^p(\xi)\, d\xi < \infty$$

and when $p = \infty$ we shall interpret $\|u\|_{\infty,\mu}$ as $\operatorname{ess\,sup} |\hat{u}(\xi)\mu(\xi)|$.

We shall assume that $N = n+1$ and a point of R_{n+1} will be denoted by (x, t), $x = (x_1, \ldots, x_n)$. Let $\varrho(t)$ be any real-valued C^∞ function of $t \in R_t$ equal to 1 for $t \geqslant 2$ and 0 for $t \leqslant 1$ and put $\varrho_{(\varepsilon)}(t) = \varrho(t/\varepsilon)$ for $\varepsilon > 0$. For any $u \in \mathscr{D}'(R_{n+1}^+)$, $R_{n+1}^+ = \{(x, t);\ t > 0\}$, we may consider $\varrho_{(\varepsilon)}u$ as a distribution extended over R_{n+1} so as to vanish for $t < \varepsilon$. If $\varrho_{(\varepsilon)}u$ converges in $\mathscr{D}'(R_{n+1})$ to $u_\sim$ for any ϱ as $\varepsilon \downarrow 0$, $u_\sim$ will be called a *canonical extension of u over $t = 0$*. Let $u \in \mathscr{D}'(R_{n+1})$. If the restriction $u \,|\, R_{n+1}^+$ has the canonical extension $(u \,|\, R_{n+1}^+)_\sim$ we shall denote it by u_+ and call it also the *canonical extension of u* in this paper.

The present paper is a continuation of the previous paper [3]. The purpose of this paper is to investigate the canonical extensions for distributions in the space $B_{p,\mu}(R_{n+1})$ in connextion with multiplication of distributions.

Given a distribution $u \in \mathscr{D}'(R_{n+1}^+)$ we understand the distributional boundary value $\lim_{t\downarrow 0} u = \alpha \in \mathscr{D}'(R_n)$ as follows: put

$$\chi = \varrho' \quad \text{and} \quad \chi_\varepsilon(t) = (1/\varepsilon)\chi(t/\varepsilon) \quad \text{for} \quad \varepsilon > 0$$

and consider $\chi_\varepsilon(t)u$ as a distribution in $\mathscr{D}'(R_{n+1})$. If the distributional limit $\lim_{\varepsilon\downarrow 0}\chi_\varepsilon u$ exists in $\mathscr{D}'(R_{n+1})$ and equals $\alpha\otimes\delta$ with $\alpha \in \mathscr{D}'(R_n)$ and Dirac measure $\delta \in \mathscr{D}'(R_t)$ for any choice of ϱ with the properties just indicated, we define $\alpha = \lim_{t\downarrow 0} u$. We can also write

$$\lim_{\varepsilon\downarrow 0}\langle u(\cdot, t), \chi_\varepsilon(t)\rangle = \alpha \quad \text{in} \quad \mathscr{D}'(R_n),$$

namely,

$$\lim_{\varepsilon\downarrow 0} u(x, \varepsilon t) = \alpha \otimes Y \quad \text{in} \quad R_{n+1}^+,$$

where Y is the Heaviside function. In view of a theorem of S. Łojasiewicz [5], p. 21, $\lim_{t\downarrow 0} u = \alpha$ means that for any bounded open non-empty subset $G\times(-\varepsilon_0, \varepsilon_0) \subset R_{n+1}$ there exist a non-negative integer k, a multi-index $s = (s_1, \ldots, s_n)$ of non-negative integers and a continuous function $F(x,t)$ on $G\times(-\infty, \varepsilon_0)$ such that $u = \alpha\otimes Y + D_x^s D_t^k F(x,t)$ on $G\times(0,\varepsilon_0)$ and $F = o(t^k)$ uniformly on G as $t\to 0$.

By a *restricted δ-sequence* in $\mathscr{D}(R_t^+)$ we understand every sequence of non-negative functions $\varrho_n \in \mathscr{D}(R_t^+)$ with the following properties [7], p. 91:

(1) $\operatorname{supp}\varrho_j$ converges to $\{0\}$ as $j\to\infty$;

(2) $\int_0^\infty \varrho_j(t)\,dt$ converges to 1 as $j\to\infty$;

(3) $\int_0^\infty t^k|D^k\varrho_j|\,dt \leqslant M_k$ (M_k being independent of j).

PROPOSITION 1. *Let $u \in \mathscr{D}'(R_{n+1}^+)$. u has the boundary value α if and only if $\lim_{j\to\infty}\langle u(\cdot, t), \varrho_j\rangle = \alpha$ for every restricted δ-sequence $\{\varrho_j\}$ in $\mathscr{D}(R_t^+)$.*

Proof. Suppose $\lim_{t\downarrow 0} u = \alpha$ and take a bounded open subset $G\times(0, \varepsilon_0)$ of R_{n+1}^+. Then we can write

$$u = \alpha\otimes Y + D_x^s D_t^k F(x,t) \quad \text{in} \quad G\times(0,\varepsilon_0)$$

with a continuous function $F(x,t)$ on $G\times(-\infty, \varepsilon_0)$ such that $F = o(t^k)$ uniformly as $t\to 0$. We have for any $\varphi\in\mathscr{D}(G)$

$$\langle u, \varphi(x)\varrho_j(t)\rangle = \langle \alpha, \varphi\rangle \int \varrho_j\,dt + (-1)^{|s|+k}\langle F, D_x^s\varphi D_t^k\varrho_j\rangle$$

and

$$|\langle F, D_x^s\varphi D_t^k\varrho_j\rangle| \leqslant M_k \sup_{x\in G}|F(x,t)t^{-k}| \int |D_x^s\varphi|\,dx,$$

which shows that $\lim_{j\to\infty}\langle u(\cdot, t), \varrho_j\rangle$ exists and is equal to α.

Conversely, suppose that $\lim_{j\to\infty}\langle u(\cdot, t), \varrho_j\rangle = \alpha$ exists for every restricted δ-sequence $\{\varrho_j\}$ in $\mathscr{D}(R_t^+)$. Let ψ be any non-negative function $\in\mathscr{D}(R_t^+)$ such that $\int\psi(t)\,dt = 1$. Put $\psi_{\lambda_j}(t) = (1/\lambda_j)\psi(t/\lambda_j)$ for a sequence $\{\lambda_j\}$ of positive numbers such that $\lambda_j\downarrow 0$ as $j\to\infty$. Then $\{\psi_{\lambda_j}\}$ is a restricted δ-sequence and therefore

$$\lim_{j\to\infty}\langle u(\cdot, \lambda_j t), \psi\rangle = \lim_{j\to\infty}\langle u(\cdot, t), \psi_{\lambda_j}(t)\rangle = \alpha,$$

which means $\lim_{t\downarrow 0} u = \alpha$.

COROLLARY 1. *Let $u\in\mathscr{D}'(R_{n+1}^+)$. u has the boundary value $\lim_{t\downarrow 0} u = \alpha$ if and only if $\lim_{j\to\infty}\varrho_j u = \alpha\otimes\delta$ for every restricted δ-sequence $\{\varrho_j\}$ in $\mathscr{D}(R_t^+)$.*

Let $u\in\mathscr{D}(R_{n+1}^+)$. If $\langle u, \varrho_j\rangle$ has the distributional limit α in $\mathscr{D}'(R_n)$ for every δ-sequence $\{\varrho_j\}$ in $\mathscr{D}(R_t^+)$, we call it the *boundary value of u in the strict sense* and denote it by $\text{s-}\lim_{t\downarrow 0} u$.

COROLLARY 2. *Let $u\in\mathscr{D}'(R_{n+1}^+)$. u has the boundary value $\text{s-}\lim_{t\downarrow 0} u = \alpha$ if and only if $\lim_{j\to\infty}\varrho_j u = \alpha\otimes\delta$ for every δ-sequence $\{\varrho_j\}$ in $\mathscr{D}(R_t^+)$.*

Let $u\in\mathscr{D}'(R_{n+1})$. If $\lim_{t\downarrow 0}(u\,|\,R_{n+1}^+)$ exists, we shall also call it in this paper the *boundary value of* u and denote it by $\lim_{t\downarrow 0} u$. The notations $\lim_{t\uparrow 0} u$, $\text{s-}\lim_{t\downarrow 0} u$ and $\text{s-}\lim_{t\uparrow 0} u$ will have obvious meaning.

Let $u, v\in\mathscr{D}'(R_{n+1})$. If the distributional limit $\lim_{j\to\infty}(u*\varrho_j)v$ exists for every δ-sequence $\varrho_j\in\mathscr{D}(R_{n+1})$, the limit is called the *product in the strict sense* and denoted by $u\cdot v$. The *partial product $w\cdot u$ in the strict sense* between $w\in\mathscr{D}'(R_t)$ and $u\in\mathscr{D}'(R_{n+1})$ means $(1\otimes w)\cdot u$ if it exists. The above two definitions with $\{\varrho_j\}$ replaced by the restricted δ-sequence in $\mathscr{D}(R_{n+1})$ [7], p. 91, yield the product uv and the partial product wu.

In accordance with S. Łojasiewicz [5], p. 15, $u\in\mathscr{D}'(R_{n+1})$ has a section $\alpha\in\mathscr{D}'(R_n)$ for $t = 0$ if $\lim_{\varepsilon\downarrow 0}\langle u, \varphi_\varepsilon\rangle = \alpha$ for any $\varphi\in\mathscr{D}(R_t)$ with $\varphi(t)\geqslant 0$ and $\int\varphi(t)\,dt = 1$. We can also write $\lim_{\varepsilon\downarrow 0} u(x, \varepsilon t) = \alpha\otimes 1_t$. If the distributional limit $\lim_{j\to\infty}\langle u, \varrho_j\rangle$ exists for every δ-sequence $\varrho_j\in\mathscr{D}(R_t)$, then the limit is called the *section of u in the strict sense.*

In our previous papers [4], [5], we have investigated the trace mapping for the space $B_{p,\mu}(R_{n+1})$ with $1\leqslant p<\infty$. If the mapping $\mathscr{D}(R_{n+1})\ni u\to u(x,0)\in\mathscr{D}'(R_n)$ can be continuously extended from $B_{p,\mu}(R_{n+1})$ into $\mathscr{D}'(R_n)$, then the extended mapping is called a *trace mapping*. The image of $u\in B_{p,\mu}(R_{n+1})$ by this mapping is called the *trace of u on $t = 0$*

and denoted by $u(x, 0)$. The trace mapping is defined if and only if $\mu^{-1}(0, \tau) \in L^{p'}$ with $p^{-1}+p'^{-1} = 1$. In this case, the trace $u(x, 0)$ of $u \in B_{p,\mu}(R_{n+1})$ belongs to the space $B_{p,\nu_{p'}}(R_n)$, where $\nu_{p'}(\xi) = \{\int \mu^{-p'}(\xi, \tau)d\tau\}^{-1/p'}$ for $p > 1$ $\inf_{\tau} \tilde{\mu}(\xi, \tau)$ for $p = 1$ with notation in [5], p. 562. We have

THEOREM 1. *For the space $B_{p,\mu}(R_{n+1})$ with $1 \leqslant p < \infty$, the following statements are equivalent:*

(1) *The trace mapping $B_{p,\mu}(R_{n+1}) \ni u \to u(x, 0) \in \mathscr{D}'(R_n)$ is defined.*

(2) *The section of u for $t = 0$ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(2)′ *Condition* (2) *holds in the strict sense.*

(3) *The partial product δu exists for every $u \in B_{p,\mu}(R_{n+1})$, where δ is the Dirac measure in R_t.*

(3)′ *The partial product $\delta \cdot u$ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(4) *The distributional limit $\lim_{j\to\infty}(1 \otimes \delta)(u * \varrho_j)$ exists for a fixed restricted δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_{n+1})$, for every $u \in B_{p,\mu}(R_{n+1})$.*

(4)′ *The distributional limit $\lim_{j\to\infty}(1 \otimes \delta)(u * \varrho_j)$ exists for a fixed δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_{n+1})$, for every $u \in B_{p,\mu}(R_{n+1})$.*

(5) *The distributional limit $\lim_{j\to\infty} \varrho_j u$ exists for a fixed restricted δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$, for every $u \in B_{p,\mu}(R_{n+1})$.*

(5)′ *The distributional limit $\lim_{j\to\infty} \varrho_j u$ exists for a fixed δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$, for every $u \in B_{p,\mu}(R_{n+1})$.*

(6) *The boundary value $\lim_{t\downarrow 0} u$ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(6)′ *The boundary value $\text{s-}\lim_{t\downarrow 0} u$ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(7) *The distributional limit $\lim_{j\to\infty} \varrho_j u$ exists for a fixed restricted δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$ with $\operatorname{supp} \varrho_j \subset (0, \infty)$.*

(7)′ *The distributional limit $\lim_{j\to\infty} \varrho_j u$ exists for a fixed δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$ with $\operatorname{supp} \varrho_j \subset (0, \infty)$.*

Proof. We have shown in [4], p. 178, that the statements (1), (2), (2)′, (3), (3)′, (4) and (5) are equivalent. The implications (3)′ ⇒ (4)′ and (3)′ ⇒ (5)′ are trivial from the definition of the product in the strict sense between distributions. The proofs of the implications (4)′ ⇒ (1) and (5)′ ⇒ (1) are carried out with any δ-sequences instead of the restricted δ-sequences in the proofs of (4) ⇒ (1) and (5) ⇒ (1) and the implications (3) ⇒ (6) ⇒ (7) and (3)′ ⇒ (6)′ ⇒ (7)′ are trivial. The implications (7) ⇒ (1) and (7)′ ⇒ (1) can be proved the same line as in the proof of (5) ⇒ (1).

Remark. We can replace $\lim_{t\downarrow 0} u$ and $\text{s-}\lim_{t\downarrow 0} u$ in (6) and (6)′ by $\lim_{t\uparrow 0} u$ and $\text{s-}\lim_{t\uparrow 0} u$, respectively, and we have $\lim_{t\downarrow 0} u = \lim_{t\uparrow 0} u = \text{s-}\lim_{t\downarrow 0} u = \text{s-}\lim_{t\uparrow 0} u$.

PROPOSITION 2. *If $\lim_{t\downarrow 0} u$ exists for every $u \in B_{p,\mu}(R_{n+1})$ with $1 \leqslant p < \infty$, then $\lim_{j\to\infty} \varrho_j u = a \otimes \delta$ in $B_{p,\varkappa}(R_{n+1})$ for every δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$, where $a = \lim_{t\downarrow 0} u$ and $\varkappa = (1+\tau^2)^{-k/2}\nu_{p'}(\xi)$ with $kp > 1$.*

Proof. We see from Proposition 4 in [4], p. 180, that u can be identified as a $B_{p,\nu_{p'}}(R_n)$-valued continuous function of t. We have for any $\psi \in \mathscr{D}(R_{n+1})$

$$|\langle \varrho_j u, \psi \rangle| \leqslant \int \varrho_j(t) \|u(\cdot, t)\|_{p,\nu_{p'}} \|\psi(\cdot, t)\|_{p',1/\nu_{p'}} dt$$
$$\leqslant \max \|u(\cdot, t)\|_{p,\nu_{p'}} \max \|\psi(\cdot, t)\|_{p',1/\nu_{p'}},$$

where $\|u(\cdot, t)\|_{p,\nu_{p'}} \leqslant C_1 \|u\|_{p,\mu}$ and

$$\|\psi(\cdot, t)\|_{p',1/\nu_{p'}} = \Big\{(2\pi)^{-n} \int_{\Xi^n} \Big|(2\pi)^{-1} \int \hat{\psi}(\xi, \tau) e^{i\langle t,\tau\rangle} d\tau\Big|^{p'} \nu_{p'}^{-p'}(\xi)\, d\xi\Big\}^{1/p'}$$
$$\leqslant \Big\{(2\pi)^{-n-p'} \iint |\hat{\psi}|^{p'} (1+\tau^2)^{kp'/2} \nu_{p'}^{-p'}(\xi)\, d\xi d\tau\Big\}^{1/p'} \Big\{\int (1+\tau^2)^{-kp/2} d\tau\Big\}^{1/p}$$
$$= C_2 \|\psi\|_{p',\varkappa}.$$

The set $\{\varphi_1 \otimes \varphi_2;\ \varphi_1 \in \mathscr{D}(R_n),\ \varphi_2 \in \mathscr{D}(R_t)\}$ is total in $B_{p,\mu}(R_{n+1})$ and

$$\|\varrho_j(\varphi_1 \otimes \varphi_2) - \varphi_2(0)\varphi_1(x)\delta\|_{p,\varkappa} = \|\varrho_j\varphi_2 - \varphi_2(0)\delta\|_{p,(1+\tau^2)^{-k/2}} \|\varphi_1\|_{p,\nu_{p'}},$$

where $\varrho_j\varphi_2$ tends to $\varphi_2(0)\delta$ in $B_{p,(1+\tau^2)^{-k/2}}$ as $j \to \infty$. By virtue of the Banach–Steinhaus theorem, we see that $\varrho_j u$ converges in $B_{p,\mu}(R_{n+1})$ to $a \otimes \delta$.

PROPOSITION 3. *If $\lim_{t\downarrow 0} u$ exists for every $u \in B_{p,\mu}(R_{n+1})$ with $1 \leqslant p < \infty$, then $(1 \otimes \delta)(u * \varrho_j)$ tends in $B_{p,\varkappa}(R_{n+1})$ to $u(x, 0)$ as $j \to \infty$ for every δ-sequence $\{\varrho_j\}$ in $\mathscr{D}(R_{n+1})$, where $\varkappa = (1+\tau^2)^{-k/2}\nu_{p'}(\xi)$ with $kp > 1$.*

Proof. The trace mapping $B_{p,\mu}(R_{n+1}) \ni u \to u(x, 0) \in \mathscr{D}'(R_n)$ is defined and $u(x, 0)$ belongs to the space $B_{p,\nu_{p'}}(R_n)$. Furthermore, we have

$$\|(u * \varrho_j)(x, 0) - u(x, 0)\|_{p,\nu_{p'}} \leqslant (2\pi)^{-1/p'} \|u * \varrho_j - u\|_{p,\mu}$$

for any $u \in B_{p,\mu}(R_{n+1})$ [4], p. 169, and $u * \varrho_j$ converges in $B_{p,\mu}(R_{n+1})$ to u ([4], p. 179). Thus $(1 \otimes \delta)(u * \varrho_j) = (u * \varrho_j)(x, 0)\delta$ tends in $B_{p,\varkappa}(R_{n+1})$ to $u(x, 0)\delta$ as $j \to \infty$, where $\varkappa = (1+\tau^2)^{-k/2}\nu_{p'}(\xi)$ with $kp > 1$.

Let P be a non-trivial polynomial of (ξ, τ) and $P(D)$ be the differential operator.

PROPOSITION 4. *If the boundary value $\lim_{t\downarrow 0} P(D)u$ exists for every $u \in B_{p,\mu}(R_{n+1})$ with $1 \leqslant p < \infty$, then the trace mapping $B_{p,\mu}(R_{n+1}) \ni u \to [P(D)u](x, 0) \in \mathscr{D}'(R_n)$ is defined.*

Proof. The distributional limit $\lim_{j\to\infty} \varrho_j P(D)u$ exists for every restricted δ-sequence $\{\varrho_j\}$ in $\mathscr{D}(R_t)$ with support $\subset (0, \infty)$ and $\lim_{j\to\infty} \varrho_j P(D)u$

$= \alpha\otimes\delta$ with $\alpha \in \mathscr{D}'(R_n)$. Since the map $\mathscr{D}'(R_{n+1}) \ni u \to \varrho_j u \in \mathscr{D}'(R_{n+1})$ is continuous, the map $B_{p,\mu}(R_{n+1}) \ni u \to \varrho_j P(D)u \in \mathscr{D}'(R_{n+1})$ is continuous. By the Banach–Steinhaus theorem the map $B_{p,\mu}(R_{n+1}) \ni u \to \lim_{j\to\infty} \varrho_j P(D)u \in \mathscr{D}'(R_{n+1})$ is continuous. From the fact that $\lim_{j\to\infty} \varrho_j P(D)u = \alpha\otimes\delta$ for any $u \in \mathscr{D}(R_{n+1})$ we can conclude that the trace mapping $B_{p,\mu}(R_{n+1}) \ni u \to [P(D)u](x, 0) \in \mathscr{D}'(R_n)$ is defined.

In our previous papers [4], p. 168; [5], p. 563, we have given

Proposition 5. *A necessary and sufficient condition that the trace mapping $B_{p,\mu}(R_{n+1}) \ni u \to [P(D)u](x, 0) \in \mathscr{D}'(R_n)$ may be defined is that one of the following conditions is satisfied: In the case $p > 1$*

(1) $\mu_{\tilde{P},p'}(\xi) = \left\{\int_{\Xi^n} \tilde{P}_{p'}^{p'}(\xi, \tau)\mu^{-p'}(\xi, \tau)d\tau\right\}^{-1/p'} > 0$ *for some* $\xi \in \Xi_n$,

(2) $\int_{\Xi^n} |P(\xi, \tau)|^{p'}\mu^{-p'}(\xi, \tau)d\tau < \infty$ *for every* $\xi \in \Xi_n$;

and in the case $p = 1$

(1) $\sup_\tau \tilde{P}_\infty(\xi, \tau)\mu^{-1}(\xi, \tau) < \infty$ *for some* $\xi \in \Xi_n$,

(2) $\sup_\tau |P(\xi, \tau)|\mu^{-1}(\xi, \tau) < \infty$ *for every* $\xi \in \Xi_n$,

where $\tilde{P}_p(\xi, \tau) = \left(\sum_{|\alpha|\geqslant 0} |P^{(\alpha)}(\xi, \tau)|^p\right)^{1/p}$ *and* $\tilde{P}_\infty(\xi, \tau) = \max_{|\alpha|\geqslant 0} |P^{(\alpha)}(\xi, \tau)|$.

Proposition 6. *If the trace mapping $B_{p,\mu}(R_{n+1}) \ni u \to [P(D)u](x, 0) \in \mathscr{D}'(R_n)$ is defined, then the partial product $\delta\cdot P(D)u$ exists for every $u \in B_{p,\mu}(R_{n+1})$, where δ is Dirac measure in R_t and $1 \leqslant p < \infty$.*

Proof. Let $\varphi \in \mathscr{D}(R_n)$. Since $P(D)(\psi\otimes\delta) \in B_{p',1/\mu}(R_{n+1})$ for every $\psi \in \mathscr{D}(R_n)$ we have for every δ-sequence $\{\varrho_j\}$ in $\mathscr{D}(R_{n+1})$

$$\langle(1\otimes\delta)(P(D)u * \varrho_j), \varphi\rangle = \langle u * \varrho_j, P^*(D)(\varphi(x, 0)\otimes\delta)\rangle_{B_{p,\mu}, B_{p',1/\mu}}.$$

From the fact that $u * \varrho_j$ converges in $B_{p,\mu}(R_{n+1})$ to u, we can conclude that the product $(1\otimes\delta)\cdot(P(D)u)$ in the strict sense exists, that is, the partial product $\delta\cdot P(D)u$ exists. Namely, $P(D)u$ has the section in the strict sense.

The above considerations yield the following

Theorem 2. *For the space $B_{p,\mu}(R_{n+1})$ with $1 \leqslant p < \infty$, the following statements are equivalent:*

(1) *The trace mapping $B_{p,\mu}(R_{n+1}) \ni u \to [P(D)u](x, 0) \in \mathscr{D}'(R_n)$ is defined.*

(2) $\lim_{t\downarrow 0} P(D)u$ *exists for every* $u \in B_{p,\mu}(R_{n+1})$.

(2)′ $\operatorname{s-lim}_{t\downarrow 0} P(D)u$ *exists for every* $u \in B_{p,\mu}(R_{n+1})$.

(3) *The partial product $\delta P(D)u$ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(3)′ *The partial product $\delta\cdot P(D)u$ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(4) *The section of $P(D)u$ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(4)′ *The section of $P(D)u$ exists in the strict sense for every $u \in B_{p,\mu}(R_{n+1})$.*

Proof. The implications $(3)'\Rightarrow(2)'\Rightarrow(2)$ and $(3)'\Rightarrow(3)\Rightarrow(2)$ are trivial. The implications $(2)\Rightarrow(1)$ and $(1)\Rightarrow(3)'$ are followed by Proposition 4 and Proposition 6 respectively. The equivalence of (3)′ and (4)′ (resp. (3) and (4)) is shown in Theorem 1.

Corollary. *Let P, Q be polynomials such that the degree of P in τ is larger than the degree of Q in τ. If $\lim_{t\downarrow 0} P(D)u$ exists for every $u \in B_{p,\mu}(R_{n+1})$, then $\lim_{t\downarrow 0} Q(D)u$ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

Proof. $\tilde{P}_{p'}(0, \tau)\mu^{-1}(0, \tau) \in L^{p'}$ by Proposition 5 and so $\tilde{Q}_{p'}(0, \tau)\mu^{-1}(0, \tau) \in L^{p'}$. Thus the trace mapping $B_{p,\mu}(R_{n+1}) \ni u \to [Q(D)u](x, 0) \in \mathscr{D}'(R_n)$ exists and therefore $\lim_{t\downarrow 0} Q(D)u$ exists for every $u \in B_{p,\mu}(R_{n+1})$.

Corollary. *Let P be a polynomial in (ξ, τ) and m be the degree of τ. Then the following statements are equivalent:*

(1) $\lim_{t\downarrow 0} P(D)u$ *exists for every* $u \in B_{p,\mu}(R_{n+1})$.

(2) $\tau^m\mu^{-1}(0, \tau) \in L^{p'}$.

(3) $\lim_{t\downarrow 0} u, \ldots, \lim_{t\downarrow 0} D_t^m u$ *exist for every* $u \in B_{p,\mu}(R_{n+1})$.

Proof. It is sufficient only to show the implication $(1)\Rightarrow(2)$. Suppose (1) holds. Owing to Proposition 4 the trace mapping

$$B_{p,\mu}(R_{n+1}) \ni u \to [P(D)u](x, 0) \in \mathscr{D}'(R_n)$$

is defined and therefore $\int |P(\xi, \tau)|^{p'}\mu^{-p'}(\xi, \tau)d\tau < \infty$ for every $\xi \in \Xi_n$. From the expression $P(\xi, \tau) = \sum_{j=0}^m \tau^j\gamma_j(\xi)$ with $\gamma_m \not\equiv 0$, we see that $\tau^m\mu^{-1}(0, \tau) \in L^{p'}$.

Let $u \in \mathscr{D}'(R_{n+1})$. If $\lim_{t\downarrow 0} u$ exists, then the canonical extension $(u\,|\,R_{n+1}^+)_\sim = u_+$ exists. If the partial product Yu exists, then u_+ exists. In fact, let $\varrho(t)$ be any real-valued C^∞ function of $t \in R_t$ equal to 1 for $t \geqslant 2$ and 0 for $t \leqslant 1$ and put $\chi = \varrho'$. Then $\chi \in \mathscr{D}'(R_t)$, $\int \chi dt = 1$ and $\varrho_{(\varepsilon)} = Y * \chi_\varepsilon$ for $\varepsilon > 0$.

Proposition 7. *The canonical extension u_+ exists for every $u \in B_{p,\mu}(R_{n+1})$ if and only if $(1+\tau^2)^{-1/2}\mu^{-1}(0, \tau) \in L^{p'}$ for $p > 1$ and $\inf (1+\tau^2)^{1/2}\mu(0, \tau) > 0$ for $p = 1$.*

Proof. Let $p > 1$. For any given $u \in B_{p,\mu}(R_{n+1})$ we can take v such that $u = D_t v - iv$ and therefore the map $v \to (D_t - i)v$ is an isomorphism from $B_{p,\varkappa}(R_{n+1})$ with $\varkappa = (1+\tau^2)^{1/2}\mu$ onto $B_{p,\mu}(R_{n+1})$. In this case, u_+ exists if and only if the distributional limit $\lim_{t\downarrow 0} v$ exists. In fact, if $\lim_{t\downarrow 0} v$ exists, we can consider $v|R_{n+1}^+$ as a solution to the equation $D_t v - iv = u$ with initial value $\lim_{t\downarrow 0} v$. By Theorem 1 in [2], p. 18, we see the existence of u_+. Conversely, suppose u_+ exists. Since $v|R_{n+1}^+$ has an extension over $t = 0$, we see from Proposition 7 in [2], p. 21, that $\lim_{t\downarrow 0} u$ exists. Thus u_+ exists for every $u \in B_{p,\mu}(R_{n+1})$ if and only if the trace mapping $B_{p,\varkappa}(R_{n+1}) \ni v \to v(x, 0) \in \mathscr{D}'(R_n)$ is defined, namely, $(1+\tau^2)^{-1/2}\mu^{-1}(0, \tau) \in L^{p'}$. Similarly for the case $p = 1$.

Proposition 8. *Let P be a polynomial in (ξ, τ) and m be the degree of P in τ. Then the following statements are equivalent:*

(1) *$(P(D)u)_+$ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(2) *$\tau^m(1+\tau^2)^{-1/2}\mu^{-1}(0, \tau) \in L^{p'}$ for $p > 1$ and $\inf|\tau|^{-m}(1+\tau^2)^{1/2}\mu(0, \tau) > 0$ for $p = 1$.*

(3) *$u_+, (D_t u)_+, \ldots, (D_t^m u)_+$ exist for every $u \in B_{p,\mu}(R_{n+1})$.*

Proof. For any given $u \in B_{p,\mu}(R_{n+1})$ we can find $v \in B_{p,\varkappa}(R_{n+1})$ with $\varkappa = (1+\tau^2)^{1/2}\mu$ such that $D_t v - iv = u$. From the equation $(D_t - i)P(D)v = P(D)u$ we see that $(P(D)u)_+$ exists for every $u \in B_{p,\mu}(R_{n+1})$ if and only if $\lim_{t\downarrow 0} P(D)v$ exists for every $v \in B_{p,\varkappa}(R_{n+1})$, which is equivalent to $\tau^m(1+\tau^2)^{-1/2}\cdot \mu^{-1}(0, \tau) \in L^{p'}$ for $p > 1$ and $\inf|\tau|^{-m}(1+\tau^2)^{1/2}\mu(0, \tau) > 0$ for $p = 1$, namely, $\lim_{t\downarrow 0} v, \ldots, \lim_{t\downarrow 0} D_t^m v$ exist for every $v \in B_{p,\varkappa}(R_{n+1})$. We can show along a similar line as in the proof of Proposition 7 that the last statement is equivalent to statement (3).

Let us consider the map $u \to u_+$ of $\mathscr{D}(R_{n+1})$ into $\mathscr{D}'(R_{n+1})$. $\mathscr{D}(R_{n+1})$ is dense in $B_{p,\mu}(R_{n+1})$ with $1 \leqslant p < \infty$. If the map is continuously extended from $B_{p,\mu}(R_{n+1})$ into $\mathscr{D}'(R_{n+1})$, we shall denote by $\tilde{Y}$ the extended map. The map $\tilde{Y}$ is defined if and only if $\varphi_+ \in B_{p',1/\mu}(R_{n+1})$ for every $\varphi \in \mathscr{D}(R_{n+1})$. In fact, if $\tilde{Y}$ is defined, then the map $\mathscr{D}(R_{n+1}) \ni \psi \to \langle \psi_+, \varphi\rangle = \langle \psi, \varphi_+\rangle$, $\varphi \in \mathscr{D}(R_{n+1})$, can be continuously extended on $B_{p,\mu}(R_{n+1})$ and therefore $\varphi_+ \in (B_{p,\mu}(R_{n+1}))' = B_{p',1/\mu}(R_{n+1})$. Conversely, if $\varphi_+ \in B_{p',1/\mu}(R_{n+1})$ for every $\varphi \in \mathscr{D}(R_{n+1})$, then the map $B_{p,\mu}(R_{n+1}) \ni u \to \langle u, \varphi_+\rangle$ is continuous. From the fact that the injection of $\mathscr{D}(R_{n+1})$ into $B_{p,\mu}(R_{n+1})$ is continuous and the map $B_{p,\mu}(R_{n+1}) \ni u \to \varphi u \in B_{p,\mu}(R_{n+1})$ is continuous with $\varphi \in \mathscr{S}(R_{n+1})$ [1], p. 39, we see that the map $\mathscr{D}_K \ni \varphi \to \varrho_{(\varepsilon)}\varphi \in B_{p,\mu}(R_{n+1})$ is continuous for any compact subset K of R_{n+1}. $\varrho_{(\varepsilon)}\varphi$ tends in $B_{p,\mu}(R_{n+1})$ to φ_+ as $\varepsilon\downarrow 0$. Owing to the Banach–Steinhaus theorem the map $\mathscr{D}(R_{n+1}) \ni \varphi \to \varphi_+ \in B_{p,\mu}(R_{n+1})$ is continuous and therefore we can write $\langle u, \varphi_+\rangle = \langle w_u, \varphi\rangle_{\mathscr{D}',\mathscr{D}}$ with $w_u \in \mathscr{D}'(R_{n+1})$. If we take $u = \alpha \in \mathscr{D}(R_{n+1})$, then $w_\alpha = \alpha_+$. Thus $\tilde{Y}$ is defined.

Let Y be the Heaviside function on R_t. Then we have

Theorem 3. *The following statements are equivalent:*

(1) *The map $\tilde{Y}$ of $B_{p,\mu}(R_{n+1})$ into $\mathscr{D}'(R_{n+1})$ is defined.*

(2) *The partial product Yu exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(2)′ *The partial product $Y\cdot u$ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(3) *The distributional limit $\lim_{j\to\infty}(u*\varrho_j)Y$ exists for a fixed restricted δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_{n+1})$, for every $B_{p,\mu}(R_{n+1})$.*

(3)′ *The distributional limit $\lim_{j\to\infty}(u*\varrho_j)Y$ exists for a fixed δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_{n+1})$, for every $u \in B_{p,\mu}(R_{n+1})$.*

(4) *The distributional limit $\lim_{j\to\infty}(Y*\varrho_j)u$ exists for a fixed restricted δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$, for every $u \in B_{p,\mu}(R_{n+1})$.*

(4)′ *The distributional limit $\lim_{j\to\infty}(Y*\varrho_j)u$ exists for a fixed δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$ for every $u \in B_{p,\mu}(R_{n+1})$.*

(5) *The canonical extension u_+ exists for every $u \in B_{p,\mu}(R_{n+1})$.*

(6) *The distributional limit $\lim_{j\to\infty}(Y*\varrho_j)u$ exists for a fixed restricted δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$ with $\operatorname{supp}\varrho_j \subset (0, \infty)$, for every $u \in B_{p,\mu}(R_{n+1})$.*

(6)′ *The distributional limit $\lim_{j\to\infty}(Y*\varrho_j)u$ exists for a fixed δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$ with $\operatorname{supp}\varrho_j \subset (0, \infty)$, for every $u \in B_{p,\mu}(R_{n+1})$.*

Proof. Since the implications $(2)\Rightarrow(3)$, (4), (5), (6); $(2)'\Rightarrow(2)$, $(3)'$, $(4)'$, $(6)'$; $(3)'\Rightarrow(3)$; $(4)'\Rightarrow(4)$ and $(6)'\Rightarrow(6)$ are trivial, it suffices to show the implications $(1)\Rightarrow(2)'$, $(3)\Rightarrow(1)$, $(4)\Rightarrow(1)$ ($(6)\Rightarrow(1)$) and $(5)\Rightarrow(2)$.

$(1)\Rightarrow(2)'$: Suppose the map $\tilde{Y}$ is defined. Then $\psi_+ \in B_{p',1/\mu}(R_{n+1})$ for every $\psi \in \mathscr{D}(R_{n+1})$. Let $u \in B_{p,\mu}(R_{n+1})$ and $\{\varrho_j\}$ be any δ-sequence in $\mathscr{D}(R_{n+1})$. Then we have

$$\langle Y(u*\varrho_j), \psi\rangle = \langle u*\varrho_j, \psi_+\rangle_{B_{p,\mu}, B_{p',1/\mu}},$$

where $u*\varrho_j$ tends in $B_{p,\mu}(R_{n+1})$ to u as $j\to\infty$.

$(3)\Rightarrow(1)$: Suppose (3) holds. Since the map $u \to u*\varrho_j$ is continuous of $\mathscr{D}'(R_{n+1})$ into $\mathscr{E}(R_{n+1})$, the map $u \to (u*\varrho_j)Y$ is continuous of $B_{p,\mu}(R_{n+1})$ into $\mathscr{D}'(R_{n+1})$. By the Banach–Steinhaus theorem the map

$$B_{p,\mu}(R_{n+1}) \ni u \to \lim_{j\to\infty}(u*\varrho_j)Y \in \mathscr{D}'(R_{n+1})$$

is continuous and therefore there exists $w_\varphi \in B_{p',1/\mu}(R_{n+1})$ such that

$$\langle \lim_{j\to\infty}(u*\varrho_j)Y, \varphi\rangle_{\mathscr{D}',\mathscr{D}} = \langle u, w_\varphi\rangle_{B_{p,\mu}, B_{p',1/\mu}}$$

for every $\varphi \in \mathscr{D}(R_{n+1})$. If we take $u = \alpha \in \mathscr{D}(R_{n+1})$, then

$$\langle \alpha, w_\varphi\rangle = \langle \lim_{j\to\infty}(\alpha*\varrho_j)Y, \varphi\rangle = \langle \alpha Y, \varphi\rangle = \langle \alpha, \varphi_+\rangle.$$

From the fact that $\mathscr{D}(R_{n+1})$ is dense in $B_{p,\mu}(R_{n+1})$, we have $\varphi_+ = w_\varphi \in B_{p',1/\mu}(R_{n+1})$.

$(4)\Rightarrow(1)$: In the same way as in the proof of $(3)\Rightarrow(1)$ the map

$$B_{p,\mu}(R_{n+1}) \ni u \to \lim_{j\to\infty}(Y*\varrho_j)u \in \mathscr{D}'(R_{n+1})$$

is continuous. If we take $u = \alpha \in \mathscr{D}(R_{n+1})$, then $\lim_{j\to\infty}(Y*\varrho_j)\alpha = \alpha_+$ and therefore the map $\tilde{Y}$ is defined. Similarly for $(6)\Rightarrow(1)$.

$(5)\Rightarrow(2)$: Suppose u_+ exists for every $u \in B_{p,\mu}(R_{n+1})$. By Proposition 7 we have $(1+\tau^2)^{-1/2}\mu^{-1}(0,\tau) \in L^{p'}$ and by Theorem 1 we see that the partial product δv exists for every $v \in B_{p,\varkappa}(R_{n+1})$ with $\varkappa = (1+\tau^2)^{1/2}\mu$. For any given $u \in B_{p,\mu}(R_{n+1})$ we take $v \in B_{p,\varkappa}(R_{n+1})$ such that $u = D_t v - iv$. By the existence of the partial product $\delta v = Y'v$ we see that Yv and $Y(D_t v)$ exist. Thus Yu exists.

PROPOSITION 9. *Assume that the canonical extension u_+ exists for every $u \in B_{p,\mu}(R_{n+1})$ with $1 < p < \infty$. If $(Y*\varrho_j)u$ belongs to a space $B_{p,\varkappa}(R_{n+1})$ for a δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$, then $(Y*\varrho_j)u$ tends in some space $B_{p,\tilde{\varkappa}}(R_{n+1})$ as $j\to\infty$.*

Proof. If we take v such that $D_t v - iv = u$ for any given $u \in B_{p,\mu}(R_{n+1})$, then $v \in B_{p,\mu_1}(R_{n+1})$ with $\mu_1 = (1+\tau^2)^{1/2}\mu$ and

$$(Y*\varrho_j)u = (D_t - i)\big((Y*\varrho_j)v\big) + i\varrho_j v,$$

where $\varrho_j v$ tends to $\lim_{t\downarrow 0} v\otimes\delta$ in $B_{p,\varkappa_1}(R_{n+1})$ with $\varkappa_1 = (1+\tau^2)^{-k/2}\nu_1(\xi)$, $\nu_1(\xi) = \{\int \mu_1^{-p'}(\xi,\tau)d\tau\}^{-1/p'}$ with $kp > 1$. For any given $\varepsilon > 0$ there exists j_0 such that $Y*(\varrho_j - \varrho_{j'}) = 0$ for $|t| \geqslant \varepsilon$ and any $j, j' \geqslant j_0$. Thus we have for any $\psi \in \mathscr{D}(R_{n+1})$ and for any $j, j' \geqslant j_0$

$$|\langle Y*\varrho_j)v - (Y*\varrho_{j'})v, \psi\rangle| \leqslant \int_{-\varepsilon}^{\varepsilon} |Y*(\varrho_j - \varrho_{j'})|\,\|v(\cdot,t)\|_{p,\nu_1}\|\psi(\cdot,t)\|_{p',1/\nu_1}dt$$

$$\leqslant \|v\|_{p,\mu_1}\|\psi\|_{p',1/\varkappa_1}\int_{-\varepsilon}^{\varepsilon}|Y*(\varrho_j - \varrho_{j'})|\,dt$$

and therefore $\{(Y*\varrho_j)v - (Y*\varrho_1)v\}$ is a Cauchy sequence in $B_{p,\varkappa_1}(R_{n+1})$ with $kp > 1$. From the fact that $(Y*\varrho_1)u \in B_{p,\varkappa}(R_{n+1})$ and $(Y*\varrho_j)u - (Y*\varrho_1)u$ is a Cauchy sequence in $B_{p,\varkappa_2}(R_{n+1})$ with $\varkappa_2 = (1+\tau^2)^{-1/2}\varkappa_1$ if we take $\tilde{\varkappa} = \min(\varkappa, \varkappa_2)$, then $(Y*\varrho_j)u$ tends in $B_{p,\tilde{\varkappa}}(R_{n+1})$ to u_+ as $j\to\infty$.

PROPOSITION 10. *If the partial product Yu exists in $B_{p,\mu}(R_{n+1})$ for every $u \in B_{p,\mu}(R_{n+1})$, then the map $B_{p,\mu}(R_{n+1}) \ni u \to Yu \in B_{p,\mu}(R_{n+1})$ is continuous, where $p > 1$.*

Proof. Let $u_j \in B_{p,\mu}(R_{n+1})$ and suppose u_j tends in $B_{p,\mu}(R_{n+1})$ to u and Yu_j tends in $B_{p,\mu}(R_{n+1})$ to v as $j\to\infty$. Then we have for any $\varphi \in \mathscr{D}(R_{n+1})$

$$\langle v, \varphi\rangle_{B_{p,\mu},B_{p',1/\mu}} = \lim_{j\to\infty}\langle Yu_j, \varphi\rangle_{B_{p,\mu},B_{p',1/\mu}} = \lim_{j\to\infty}\langle Yu_j, \varphi\rangle_{\mathscr{D}',\mathscr{D}}$$

and for every δ-sequence $\{\varrho_j\}$, $\varrho_j \in \mathscr{D}(R_t)$

$$\langle Yu_j, \varphi\rangle_{\mathscr{D}',\mathscr{D}} = \lim_{i\to\infty}\langle u_j, (Y*\varrho_i)\varphi\rangle_{\mathscr{D}',\mathscr{D}} = \lim_{i\to\infty}\langle u_j, (Y*\varrho_i)\varphi\rangle_{B_{p,\mu},B_{p',1/\mu}}$$

$$= \langle u_j, \varphi_+\rangle_{B_{p,\mu},B_{p',1/\mu}}$$

and therefore

$$\langle v, \varphi\rangle_{B_{p,\mu},B_{p',1/\mu}} = \langle u, \varphi_+\rangle_{B_{p,\mu},B_{p',1/\mu}}.$$

Moreover, we can write

$$\langle u, \varphi_+\rangle_{B_{p,\mu},B_{p',1/\mu}} = \lim_{i\to\infty}\langle u, (Y*\varrho_i)\varphi\rangle_{B_{p,\mu},B_{p',1/\mu}}$$

$$= \lim_{i\to\infty}\langle (Y*\varrho_i)u, \varphi\rangle_{\mathscr{D}',\mathscr{D}} = \langle Yu, \varphi\rangle_{\mathscr{D}',\mathscr{D}}$$

$$= \langle Yu, \varphi\rangle_{B_{p,\mu},B_{p',1/\mu}}.$$

Thus we obtain $v = Yu$, which completes the proof.

Assume that the map $B_{p,\mu}(R_{n+1}) \ni u \to Yu \in B_{p,\mu}(R_{n+1})$, $1 < p < \infty$, is continuous. Then, if we consider the adjoint map, then we see that the partial product Yv exists for every $v \in B_{p',1/\mu}(R_{n+1})$ and $Yv \in B_{p',1/\mu}(R_{n+1})$. In this case

$$\int (1+\tau^2)^{-p'/2}\mu^{-p'}(\xi,\tau)d\tau < \infty \quad \text{and} \quad \int (1+\tau^2)^{-p/2}\mu^{p}(\xi,\tau)d\tau < \infty.$$

We denote by $B^+_{p,\mu}$ the closed subspace of $B_{p,\mu}(R_{n+1})$ consisting of the elements of $B_{p,\mu}(R_{n+1})$ with support in $\bar{R}^+_{n+1}$, where $1 < p < \infty$. $\mathscr{D}(R^+_{n+1})$ is dense in $B^+_{p,\mu}$. Similarly we can consider the space $B^-_{p,\mu}$.

PROPOSITION 11. *Assume that the partial product Yu exists for every $u \in B_{p,\mu}(R_{n+1})$, $p > 1$. Then the following statements are equivalent:*

(1) *$Yu \in B_{p,\mu}(R_{n+1})$ for every $u \in B_{p,\mu}(R_{n+1})$.*

(2) *$Yu \in B_{p',1/\mu}(R_{n+1})$ for every $u \in B_{p',1/\mu}(R_{n+1})$.*

(3) *$Y\varphi \in B_{p',1/\mu}(R_{n+1})$ for every $\varphi \in \mathscr{D}(R_{n+1})$ and $B_{p,\mu}(R_{n+1}) = B^+_{p,\mu} + B^-_{p,\mu}$ (topological sum).*

(4) *$Y\varphi \in B_{p,\mu}(R_{n+1})$ for every $\varphi \in \mathscr{D}(R_{n+1})$ and $B_{p',1/\mu}(R_{n+1}) = B^+_{p',1/\mu} + B^-_{p',1/\mu}$ (topological sum).*

Proof. It suffices to prove the equivalence between (1) and (3).

$(1)\Rightarrow(3)$: Suppose (1) holds. If we put $u_+ = Yu$, $u_- = (1-Y)u$, then $u_+ \in B^+_{p,\mu}$ $u_- \in B^-_{p,\mu}$ and $u = u_+ + u_-$. If we assume $B^+_{p,\mu}\cap B^-_{p,\mu}$ has a non-zero element, then $\mu(0,\tau) \in L^p$ and every $u \in B_{p',1/\mu}(R_{n+1})$ will be identified with $B_{p',\nu}(R_n)$-valued continuous function of t with $\nu(\xi) = \{\int \mu^p(\xi,\tau)d\tau\}^{-1/p}$, which contradicts with that $\varphi_+ \in B_{p',1/\mu}(R_{n+1})$ for every $\varphi \in \mathscr{D}(R_{n+1})$. Thus $B^+_{p,\mu}\cap B^-_{p,\mu} = \{0\}$. By the preceding proposition the maps $B_{p,\mu}(R_{n+1}) \ni u \to u_+ \in B^+_{p,\mu}$ and $B_{p,\mu}(R_{n+1}) \ni u \to u_- \in B^-_{p,\mu}$ are continuous.

(3) ⇒ (1): Suppose (3) holds. $\mathscr{D}(R^+_{n+1})$ and $\mathscr{D}(R^-_{n+1})$ are dense in $B^+_{p,\mu}$ and $B^-_{p,\mu}$, respectively. The map l: $\varphi_1+\varphi_2 \to \varphi_1$ of $\mathscr{D}(R^+_{n+1})+\mathscr{D}(R^-_{n+1})$ into $\mathscr{D}(R^+_{n+1})$ is continuous, that is, $\|\varphi_1\|_{p,\mu} \leqslant C\|\varphi_1+\varphi_2\|_{p,\mu}$ with a positive constant C. Since $\mathscr{D}(R^+_{n+1})+\mathscr{D}(R^-_{n+1})$ is dense in $B_{p,\mu}(R_{n+1})$, it suffices to $l(\varphi) = \varphi_+$ for any $\varphi \in \mathscr{D}(R^+_{n+1})+\mathscr{D}(R^-_{n+1})$, which is an immediate consequence of the definition of the map l.

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Factorization in some Fréchet algebras of differentiable functions

by

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Dedicated to Professor Jan Mikusiński on the occasion of his 70th birthday

Abstract. It is shown that for each compact set $B \subset \mathscr{S}(\boldsymbol{R}^n)$ there exist $u \in \mathscr{S}(\boldsymbol{R}^n)$ and a compact set $B' \subset \mathscr{S}(\boldsymbol{R}^n)$ such that $uB' = B$ holds (essential part of the "compact strong factorization property"). The same property is shown for s (rapidly decreasing sequences) and $\mathring{\mathscr{B}}(\Omega)$.

Introduction. The starting point of this paper was the question of Kamiński whether $\mathrm{lin}(\mathscr{S}*\mathscr{S}) = \mathscr{S}$, or equivalently, $\mathrm{lin}(\mathscr{S}\cdot\mathscr{S}) = \mathscr{S}$ holds ([11], Problems 4 and 5, p. 282). We give an affirmative answer by showing that $\mathscr{S}\cdot\mathscr{S} = \mathscr{S}$ holds. More precisely, we show that $\mathscr{S}$ has the compact strong factorization property, i.e., roughly speaking, that a compact $B \subset \mathscr{S}$ can be written as uB', with $u \in \mathscr{S}$ and compact $B' \subset \mathscr{S}$. (Let us mention that $\mathrm{lin}(\mathscr{D}*\mathscr{D}) = \mathscr{D}$ is known from [15], [7].)

Factorization properties are known for Fréchet algebras having a uniformly bounded approximate unit. From the known factorization theorems for Fréchet and Banach algebras we have extracted a rather strong concept of factorization property (cf. Definition 1.1), which is satisfied in Fréchet algebras having a uniformly bounded left approximate unit. We show that this factorization property is also satisfied in a certain class of Fréchet algebras of differentiable functions, which do not have a bounded approximate unit. $\mathscr{S}(\boldsymbol{R}^n)$ and $\mathring{\mathscr{B}}(\Omega)$ belong to this class, for quasi-bounded $\Omega = \mathring{\Omega} \subset \boldsymbol{R}^n$.

In Section 1 we define the concepts of Fréchet algebra, strong factorization property, (uniformly bounded) left approximate unit. We state the factorization theorem, and we note that reflexive Fréchet algebras having no unit cannot have a bounded approximate unit. We introduce $\mathring{\mathscr{B}}(\Omega)$ and show that $\mathring{\mathscr{B}}(\boldsymbol{R}^n)$ has a uniformly bounded approximate unit.

In Section 2 we define a class of Fréchet algebras $\mathscr{B}^m_\gamma(\Omega)$ of m-times differentiable functions on $\Omega = \mathring{\Omega} \subset \boldsymbol{R}^n$ $(0 \leqslant m \leqslant \infty)$. If the weight function $\gamma \in C(\Omega)$ is such that there exists a certain kind of partition