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MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNCHEN
Theresienstr. 39, D-8000 München 2, Federal Republic of Germany

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Completeness type properties of locally solid Riesz spaces

by

IWO LABUDA (Poznań)

Dedicated to Professor Jan Mikusiński on the occasion of his 70th birthday

Abstract. The main results, in the terminology of Aliprantis and Burkinshaw [A&B], and Fremlin [F] (this terminology has been changed for the reasons 'intrinsic' to this paper), are as follows. Let (L, τ) be a Hausdorff locally solid Riesz space. It embeds order densely into a Nakano space $(L^{\#}, \tau^{\#})$ if (and only if) τ is Fatou; this embedding is unique. A Dedekind complete (L, τ) embeds order densely into a Hausdorff locally solid Dedekind complete Riesz space $(L^{\#}, \tau^{\#})$ having the Monotone Completeness Property if (and only if) τ is pseudo-Lebesgue.

Let Ω be an extremally disconnected topological space, $C^{\infty}(\Omega)$ the Riesz space of continuous functions, from Ω into the extended real line, which take finite values on dense subsets of Ω.

In the first part of this paper a theory which parallels the one of Banach function spaces by Luxemburg and Zaanen [4], is initiated on $C^{\infty}(\Omega)$. Function filters and their topological vector cores replace function norms and their Banach function spaces. This permits to treat the general locally solid case. In § 1 the (topological) completness properties of vector cores are investigated.

In the second part, to an order dense Riesz subspace of C^{∞} with a locally solid vector topology appropriate function filters are associated.

In the third part the previous results are applied, via the Maeda–Ogasawara representation theorem, to general locally solid Riesz spaces. The main results are as follows.

Let (L, τ) be a Hausdorff topological Riesz space.

(L, τ) embeds order densely into a Hausdorff locally solid-boundedly order-complete $(L^{\#}, \tau^{\#})$ if (and only if) τ is locally solid-order-closed; this embedding is unique up to an isomorphism.

A Dedekind complete (L, τ) embeds order densely into a Hausdorff locally solid Dedekind complete Riesz space $(L^{\#}, \tau^{\#})$ having the Monotone Completeness Property if (and only if) τ is locally solid-pseudo-order-closed.

A metrizable locally solid (L, τ) embeds order densely into an F-lattice $(L^{\delta\#}, \tau^{\delta\#})$ iff the following condition is satisfied

$$(*)\qquad 0 \leqslant x_n\uparrow \leqslant y \text{ in } L,\ (x_n)\ \tau\text{-Cauchy} \Rightarrow x_n \to x \text{ in } (L^\delta, \tau^\delta),$$

where τ^δ is the 'solid hull' of τ in the Dedekind completion L^δ of L.

§ 0. Vector cores of function filters. Let R^∞ denote the real line compactified by $+\infty$ and $-\infty$. The rules for addition and multiplication are extended to $\pm\infty$ in the usual way, in particular

$$(\pm\infty)\cdot 0 = 0\cdot(\pm\infty) = 0, \qquad (\pm\infty)+(\mp\infty) = (\pm\infty)-(\pm\infty) = 0.$$

In what follows Ω is an extremally disconnected topological space, $C^\infty = C^\infty(\Omega) = C^\infty(\Omega, R^\infty)$ the Riesz space of continuous functions from Ω into R^∞ which take finite values on dense subsets of Ω (see [L&Z], p. 322), $C^{\overline{\infty}} = C(\Omega, R^\infty)$ the collection of all continuous functions from Ω into R^∞.

C^∞ is a universally complete Riesz space, i.e., it is laterally complete (any disjoint family of positive elements has the supremum) and Dedekind complete. Clearly, $C^{\overline{\infty}}$ is order isomorphic to $C(\Omega, [-1, 1])$, an order interval in C^∞, and therefore is *order-complete* (in the sense that for any $x_a\uparrow$, (x_a) has the supremum x i.e., $x_a\uparrow x$). There are no difficulties with multiplying the elements of $C^{\overline{\infty}}$ by real, or even extended real, numbers. The situation is, however, a bit precarious with the addition so that $C^{\overline{\infty}}$ is *not* a vector space. Throughout this paper the Riesz space terminology and notation, if not explicitly changed, follows [A&B]. L will be the generic notation for a Riesz space.

Let X be a vector space. Appropriate filter bases around 0 define uniquely group or vector topologies on X. For instance, if $\mathscr{U}$ is a filter base in X such that it is

(i) *symmetric*: $\forall V \in \mathscr{U},\ V = -V$,

(ii) *summative*: $\forall V \in \mathscr{U}\ \exists U \in \mathscr{U}$ so that $U+U \subset V$,

then $\mathscr{U}$ defines a unique group (i.e., compatible with the additive structure of X) topology on X for which $\mathscr{U}$ is a (filter) base of neighbourhoods of the origin. Conversely, any group topology τ on X has a symmetric summative base of neighbourhoods of 0.

It turns out that for purposes of the present research it will be very convenient to speak and to think in terms of filters. Accordingly some terminology will be introduced.

Filters will be considered in $C^{\overline{\infty}}$ (hence, in particular, in C^∞, $C^{\overline{\infty}}_+$, a Riesz subspace of C^∞, etc.). A filter base is *central* if any of its members contains 0. In what follows only such filter bases (filters) are considered.

Let $\mathscr{U}$ be a filter base, and $\mathscr{F}$ a filter in $C^{\overline{\infty}}$. Suppose the elements $V \in \mathscr{U}$ have a property P. Then $\mathscr{U}$ is called a *P-filter base*. $\mathscr{F}$ is a *P-filter* if it is generated by a P-filter base. For instance, $\mathscr{F}$ is solid in $C^{\overline{\infty}}$ if it has a base $\mathscr{U}$ so that each $V \in \mathscr{U}$ is solid in $C^{\overline{\infty}}$.

For obvious reasons, a balanced summative absorbent (resp. symmetric summative) filter base is called shortly a vector (resp. group) filter base. If a vector (group) filter base is moreover P on L, then the topology τ that it defines is *locally P* vector (group) topology and (L, τ) is a locally P tvs (topological vector space) or tg (topological group) or even tRs (topological Riesz space). For instance, if $\mathscr{U}$ is solid vector, then τ it defines is a locally solid vector topology, (L, τ) is a locally solid tRs, and the filter of neighbourhoods of 0 for τ is solid and vector.

A net (x_a) *is converging to* 0 *in* $\mathscr{U}$ if for each $V \in \mathscr{U}$ there exists a so that $\{x_\beta\colon \beta \geqslant a\} \subset V$.

A set B is *$\mathscr{U}$-bounded* if it is absorbed by any $V \in \mathscr{U}$. $\mathscr{U}$ is *Hausdorff* if $\bigcap\{V\colon V \in \mathscr{U}\} = \{0\}$.

Recall that a *Riesz F-semi-norm* (F-norm) ϱ on L is a monotone F-semi-norm (F-norm), i.e., such that $|x| \leqslant |y| \Rightarrow \varrho(x) \leqslant \varrho(y)$. In particular $\varrho(x) = \varrho(|x|)$ for all $x \in L$. If ϱ is homogeneous, then it is a semi-norm (norm).

On $C^{\overline{\infty}}$ (similarly as in [4], where it is done on M) *function semi-norms* and norms can be defined. It happens so, however, that function F-semi-norms (F-norms) do not really exist. When the assumption of homogeneity is dropped, the resulting object turns out to be a function group semi-norm (norm). Furthermore, a 'non-metric' generalization of a function group norm can be given in terms of a filter in $C^{\overline{\infty}}$.

0.1. Definitions. A filter base $\mathscr{U}^{\overline{\infty}}_+$ on $C^{\overline{\infty}}_+ = C(\Omega, R^\infty_+)$ is said to be a *pre-function filter base* if it is solid and summative.

A functional $\varrho\colon C^{\overline{\infty}}_+ \to R^\infty_+$ is said to be a *pre-function group semi-norm* if

(i) $\varrho(0) = 0$,

(ii) $\varrho(x+y) \leqslant \varrho(x)+\varrho(y)$,

(iii) $x \leqslant y \Rightarrow \varrho(x) \leqslant \varrho(y)$;

it is a *pre-function group norm* if

(iv) $\qquad \varrho(x) = 0 \Rightarrow x = 0.$

One extends $\mathscr{U}^{\overline{\infty}}_+$ and ϱ on the whole $C^{\overline{\infty}}$ by defining

$$\mathscr{U}^{\overline{\infty}} = \{V^{\overline{\infty}} = (x \in C^{\overline{\infty}}\colon |x| \in V^{\overline{\infty}}_+)\colon V^{\overline{\infty}}_+ \in \mathscr{U}^{\overline{\infty}}_+\},$$

$$\varrho(x) = \varrho(|x|) \quad \text{for} \quad x \in C^{\overline{\infty}}.$$

Then $\mathscr{U}^{\overline{\infty}}$ is said to be a *function filter base*, and ϱ a *function group (semi) norm*. The filter $\mathscr{F}$ generated by $\mathscr{U}^{\overline{\infty}}$ is a *function filter*.

Note that one has indeed $\mathscr{U}^{\overline{\infty}}_+ = \mathscr{U}^{\overline{\infty}} \cap C^{\overline{\infty}}_+$ and that $\mathscr{U}^{\overline{\infty}}$ is solid.

For any (central) filter base $\mathscr{U}$ in $C^{\overline{\infty}}$ set

$$v(C^{\overline{\infty}}, \mathscr{U}) = \{x \in C^{\overline{\infty}}\colon \lim(1/n)x = 0 \text{ in } \mathscr{U}\},$$

$$v(C^{\infty}, \mathscr{U}) = \{x \in C^{\infty}\colon \lim(1/n)x = 0 \text{ in } \mathscr{U}\}.$$

0.2. Proposition. *Let $\mathscr{U}$ be a Hausdorff solid base of a filter $\mathscr{F}$ on $C^{\overline{\infty}}$. Then $v(C^{\overline{\infty}}, \mathscr{F}) = v(C^{\overline{\infty}}, \mathscr{U}) = v(C^{\infty}, \mathscr{U})$.*

Proof. Take $x \in v(C^{\overline{\infty}}, \mathscr{U})$. If $x \in C^{\infty}$ there is nothing to prove and so suppose $x \in C^{\overline{\infty}} \setminus C^{\infty}$. Replacing x by $|x|$, it can be assumed that x is positive. Then $\{t \in \Omega\colon x(t) = \infty\}$ is closed and not nowhere dense, hence contains an open-closed set $E \neq \emptyset$ since Ω is extremally disconnected (closure of any open set is open). For each $n \in N$, $(1/n)x \geqslant (1/n)x\chi_E = x\chi_E \geqslant \chi_E \notin V$ for some V in $\mathscr{U}$, as $\mathscr{U}$ is Hausdorff. This means that $x \notin v(C^{\overline{\infty}}, \mathscr{U})$ and implies the result.

0.3. Remark. $v(C^{\infty}, \mathscr{U})$ is symmetric (resp. balanced, solid, 'summative') subset of C^{∞} provided $\mathscr{U}$ is such. In particular, it is visibly a vector subspace of C^{∞} if it is generated by a balanced summative filter base.

Let $\mathscr{F}$ be a function filter and let $\mathscr{U}^{\overline{\infty}}$ be a base generating $\mathscr{F}$. Then $\mathscr{F}$ defines uniquely a locally solid group topology μ^{∞} on C^{∞} for which

$$\mathscr{U}^{\infty} := \mathscr{U}^{\overline{\infty}} \cap C^{\infty}$$

is a base of neighbourhoods of 0.

Indeed, taking the pre-function filter base $\mathscr{U}^{\overline{\infty}}_{+}$,

$$\mathscr{U}^{\infty}_{+} := \mathscr{U}^{\overline{\infty}}_{+} \cap C^{\infty}$$

is summative since $\mathscr{U}^{\overline{\infty}}_{+}$ is such. It follows that

$$\mathscr{U}^{\infty} := \{(x \in C^{\infty}\colon |x| \in V^{\infty}_{+})\colon V^{\infty}_{+} \in \mathscr{U}^{\infty}_{+}\}$$

is summative. It is clear that $\mathscr{U}^{\infty}$ is also solid. As 'solid + summative' = 'solid+group', $\mathscr{U}^{\infty}$ defines μ^{∞}.

Consequently, it is natural to write

$$v(C^{\infty}, \mu^{\infty}) = \{x \in C^{\infty} := \lim(1/n)x = 0\,(\mu^{\infty})\} = v(C^{\infty}, \mathscr{U}^{\infty})$$

and it is clear that $v(C^{\infty}, \mu^{\infty})$ is the largest vector subspace of C^{∞} on which μ^{∞} defines the structure of a topological vector space.

The pair $\big(v(C^{\infty}, \mu^{\infty}), \mu^{\infty} \cap v(C^{\infty}, \mu^{\infty})\big)$ is called the *topological vector core* (TVC) of $\mathscr{F}$ or μ^{∞}. Denoting it by (M, μ), $M = v(C^{\infty}, \mathscr{F})$ is the *vector core* (VC) of $\mathscr{F}$ (or μ^{∞}) and $\mu = \mu^{\infty} \cap M$.

As μ^{∞} is locally solid, μ is so, too. Moreover, M is regular in C^{∞} and is a solid vector subspace = an order ideal of C^{∞}, i.e., M is a Dedekind complete Riesz space on its own right. Taking into account 0.2, a remarkable characterization of the vector core M in terms of $C^{\overline{\infty}}$ holds in the Hausdorff situation.

0.4. Proposition. *Let $\mathscr{F}$ be a Hausdorff function filter with a base $\mathscr{U}^{\overline{\infty}}$. Then*

$$M = v(C^{\overline{\infty}}, \mathscr{F}) \;\; (= v(C^{\overline{\infty}}, \mathscr{U}^{\overline{\infty}}))$$

i.e., any $x \in v(C^{\overline{\infty}}, \mathscr{F})$, a priori a member of $C^{\overline{\infty}}$, is in M. M is an order ideal of C^{∞} with a Hausdorff locally solid vector topology defined by the base $\mathscr{U} = \mathscr{U}^{\overline{\infty}} \cap M$ of neighbourhoods of 0.

Let L be an arbitrary Riesz space. A topology τ on L is (locally solid) *order-continuous* (oc) if $x_a \uparrow x \Rightarrow x_a \to x(\tau)$.

A Riesz F-semi-norm ϱ on L is *semi-order-continuous* (soc) if $x_a \uparrow x \Rightarrow \varrho(x_a) \uparrow \varrho(x)$. Let $\mathscr{R} = \{\varrho\}$ be a family of such Riesz F-semi-norms. Then the canonical filter base of closed neighbourhoods of 0 that $\mathscr{R}$ defines, is solid order-closed. The corresponding topology is locally solid-order-closed (locally s-o-c(1)). By a result of Fremlin ([F], 23 B) any locally s-o-c topology arises in that way (i.e., may be defined by a family of soc Riesz F-semi-norms).

A property P having the form of a condition on the behaviour of τ or ϱ with respect to a monotone net (x_a) may often be reformulated by imposing the additional requirement that (x_a) be Cauchy. Then the resulting property is qualified as *pseudo-P*. In particular, τ on L is (locally solid) *pseudo-order-continuous* (pseudo-oc) if $x_a \uparrow x$, (x_a) Cauchy $\Rightarrow x_a \to x(\tau)$. Similarly, τ on L is *locally solid-pseudo-order-closed* (locally pseudo-s-o-c(2)) if it admits a (solid vector) filter base $\mathscr{V}$ of neighbourhoods of 0 so that $x_a \uparrow x$, (x_a) τ-Cauchy, $(x_a) \subset V \Rightarrow x \in V$, i.e., $V \in \mathscr{V}$ are *pseudo-order-closed*.

§ 1. Completeness properties of topological vector cores. Assume that a function filter $\mathscr{F}$ on $C^{\overline{\infty}}$ is given and let (M, μ) be its TVC, i.e., $M = v(C^{\infty}, \mathscr{F})$ and $\mu = M \cap \mu^{\infty}$, where μ^{∞} is a locally solid group topology defined by $\mathscr{F}$ on C^{∞}.

1.1. Definitions. A solid subset V of $C^{\overline{\infty}}$ is said to be

(i) *Fatou* if $0 \leqslant x_a \uparrow x$ in $C^{\overline{\infty}}$, $(x_a) \subset M \cap V \Rightarrow x \in V$.

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(1) The double possible meaning of the abbreviation 's-o-c' will not cause any confusion as 'locally semi-order-continuous' has no sense. On the other hand, it is nice to have locally s-o-c topologies generated by soc F-semi-norms!

(2) At first glance, the 'locally solid pseudo-order-continuous' could also abbreviate — by the same transposition — as 'locally pseudo-s-o-c'. Even this, miraculously enough, would not cause any real danger as in 1.4 below it is shown that both notions coincide. In reality the adjective 'locally' does not apply to 'pseudo-oc' which is not a property of any filter base of neighbourhoods of the origin (a closer look revels the lack of a hyphen between 'solid' and 'pseudo') thus 'locally solid pseudo-oc' is the correct version.

(ii) *boundedly Fatou* if $0 \leqslant x_\alpha \uparrow x$ in C^∞, $(x_\alpha) \subset M \cap V$ and is μ-bounded $\Rightarrow x \in V$.

(iii) *pseudo-Fatou* if $0 \leqslant x_\alpha \uparrow x$ in C^∞, $(x_\alpha) \subset M \cap V$ and is μ-Cauchy $\Rightarrow x \in V$.

According to the grammar adopted in § 1, the notions of a Fatou, boundedly Fatou or pseudo-Fatou filter and filter base are clear. In particular $\mathscr{F}$ itself may be such.

1.2. Definitions. Let (L, τ) be an arbitrary locally solid tRs. A solid subset V of L is said to be

(i) *order-complete* if $0 \leqslant x_\alpha \uparrow \subset V \Rightarrow x = \sup x_\alpha \in V$.

(ii) *boundedly order-complete* if $0 \leqslant x_\alpha \uparrow \subset V$, (x_α) τ-bounded $\Rightarrow x = \sup x_\alpha \in V$.

(iii) *pseudo-order-complete* if $0 \leqslant x_\alpha \uparrow \subset V$, (x_α) τ-Cauchy $\Rightarrow x = \sup x_\alpha \in V$.

Again, the notions of order-complete, boundedly-order-complete and pseudo-order-complete filter or filter base are clear. Furthermore, by analogy with the 'locally s-o-c' terminology the following abbreviations are adopted for a topology: it is locally S-O-C = locally solid-order-complete; or locally boundedly S-O-C = locally solid-boundedly-order-complete; or locally pseudo-S-O-C- = locally solid-pseudo-order complete.

1.3. Remarks. (1) To avoid any confusion in 1.2, it would be perhaps better to write in these conditions openly that $x \in V$ means that $x = \sup x_\alpha$ exists in L and moreover belongs to V.

(2) It is immediate that if (L, τ) is a regular Riesz subspace in the TVC $M \subset C^\infty$, $\tau = \mu \cap L$ and $V \subset L$, then V is Fatou (resp. boundedly Fatou, resp. pseudo-Fatou) iff it is order-complete (resp. boundedly order-complete, resp. pseudo-order-complete), i.e., 'Fatou sets become order-complete sets'.

(3) It is also clear that a boundedly Fatou (thus, a Fatou) filter induces an order-closed filter $\mathscr{F} \cap L$ in L. Similarly, a pseudo-Fatou $\mathscr{F}$ induces a pseudo-order-closed filter in L.

(4) It may happen that L itself is boundedly-order-complete. Then (L, τ) will also be qualified as having the *bounded order-completeness property* (BOC property). Note that this is the weak Fatou property of Luxemburg and Zaanen [4] and the Levi property of [A&B].

(5) One can thus also call (L, τ) as having the *pseudo-order-completeness property* (POC-property) when L is pseudo-order-complete: $0 \leqslant x_\alpha \uparrow$ Cauchy in $L \Rightarrow x = \sup x_\alpha \in L$.

(6) Finally, note that $V \subset C^\infty$ is order-closed $(0 \leqslant x_\alpha \uparrow x, (x_\alpha) \subset V \Rightarrow x \in V)$ in C^∞ iff it is order-complete therein $(0 \leqslant x_\alpha \uparrow \subset V \Rightarrow x \in V)$. This is due to the fact that C^∞ itself is order-complete.

1.4. Proposition. *Let τ be a vector topology on a Riesz space L. The following are equivalent*:

(i) *τ is locally solid-pseudo-order-closed.*

(ii) *τ is locally solid pseudo-order-continuous.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathscr{V}$ be a solid pseudo-order-closed filter base of neighbourhoods of 0 in τ. As τ is locally solid, N = the closure of 0 in τ, is a closed order-ideal in L. Consider the (Riesz) quotient $K = L/N$. Let $\varkappa$ be the quotient topology on K and let q be the quotient map. $(K, \varkappa)$ is a Hausdorff locally solid tRs and let $(\hat{K}, \hat{\varkappa})$ denote the Riesz topological completion of $(K, \varkappa)$ (see [A&B], p. 43). Suppose (i) holds and let $x_\alpha \uparrow x$, (x_α) τ-Cauchy in L be given. Then $q(x_\alpha)\uparrow$, $q(x_\alpha)$ is $\varkappa$-Cauchy and $q(x_\alpha) \leqslant q(x)$. Let $\hat{x} = \hat{\varkappa}\text{-}\lim q(x_\alpha) = \sup q(x_\alpha)$ in $\hat{K}$. Clearly, $\hat{x} \leqslant q(x)$ and suppose $\hat{x} \neq q(x)$. Set

$$y_\alpha = q(x) - q(x_\alpha), \qquad z_\alpha = \hat{x} - q(x_\alpha);$$

$y_\alpha \downarrow$ and $z_\alpha \downarrow 0$ in $\hat{K}$. Then $y = \hat{\varkappa}\text{-}\lim y_\alpha \geqslant 0$ and in fact $y > 0$ since otherwise $q(x) = \hat{x}$. Find $V \in \mathscr{V}$ so that $q(V)$, a neighbourhood of 0 for $\varkappa$, is such that $\overline{q(V)}$ (:= the closure of $q(V)$ in $(K, \varkappa)$) does not contain y $(\{\overline{q(V)}: V \in \mathscr{V}\}$ is a base of neighbourhoods of 0 for $\hat{\varkappa}$). As $\overline{q(V)}$ is solid and $y_\alpha \downarrow y$, for all α, $y_\alpha \notin \overline{q(V)}$. On the other hand, (x_α) being Cauchy, it is possible to find an α so that $(x_\beta - x_\alpha)_{\beta \geqslant \alpha} \subset V$. Note that $(x_\beta - x_\alpha) \uparrow x - x_\alpha$ (in β) and is τ-Cauchy. As V is pseudo-order-closed, $x - x_\alpha \in V$; hence $y_\alpha = q(x) - q(x_\alpha) \in q(V) \subset \overline{q(V)}$; a contradiction. This shows that $q(x) = \hat{x}$ which means that (ii) holds.

(ii) $\Rightarrow$ (i): A Riesz F-semi-norm ϱ is pseudo-soc if (of course) $x_\alpha \uparrow x$, (x_α) τ-Cauchy $\Rightarrow \varrho(x_\alpha) \uparrow \varrho(x)$. Note that (ii) implies that any continuous Riesz F-semi-norm ϱ on (L, τ) is pseudo-soc. Thus the family $\mathscr{R}$ of all such ϱ defines τ, and the canonical base $\mathscr{V}$ of closed neighbourhoods of 0 defined by $\mathscr{R}$ is obviously solid pseudo-order-closed.

Remark. The notion of a locally pseudo-s-o-c topology seems to be new. Even though it coincides with the 'locally solid pseudo-oc' (= pseudo-Lebesgue of [A&B]) I will use both names to stress certain analogies. For instance, it is the use of a locally pseudo-s-o-c topology which leads to the pseudo-Fatou property and I do not see how the pseudo-order continuity in this context could be relevant.

1.5. Proposition. *The following are equivalent*:

(i) *(L, τ) has* BOC *property and τ is locally* s-o-c, *i.e., (L, τ) is a Nakano space* [A&B].

(ii) *(L, τ) is locally boundedly* S-O-C.

Proof. Only (i) $\Rightarrow$ (ii) needs the proof. Let $V \in \mathscr{V}$ = a solid order-closed base of neighbourhoods of 0 for τ. If $0 \leqslant x_\alpha \uparrow \subset V$ and (x_α) is

τ-bounded, then $x = \sup x_\alpha \in L$ by BOC, hence $x \in V$ using the order-closedness of V. This means that V is boundedly order-complete.

Note that Nakano spaces are necessarily Hausdorff and τ-complete [F], [A&B].

1.6. Proposition. *The following are equivalent:*

(i) (L, τ) *has* POC *property and is locally pseudo*-s-o-c.

(ii) (L, τ) *is locally pseudo*-S-O-C.

(iii) (L, τ) *has the Monotone Completeness Property* (MCP [A&B]: $0 \leqslant x_\alpha \uparrow \subset L$, (x_α) τ*-Cauchy* $\Rightarrow x_\alpha \to x(\tau)$).

Proof. (iii)$\Rightarrow$(i) is clear. (i)$\Rightarrow$(ii): Analogously as in 1.15 above. (ii)$\Rightarrow$(iii): If $0 \leqslant x_\alpha \uparrow$ τ-Cauchy is given in L, then $x_\alpha \uparrow x = \sup x_\alpha$ by (ii). Now, applying 1.4, $x_\alpha \to x(\tau)$.

1.7. Theorem. *Let $\mathscr{F}$ be a Hausdorff boundedly Fatou (resp. pseudo-Fatou) function filter in C^∞. Then its* TVC (M, μ) *is locally boundedly* S-O-C (*resp. locally pseudo*-S-O-C).

Proof. Let $\mathscr{U}^\infty$ be a Hausdorff boundedly (resp. pseudo-) Fatou filter base generating $\mathscr{F}$. Then $\mathscr{U} = \mathscr{U}^\infty \cap M$ is an order-closed (1.3 (3) above) (resp. pseudo-order-closed) base of neighbourhoods of 0 for μ. Take $0 \leqslant x_\alpha \uparrow \subset M$, (x_α) μ-bounded (resp. Cauchy). Then $x_\alpha \uparrow x$ in C^∞ (as it is order-complete). By boundedness, for any $U \in \mathscr{U}$ there exists $n \in N$ so large that $((1/n)x_\alpha) \subset U$. Since $(1/n)x_\alpha \uparrow (1/n)x$, $(1/n)x \in U^\infty$ as U^∞ is boundedly Fatou. In the pseudo-Fatou case, given $U \in \mathscr{U}$, choose $V \in \mathscr{U}$ so that $V + V \subset U$. As (x_α) is μ-Cauchy, there exists α_0 so that for any $\beta, \alpha \geqslant \alpha_0$, $x_\beta - x_\alpha \in V$. Find $n \in N$ so large that $x_{\alpha_0} \in nV$. Then $x_\beta - x_{\alpha_0} + x_{\alpha_0} \in V + nV \subset nU$ i.e., $((1/n)x_\beta)_{\beta \geqslant \alpha_0} \subset U \subset U^\infty$. But $((1/n)x_\beta)_{\beta \geqslant \alpha_0}$ is μ-Cauchy and increases to $(1/n)x$, whence $(1/n)x \in U^\infty$ as the latter set is pseudo-Fatou. Thus in both cases $x \in v(C^\infty, \mathscr{U}^\infty) = M$ by 0.4. This shows that (M, μ) has BOC (resp. POC). As it is locally s-o-c (resp. pseudo-s-o-c), the result follows by 1.5 and 1.6, respectively.

1.8. Corollary. *Let $\mathscr{F}$ be a Hausdorff pseudo-Fatou function filter in C^∞ and (M, μ) its* TVC. *Suppose (M, μ) is intervally complete. Then (M, μ) is complete.*

Indeed, the interval completeness +MCP implies completeness ([A&B], 7.7).

1.9. Corollary. *Let $\mathscr{F}$ be a Hausdorff σ-pseudo-Fatou function filter in C^∞ which admits a countable base. Then its* TVC (M, μ) *is (metrizable and) complete.*

Arguing as in 1.7, (M, μ) is locally, σ-pseudo-order-complete. Thus (cf. 1.6) has σ-MCP. This implies completeness in metrizable spaces (cf. 5.3 below).

1.10. Corollary. *Let $\mathscr{F}$ be a Hausdorff boundedly Fatou function filter in C^∞ and suppose moreover that its* TVC (M, μ) *is locally bounded (i.e., has a base of bounded neighbourhoods of* 0). *Then (M, μ) is locally* S-O-C.

Indeed, a bounded boundedly-order-complete filter base is order-complete.

§ 2. Generating function filters. In the preceding section to a given function filter $\mathscr{F}$ its vector core, i.e., the largest solid vector subspace M of C^∞ so that $M \cap \mathscr{F}$ is a (solid) vector filter was associated. A converse question is studied now. Given a solid vector subspace L of C^∞ equipped with a solid vector filter of neighbourhoods for a topology τ, a function filter $\mathscr{F}_L$ *inducing* the original one on L is produced so that its vector core is the largest solid vector subspace in C^∞ containing L. The results are also given without assuming that L is solid.

Let L be a solid subspace of $C^\infty = C^\infty(\Omega, R^\infty)$. Consider a subset Ω' of Ω defined as follows

$$\Omega' = \mathrm{closure}\big(\bigcup(\{t\colon\ x(t) > 0\}\colon\ x \in L_+)\big).$$

Then Ω' is an open-closed subset of Ω, in particular, it is an extremally disconnected topological space on its own right, and $L \subset C^\infty(\Omega') = \{x\chi_{\Omega'}\colon x \in C^\infty\}$. Furthermore, observing that the disjoint complement of L in $C^\infty(\Omega')$ is $\{0\}$, L is an *order dense* subspace of $C^\infty(\Omega')$. Thus, more generally, from now on the assumptions are made as follows.

L is an *order dense* Riesz subspace of $C^\infty(\Omega)$ and $\mathscr{V} = \{V\}$ is a solid base of neighbourhoods of 0 for a (locally solid) vector topology τ on L.

The process of enlarging $\mathscr{V}$ will be done in two steps. The first step is nearly obvious. As the search is for a *solid* in C^∞ filter base and $\mathscr{V}$ is not so unless L is solid in C^∞, define

$$\mathrm{sol}\,\mathscr{V} = \{\mathrm{sol}\,V\colon\ V \in \mathscr{V}\}.$$

i.e., $\mathrm{sol}\,\mathscr{V}$ is the filter base consisting of solid hulls in C^∞ of members of $\mathscr{V}$.

As easily checked, $\mathrm{sol}\,V + \mathrm{sol}\,V \subset \mathrm{sol}(V+V)$. Thus $\mathrm{sol}\,\mathscr{V}$ is a solid summative filter base in (C^∞ and in fact in) $\mathrm{sol}\,L$. Since for any $x \in \mathrm{sol}\,L$ there exists $y \in L_+$ with $|x| \leqslant y$, it is clear that $\mathrm{sol}\,\mathscr{V}$ is absorbent in $\mathrm{sol}\,L$. Thus, finally we have

2.1. Proposition. $\mathrm{sol}\,\mathscr{V}$ *is a solid vector filter base in* $\mathrm{sol}\,L$ *which induces* $\mathscr{V}$ *on* L (*i.e.*, $\mathrm{sol}\,\mathscr{V} \cap L = \mathscr{V}$) *and defines a (locally solid vector) topology* $\mathrm{sol}\,\tau$ *on* $\mathrm{sol}\,L$. *If* $\mathscr{V}$ *is Hausdorff then* $\mathrm{sol}\,\mathscr{V}$ (*hence* $\mathrm{sol}\,\tau$) *is so, too.*

The latter statement is an easy consequence of the order denseness of L in C^∞ and follows also from 2.3 in view of 2.4.

Let now ϱ be a Riesz F-semi-norm (resp. F-norm) on L. Then, if $\mathscr{V} = \{V = \{x\colon\ \varrho(x) \leqslant a\}\colon\ a \in R_+\}$,

$$\mathrm{sol}\,\varrho(x) = \inf\{\varrho(y)\colon\ |x| \leqslant y\},\ x \in \mathrm{sol}\,L$$

is the F-semi-norm (F-norm) so that

$$\operatorname{sol} V = \{x\colon \operatorname{sol}\varrho(x) \leqslant a\}.$$

If ϱ is homogeneous, $\operatorname{sol}\varrho$ is such. If $\mathscr{V}$ defines the topology τ and $\mathscr{R} = \{\varrho\}$ is a family of Riesz F-semi-norms defining τ, then $\operatorname{sol}\mathscr{R} = \{\operatorname{sol}\varrho\}$ defines $\operatorname{sol}\tau$.

In the enlargement $(\operatorname{sol} L, \operatorname{sol}\tau)$ the new elements are added to the members V of $\mathscr{V}$ 'from below' since $\operatorname{sol} V = \bigcup\{\{y \in C^{\infty}\colon |y| \leqslant x\}\colon x \in V\}$. As we have seen, all and only the elements of $\operatorname{sol} L$ are added. Another, in a sense *complementary*, procedure in which the new elements are added 'from above' can be defined as follows.

Let H be an arbitrary subset in $C^{\overline{\infty}}$ (in particular, L, C^{∞}, $C^{\overline{\infty}}$ itself, etc.). By definition, for $x \in C^{\overline{\infty}}$,

$$H_x = H_{|x|} = \{y \in H\colon |y| \leqslant |x|\}.$$

With this notation, for a set $W \subset C^{\overline{\infty}}$ define

2.2.
$$W_H^{\overline{\infty}} = \{x \in C^{\overline{\infty}}\colon H_x \subset W\},$$
$$W_H^{\infty} = \{x \in C^{\infty}\colon H_x \subset W\} = W^{\overline{\infty}} \cap C^{\infty}.$$

In what follows when $H = L$ is understood, the notation $W^{\overline{\infty}}$ and W^{∞} is used.

2.3. *$\mathscr{V}^{\overline{\infty}} = \{V^{\overline{\infty}}\colon V \in \mathscr{V}\}$ is a solid filter base in $C^{\overline{\infty}}$. In particular,* $\operatorname{sol}\mathscr{V}$ *is finer than* $\mathscr{V}^{\overline{\infty}}$.

Indeed, suppose $x \in V^{\overline{\infty}}$ and $y \in C^{\overline{\infty}}$, $|y| \leqslant |x|$. Then $L_y \subset L_x$ implies $L_y \subset V$, hence $y \in V^{\overline{\infty}}$.

2.4. *$\mathscr{V}^{\overline{\infty}}$ is Hausdorff provided $\mathscr{V}$ is such.*

Indeed, let $x \in C^{\overline{\infty}}$ and $x \in \bigcap\{V^{\overline{\infty}}\colon V \in \mathscr{V}\}$. Then $L_x \subset \bigcap\{V\colon V \in \mathscr{V}\} = \{0\}$. Hence for each $y \in C^{\infty}$, $|y| \leqslant |x|$, $L_y \subset L_x = \{0\}$. This means that $(C^{\infty})_x = \{0\}$. It is obvious that if $x \in C^{\overline{\infty}}$ is non-zero, then one can find a non-zero $y \in C^{\infty}$ with $|y| \leqslant |x|$. Thus $(C^{\infty})_x = \{0\}$ implies $x = 0$.

The following characterization of $v(C^{\infty}, \mathscr{V}^{\overline{\infty}})$ in the Hausdorff situation will be very useful.

2.5. *Suppose $\mathscr{V}$ is Hausdorff. Then we have* $v(C^{\infty}, \mathscr{V}^{\overline{\infty}}) = v(C^{\overline{\infty}}, \mathscr{V}^{\overline{\infty}}) = \{x \in C^{\overline{\infty}}\colon L_x \text{ is } \mathscr{V}\text{-bounded}\}$.

Indeed, the $\mathscr{V}$-boundedness of L_x implies that for any $V \in \mathscr{V}$ there is $n \in N$ so large that $(1/n)L_x = L_{(1/n)x} \subset V$. It follows that $(1/n)x \in V^{\overline{\infty}}$ i.e., $x \in v(C^{\overline{\infty}}, \mathscr{V}^{\overline{\infty}})$. But $v(C^{\overline{\infty}}, \mathscr{V}^{\overline{\infty}}) = V(C^{\infty}, \mathscr{V}^{\overline{\infty}})$ by 0.4.

When one looks '$\overline{\infty}$' and '∞' as procedures by means of which the original set V is extended, on the first extension of V again such procedure may be applied and different iterations can be considered. The next result tells, roughly speaking, that all the new elements has been added already in the first step.

2.6. *Assume that the first extension $V_L^{\overline{\infty}} = V^{\overline{\infty}}$ has been performed and set*

$$L^{\#} = v(C^{\infty}, \mathscr{V}^{\overline{\infty}}), \qquad V^{\#} = V^{\overline{\infty}} \cap L^{\#}.$$

Then

(i) $(V^{\#})^{\overline{\infty}}_{L^{\#}} = V^{\overline{\infty}}$ (*hence* $L^{\#\#} = L^{\#}$),

(ii) $(V^{\infty})^{\overline{\infty}}_{C^{\infty}} = V^{\overline{\infty}}$ (*hence* $(V^{\infty})^{\infty}_{C^{\infty}} = V^{\infty}$),

(iii) $(V^{\overline{\infty}})^{\overline{\infty}}_{C^{\overline{\infty}}} = V^{\overline{\infty}}$.

A moment's reflection shows that (iii) implies (ii) and (i). Now, observe that for any $z \in (C^{\overline{\infty}})_x$, $L_z \subset L_x$, hence $L_x = \bigcup\{L_z\colon z \in (C^{\overline{\infty}})_x\}$. Take $x \in (V^{\overline{\infty}})^{\overline{\infty}}_{C^{\infty}}$, i.e., x so that $(C^{\overline{\infty}})_x \subset V^{\overline{\infty}}$. Then for any $z \in (C^{\overline{\infty}})_x$, $z \in V^{\overline{\infty}}$ which means precisely that $L_z \in V$. Hence $V \supset \bigcup\{L_z\colon z \in (C^{\overline{\infty}})_x\} = L_x$ i.e., $x \in V^{\overline{\infty}}$.

Let τ^{∞} be the (locally solid group) topology defined by $\mathscr{V}^{\infty}$ in the case $\mathscr{V}^{\infty}$ is summative. Then

2.7. *The vector core $L^{\#} = v(C^{\infty}, \tau^{\infty})$ is a closed subset of $(C^{\infty}, \tau^{\infty})$.*

Indeed, take $x_{\alpha} \to x$ (τ^{∞}). Choose $W^{\infty}, V^{\infty} \in \mathscr{V}^{\infty}$ so that $\overline{W^{\infty}} + \overline{W^{\infty}} \subset V^{\infty}$. Furthermore, take α so that $x_{\beta} - x_{\alpha} \in W^{\infty}$ for all $\beta \geqslant \alpha$, and then $n \in N$ so large that $(1/n)x \in W^{\infty}$. Note that $(1/n)x = (1/n)(x - x_{\alpha}) + (1/n)x_{\alpha} \in \overline{W^{\infty}} + W^{\infty} \subset V^{\infty}$. Hence $x \in v(C^{\infty}, \tau^{\infty})$.

Remark. One can check that in fact $W^{\infty} = \overline{W^{\infty}}$, i.e., $\mathscr{V}^{\infty}$ is a closed filter base.

A question arises what are conditions characterizing those L for which (starting from a solid vector filter base on L), one obtains $\mathscr{V}^{\overline{\infty}}$ generating a function filter or, equivalently, those L for which $\mathscr{V}^{\overline{\infty}}_{+}$ is summative. A sufficient condition is that L be solid in C^{∞}.

2.8. Remark. As L is assumed automatically to be order dense, L is solid in C^{∞} if and only if L is Dedekind complete.

2.9. *Assume that L is solid (Riesz subspace of C^{∞}). Then $\mathscr{V}^{\overline{\infty}}_{+}$ is summative.*

The proof will be done in two steps. First note that

(i) Given $x, y \in C^{\overline{\infty}}_{+}$ and $z \in L_{+}$, there exist $x', y' \in L_{+}$ such that $z = x' + y'$, $x' \leqslant x$, $y' \leqslant y$.

To see this, recall that any Riesz space M has the so-called *decomposition property*: if $z, x, y \in M_{+}$, $z \leqslant x + y$ implies the existence of positive x', y' such that $x' \leqslant x$, $y' \leqslant y$ and $z = x' + y'$. Define $E = \Omega \setminus$ closure $\{t\colon x(t) < \infty\}$, $F = \Omega \setminus$ closure $\{t\colon y(t) < \infty\}$, $\Omega' = \Omega \setminus E \cup F$. These are open-closed sets. Set

$$z_1 = z\chi_{\Omega'}, \qquad x_1 = x\chi_{\Omega'}, \qquad y_1 = y\chi_{\Omega'}.$$

These are functions in $C^{\infty}(\Omega')$ thus by the decomposition property of

$C^\infty(\Omega')$ there exist x_1', y_1' such that

(1) $$x_1' \leqslant x\chi_{\Omega'}, \qquad y_1' \leqslant y\chi_{\Omega'} \quad \text{and} \quad z_1' = x_1' + y_1'.$$

In particular, as L is solid in C^∞, $L(\Omega') = \{x\chi_{\Omega'}\colon x \in L\}$ is solid in $C^\infty(\Omega')$; hence

(2) $$x_1', y_1' \text{ are in } L(\Omega')$$

as they are majorized by z_1 which belongs to $L(\Omega')$. Consider $z_2 = z\chi_{E\cup F}$, $x_2 = x\chi_{E\cup F}$, $y_2 = y\chi_{E\cup F}$, and define

$$x_2'(t) = \begin{Bmatrix} (1/2)z_2(t) & \text{on} & E\cap F & (1/2)z_2(t) \\ z_1(t) & \text{on} & E\setminus F & 0 \\ 0 & \text{on} & F\setminus E & z_1(t) \end{Bmatrix} = y_2'(t).$$

By the same argument as above, these are continuous functions in $L(E\cup F)$, $x_2' \leqslant x_2$, $y_2' \leqslant y_2$ and $z_2 = x_2' + y_2'$. It is visible that $x' = x_1' + x_2'$ and $y' = y_1' + y_2'$ are as required in (i).

Now, take $w \in V_+^{\overline{\infty}} + V_+^{\overline{\infty}}$. Then $w = x + y$ with $x, y \in V_+^{\overline{\infty}}$. Take $0 \leqslant z \in L_w$. Then $z \leqslant x + y$ and by (i) $z = x' + y' \in V^{\overline{\infty}} \cap L + V^{\overline{\infty}} \cap L = V + V$. Hence $w \in (V+V)^{\overline{\infty}}$.

Recapitulating 2.2–2.9 we obtain

2.10. Proposition. *Let L be a solid (order dense) Riesz subspace of C^∞ and $\mathscr{V}$ a solid base of neighbourhoods of 0 for a (locally solid) vector topology τ on L. Then the base $\mathscr{V}^{\overline{\infty}}$ generates a function filter, denoted by $\mathscr{T}_L$, on $C^{\overline{\infty}}$; its vector core $(L^\#, \tau^\#)$ is Hausdorff provided τ is so on L and $\tau^\# \cap L = \tau$.*

Assume now that $(L, \varrho) \subset C^\infty$ is a Riesz space with a Riesz F-semi-norm (F-norm) ϱ. Then if $\mathscr{V} = \{V = \{x\colon \varrho(x) \leqslant a\}\colon a \in R_+\}$,

$$\varrho^{\overline{\infty}}(x) = \sup\{\varrho(y)\colon |y| \leqslant |x|\}, \qquad x \in C^{\overline{\infty}},$$

is an extended real valued functional defining $\mathscr{V}^{\overline{\infty}} = \{V^{\overline{\infty}}\colon V \in \mathscr{V}\}$ by putting

$$V^{\overline{\infty}} = \{x \in C^{\overline{\infty}}\colon \varrho^{\overline{\infty}}(x) \leqslant a\}.$$

If L is solid in C^∞, then $\varrho^{\overline{\infty}}$ is a function group semi-norm (norm). If ϱ is homogeneous, $\varrho^{\overline{\infty}}$ is such.

As a corollary one infers

2.11. *The function filter $\mathscr{T}_L$ can be generated by*

(i) *a function group semi-norm (resp. norm) if and only if it admits a countable base (resp. Hausdorff base);*

(ii) *a family of function group semi-norms of the form $\varrho^{\overline{\infty}}$, where ϱ is a Riesz F-semi-norm in the corresponding family (of Riesz F-semi-norms) defining the topology τ in L.*

Assume $\mathscr{V}$ is moreover (pseudo) order-closed. Then it is natural to ask for (pseudo) order-closed enlargements of $\mathscr{V}$. Clearly, sol $\mathscr{V}$ is no more appropriate a procedure of enlarging $\mathscr{V}$. An idea would be to replace solid hulls by something like their (pseudo) order-closures, as follows

$$\underline{\mathrm{sol}\,V} = \{x \in C^{\overline{\infty}}\colon \exists x_a \uparrow x,\ (x_a) \subset \mathrm{sol}\,V\},$$

$$\underline{\text{pseudo}}\text{-sol}\,V = \{x \in C^{\overline{\infty}}\colon \exists x_a \uparrow x,\ (x_a) \text{ Cauchy},\ (x_a) \subset \mathrm{sol}\,V\}.$$

The underlining bar is here an ad hoc notation for the 'one step order closure in $C^{\overline{\infty}}$', and '(x_a) Cauchy' is with respect to sol τ on sol L.

Let us examine what these definitions really give.

The 'order-closed case'. If $0 \leqslant x \in \underline{\mathrm{sol}\,V}$ and $0 \leqslant y \leqslant x$, $y \in L$ then $x_a \wedge y \uparrow y$. But each $x_a \wedge y \in \mathrm{sol}\,V$, so by definition of the solid hull for each $x_a \wedge y$ there exists $u_a \geqslant x_a \wedge y$, $u_a \in V$. Taking finite suprema of u_a's an increasing net (w_β) is defined in L. Then $(w_\beta \wedge y) \subset V$ and $w_\beta \wedge y \uparrow y$. Hence $y \in V$ by order-closedness of V. This implies that $L_x \subset V$ i.e., $x \in V^{\overline{\infty}}$.

Furthermore, defining $\underline{V}$ in an analogous way to $\underline{\mathrm{sol}\,V}$ one infers

2.12. $\underline{V} \subset \underline{\mathrm{sol}\,V} \subset V^{\overline{\infty}} \subset \underline{V}$ *or, in other words, the extension procedures '$\underline{\ \ }$', '$\underline{\mathrm{sol}}$' and '$\overline{\infty}$' coincide.*

This coincidence has a remarkable consequence.

2.13. *Suppose $V, W \in \mathscr{V}$ are such that $V + V \subset W$. Then $V_+^{\overline{\infty}} + V_+^{\overline{\infty}} \subset W_+^{\overline{\infty}}$.*

Indeed, $V_+^{\overline{\infty}} + V_+^{\overline{\infty}} = \underline{V_+} + \underline{V_+} \subset \underline{W_+} = W_+^{\overline{\infty}}$.

2.14. Proposition. *Let L be a Riesz subspace of C^∞ with a locally solid-order-closed vector topology τ. Then $\mathscr{T}_L$ is an order-complete function filter in $C^{\overline{\infty}}$. In particular, $\mathscr{T}_L$ is Fatou.*

Proof. Noting that $(V^{\overline{\infty}})_+ = V_+^{\overline{\infty}}$, $\mathscr{V}_+^{\overline{\infty}}$ is summative by 2.13 i.e., $\mathscr{T}_L$ is a function filter. To see that it is order-complete in $C^{\overline{\infty}}$, take $0 \leqslant x_a \uparrow \subset V^{\overline{\infty}}$ and let $x = \sup x_a$ in $C^{\overline{\infty}}$. For any $0 \leqslant y \in L_x$, $x_a \wedge y \uparrow y$ and $(x_a \wedge y) \subset \mathrm{sol}\,L$. Hence by the argument used before 2.12, $y \in V$. It follows that $x \in V^{\overline{\infty}}$.

The 'pseudo-order-closed case'. Assuming more about L, an analogue of 2.14 holds true:

2.15. Proposition. *Let L be a solid Riesz subspace of C^∞ with a locally solid-pseudo-order-closed vector topology τ. Then $\mathscr{T}_L$ is a pseudo-Fatou function filter.*

Proof. $\mathscr{V}^\infty$ is summative by 2.9. Thus it has to be shown that $\underline{\text{pseudo}}\text{-}V^\# \subset V^{\overline{\infty}}$. To this end, take $0 \leqslant x_a \uparrow \subset V^\#$, (x_a) $\tau^\#$-Cauchy. Let $x = \sup x_a$ in $C^{\overline{\infty}}$ and $0 \leqslant y \in L_x$. Then $x_a \wedge y \uparrow y$, $(x_a \wedge y) \subset V^\# \cap L = V$ and $(x_a \wedge y)$ is $\tau^\# \cap L = \tau$-Cauchy. As V is pseudo-order-closed, $y \in V$. It follows that $x \in V^{\overline{\infty}}$; hence $\underline{\text{pseudo}}\text{-}V^\# \subset V^{\overline{\infty}}$.

I have asserted at the beginning of this section that $L^\#$ is the largest vector space associated with (L, τ). This needs some discussion which will be postponed to the following section.

§ 3. Enlargements of Riesz subspaces in C^∞. Let L be a Riesz subspace of C^∞ equipped with a solid vector filter base $\mathscr{V}$. According to the convention adopted in § 2, L is automatically assumed to be *order dense* in C^∞.

3.1. DEFINITIONS. (i) A solid subset M of C^∞ with a solid absorbent filter base $\mathscr{U}$ is said to *quasi-enlarge* $(L, \mathscr{V})$ or to be a *quasi-enlargement* of $(L, \mathscr{V})$ if $L \subset M$ and $\mathscr{U} \cap L \sim \mathscr{V}$ (i.e., these filter bases are equivalent).

(ii) A Riesz subspace M with a solid vector filter base $\mathscr{U}$ is said to *enlarge*, or to be an *enlargement* of $(L, \mathscr{V})$ if

(1) $L \subset M$,
(2) M is an order ideal (= solid vector subspace) of C^∞,
(3) $\mathscr{U} \cap L \sim \mathscr{V}$.

Remark. Thus an enlargement is a 'summative' quasi-enlargement. When vector filter bases $\mathscr{V}$, $\mathscr{U}$ on Riesz subspaces L, M are concerned, the more familiar language of topologies can and will also be used. For instance, if $\mathscr{V}$ defines τ and $\mathscr{U}$ defines μ, then, clearly, (M, μ) enlarges (L, τ) iff $(M, \mathscr{U})$ enlarges $(L, \mathscr{V})$, etc.

Generalizing this definition in another direction yet, instead of one space (L, τ) (where τ is the topology corresponding to $\mathscr{V}$) and its enlargements in C^∞, one can consider compatible families of subspaces.

A family $\{L_a\}$ of order dense solid subspaces of C^∞, each L_a equipped with a locally solid vector topology τ_a is *compatible* if τ_a coincides with τ_β on $L_{a\beta} = L_a \cap L_\beta$. Denoting by $\tau_{a\beta}$ the common topology on $L_{a\beta}$, it is clear that

(i) $L_{a\beta}$ is solid order dense in C^∞,
(ii) (L_a, τ_a) and (L_β, τ_β) are enlargements of $(L_{a\beta}, \tau_{a\beta})$.

One can consider compatible families assuming, e.g. that $L_{a\beta}$ is τ_a and τ_β dense in L_a and L_β, respectively. The latter happens automatically in the order-continuous case. One can also assume that τ_a are locally s-o-c, then in view of 2.13 there is no need to assume L_a to be solid, etc.

There is a natural ordering $\dashv$ between the elements of a compatible family and also between quasi-enlargements of (L, τ). Notably, if $(M_i, \mathscr{U}_i)$, $i = 1, 2$, are in the family or are quasi-enlargements, then

$$(M_1, \mathscr{U}_1) \dashv (M_2, \mathscr{U}_2)$$

if $M_1 \subset M_2$ and $\mathscr{U}_1$ is finer than $\mathscr{U}_2$.

The adjectives smaller and larger may be used with respect to $\dashv$ in a natural way. It will be shown now that given an (L, τ), $(\mathrm{sol}L, \mathrm{sol}\tau)$ is its smallest or *minimal* enlargement and (L, τ) is the largest or *maximal* quasi-enlargement of (L, τ). With this terminology, the enlargement $\big((\mathrm{sol}L)^\#, (\mathrm{sol}\,\tau)^\#\big)$ is the *min-max* enlargement.

Remark. An interesting result about the min-max enlargement is 4.10 below. A study of a compatible family is done in [2]. It is easy to see that for each a, $(L_a^\#, \tau_a^\#)$ is truly a maximal element of the compatible family $\{(L_a, \tau_a)\}$ mentioned above.

In what follows L, M are Riesz (order dense) subspaces in C^∞ with locally solid vector topologies τ and μ, respectively; $\mathscr{V}$ and $\mathscr{U}$ are respective bases of solid neighbourhoods of 0 for τ and μ.

3.2. PROPOSITION. *Suppose $(M, \mathscr{U})$ enlarges $(L, \mathscr{V})$. Then we have* $(\mathrm{sol}L, \mathrm{sol}\,\mathscr{V}) \dashv (M, \mathscr{U})$.

Proof. $(\mathrm{sol}L, \mathrm{sol}\mathscr{V})$ enlarges $(L, \mathscr{V})$. Clearly, as M is solid in C^∞ and contains L, $\mathrm{sol}L \subset M$. Since $\mathscr{U} \cap L \sim \mathscr{V}$, there is $V \subset U \cap L$. As U may be taken solid, $U \supset \mathrm{sol}V$; whence $\mathrm{sol}\mathscr{V}$ is finer than $\mathscr{U}$.

3.3. PROPOSITION. *Suppose $(M, \mathscr{U})$ quasi-enlarges $(L, \mathscr{V})$. Then $(M, \mathscr{U}) \dashv (L^\#, \mathscr{V}^\#)$. In particular, if $\mathscr{V}^\#$ is summative, then $(L^\#, \tau^\#)$ is the largest enlargement of (L, τ).*

Proof. Given $x \in U$, $L_x = M_x \cap L \subset U \cap L$ which means that $x \in (U \cap L)_L^\infty$. Hence $U \subset (U \cap L)_L^\infty$, i.e., $\mathscr{U}$ is finer than $\{(U \cap L)_L^\infty : U \in \mathscr{U}\}$. The latter filter base is, however, equivalent to $\{V_L^\infty : V \in \mathscr{V}\}$ which implies the result.

3.4. PROPOSITION. *Suppose $(L, \tau) \subset (M, \mu)$ with the continuous inclusion and, moreover, either*

(i) *μ is locally solid-order-closed, or*
(ii) *L is solid in M and L is μ-dense in M.*

Then $(L^\#, \mathscr{V}^\#) \dashv (M^\#, \mathscr{U}^\#)$. In particular, if both $(L^\#, \tau^\#)$ and $(M^\#, \mu^\#)$ are the respective enlargements of (L, τ) and (M, μ) (i.e., $\mathscr{V}^\#$, $\mathscr{U}^\#$ are summative or equivalently $\mathscr{V}^\infty$, $\mathscr{U}^\infty$ are function filter bases), then $(L^\#, \tau^\#) \subset (M^\#, \mu^\#)$ with the continuous inclusion.

Proof. For each $U \in \mathscr{U}$ there exists $V \in \mathscr{V}$ with $V \subset U$. Take $x \in V_L^\infty$ and consider $0 \leqslant y \in M_x$.

The (i) case: $\mathscr{U}$ may be assumed to be order-closed, and by order denseness $y = \sup L_y$. Clearly, $L_y \subset L_x \subset V \subset U$. But U is order-closed, hence $y \in U$. It follows that $x \in U_M^\infty$; this means that $\mathscr{V}_L^\infty$ is finer than $\mathscr{U}_M^\infty$ and implies the result.

The (ii) case: By μ-denseness it is possible to find $(y_a) \subset L_+$, $y_a \to y\,(\mu)$. Replacing (y_a) by $(y_a \wedge y)$ if needed, it may be assumed that $(y_a) \subset V$. It follows that $y \in \overline{V} = \mu$-closure of V in M. But $\overline{V} \subset U$, as U may be assumed to be μ-closed. Hence $M_x \subset U$, $x \in U_M^\infty$, etc.

Remark. Note that if Dedekind completeness is assumed, then the assumption that L be solid in M is automatically satisfied. Compare 2.8.

3.5. COROLLARY. *Suppose (L, τ) is a solid subspace in (M, μ) (i.e., L is an order ideal in M and $\mu \cap L = \tau$). Consider the following conditions:*

(i) *$(L^\#, \mathscr{V}^\#)$ is a quasi-enlargement of $(M, \mathscr{U})$.*
(ii) *$(L^\#, \mathscr{V}^\#) = (M^\#, \mathscr{U}^\#)$ (i.e., $L^\# = M^\#$ and $\mathscr{V}^\# \sim \mathscr{U}^\#$).*
(iii) *L is μ-dense in M.*

Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

Proof. (a) Observe that $(M^{\#}, \mathscr{U}^{\#})$ is a quasi-enlargement of $(L, \mathscr{V})$, whence $(M^{\#}, \mathscr{U}^{\#}) \dashv (L^{\#}, \mathscr{V}^{\#})$ is always true by 3.3.

(b) (i)$\Rightarrow$(ii): By (a) only $(L^{\#}, \mathscr{V}^{\#}) \dashv (M^{\#}, \mathscr{U}^{\#})$ has to be shown. But if (i) holds, this follows by 3.3 again. (ii)$\Rightarrow$(i) is trivial.

(c) (iii)$\Rightarrow$(ii): By (a) only $(L^{\#}, \mathscr{V}^{\#}) \dashv (M^{\#}, \mathscr{U}^{\#})$ has to be shown. This is a consequence of 3.4 (ii).

3.6. Corollary. *Suppose (L, τ) is a locally solid-order-closed subspace of (M, μ) (i.e., $L \subset M$ and $\mu \cap L = \tau$ is locally* s-o-c *on L.) The following are equivalent*:

(i) *$(L^{\#}, \tau^{\#})$ is an enlargement of (M, μ).*

(ii) $(L^{\#}, \tau^{\#}) = (M^{\#}, \mu^{\#})$.

(iii) *(M, μ) is locally solid-order-closed.*

Proof. The argument in 3.5 gives $(L^{\#}, \mathscr{V}^{\#}) = (M^{\#}, \mathscr{U}^{\#})$. But now, by 2.13, $\mathscr{V}^{\#}$ is vector, hence $\mathscr{U}^{\#}$ is such. This gives (i)$\Leftrightarrow$(ii).

(ii) $\Rightarrow$(iii): $(L^{\#}, \tau^{\#})$ is locally s-o-c by 2.14 and 1.3 (3). Hence $(M^{\#}, \mu^{\#})$ is such by (ii). It follows that (M, μ) is locally s-o-c (order denseness implies regularity).

(iii)$\Rightarrow$(ii) can be shown as in 3.5 using 3.4 (i) instead of 3.4 (ii).

3.7. Proposition. *Suppose (L, τ) is a solid Hausdorff* tvs *contained in C^{∞}. The following are equivalent*:

(i) *(L, τ) is locally solid-pseudo-order-closed.*

(ii) *(L, τ) is a subspace of the* TVC *of a Hausdorff pseudo-Fatou function filter $\mathscr{F}$ in C^{∞}.*

(iii) *(L, τ) is a subspace of the* TVC *of the Hausdorff pseudo-Fatou function filter $\mathscr{F}_L$ in C^{∞}.*

Indeed, (iii)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i) are clear, (i)$\Rightarrow$(iii) is the contents of 2.10+2.15.

By a similar argument, one also has

3.8. Proposition. *Suppose (L, τ) is a Hausdorff* tRs *contained in C^{∞}. The following are equivalent*:

(i) *(L, τ) is locally solid-order-closed.*

(ii) *(L, τ) is a subspace of the* TVC *of a Hausdorff boundedly Fatou function filter $\mathscr{F}$ in C^{∞}.*

(iii) *(L, τ) is a subspace of the* TVC *of the Hausdorff order-complete function filter $\mathscr{F}_L$ on C^{∞}.*

3.9. Definition. Let $\mathscr{F}, \mathscr{G}$ be function filters on C^{∞}. They are said to be *core-equivalent* if they have the common TVC.

In 3.7, in view of 3.3, $(M, \mu) \subset (L^{\#}, \tau^{\#})$ continuously (where (M, μ) is the TVC of $\mathscr{F}$). I do not know how to characterize the situation in which (M, μ) becomes a subspace of $(L^{\#}, \tau^{\#})$ (i.e., when $(M^{\#}, \mu^{\#}) = (L^{\#}, \tau^{\#})$). A sufficient condition is in 3.3.

3.10. Proposition. *Given a Hausdorff (order dense) (L, τ) in C^{∞}, all boundedly Fatou function filters whose* TVC's *enlarge (L, τ) are equivalent.*

Proof. By 3.6 $(M^{\#}, \mu^{\#}) = (L^{\#}, \tau^{\#})$. Furthermore, as (M, μ) has the BOC property (cf. 1.3 (4)), it is clear that $(M^{\#}, \mu^{\#}) = (M, \mu)$ (see 2.5). Thus, finally $(M, \mu) = (L^{\#}, \tau^{\#})$, i.e., the TVC of $\mathscr{F}$, and the one of $\mathscr{F}_L$ coincide.

§ 4. Completeness properties of locally solid Riesz spaces. Given an Archimedean Riesz space L, let $\mathscr{B}_L$ be its Boolean algebra of projections. $\mathscr{B}_L$ is Dedekind complete ([L&Z]), 22.7 and 22.8), and therefore by a theorem of Stone ([L&Z], 47.5) can be represented as the algebra of open-closed subsets of an extremally disconnected compact Hausdorff space Ω_L. By a theorem of Maeda and Ogasawara ([L&Z], 50.8) L can be embedded order densely into $C^{\infty}(\Omega_L)$, i.e., there exists a Riesz isomorphism

$$S\colon L \to C^{\infty}(\Omega_L)$$

so that $S(L)$ is order dense in $C^{\infty}(\Omega_L)$ (which means also that S is order continuous). Therefore, $C^{\infty}(\Omega_L)$ is a realization of what is called a universal completion L^u of L ([A&B], 23.19). Any two universal completions of L are Riesz isomorphic. Thus, identifying Riesz isomorphic spaces, it will be convenient to think of L as an order dense Riesz subspace of *the* universal completion $L^u = C^{\infty}(\Omega_L)$. With this convention, the solid hull sol$L$ of L in $C^{\infty}(\Omega_L)$ is precisely *the* Dedekind completion L^{δ} of L ([L&Z], 50. 8). Furthermore, if L is an order dense subspace of some Archimedean Riesz space M, then M is Riesz isomorphic to an order dense Riesz subspace of L^u. Hence if L may be embedded order densely into M, then still $M^u = L^u$ provided L is identified with its image in M ([A&B], 23.21). This establishes a one-one correspondence of the notion of the enlargement of $L \subset C^{\infty}(\Omega_L)$ as in 3.1 and the following

4.1. Definition. Let L, M be Archimedean Riesz spaces with locally solid vector topologies τ and μ, respectively. (M, μ) is an *enlargement* or *enlarges* (L, τ) if

(i) there exists a Riesz isomorphism $S\colon L \to M$ so that $S(L)$ is order dense in M,

(ii) $S\colon (L, \tau) \to (M, \mu)$ is a homeomorphism ono its image,

(iii) M is Dedekind complete.

Note that (iii) comes from the fact that an order dense $L \subset C^{\infty}$ is DC iff it is solid.

Agreeing that an *isomorphism* is a map which preserves all structures and an *embedding* is an isomorphism into[3], 4.1 may be expressed by

[3] That is, for instance, in (i) an embedding = linear injection preserving sup and inf = Riesz isomorphism into; however, in (ii) it preserves moreover the topological structures involved.

saying that (M, μ) is an enlargement of (L, τ) if it is a DC locally solid tRs so that (L, τ) can be embedded order densely into.

Similarly, $(L^{\#}, \tau^{\#})$ is just a locally solid tRs isomorphic with $(L^{\#}, \tau^{\#})$ constructed in § 2. Since $\mathrm{sol}L = L^{\delta}$, a symbol usually connected with the Dedekind completion, $(L^{\delta}, \tau^{\delta})$ replaces $(\mathrm{sol}L, \mathrm{sol}\tau)$; the min-max enlargement is $(L^{\delta\#}, \tau^{\delta\#})$, etc.

1.7 and 2.14 imply directly

4.2. Theorem. *A Hausdorff locally solid* tRs (L, τ) *can be embedded order densely into a locally boundedly* S-O-C *space* $(L^{\#}, \tau^{\#})$ *if and only if* τ *is locally solid-order-closed. This embedding is unique up to an isomorphism.*

The uniqueness statement follows by 3.10 and means that if (L, τ) embeds order densely into another locally boundedly S-O-C (M, μ), then $(L^{\#}, \tau^{\#}) \cong (M, \mu)$.

4.3. Theorem. *A Hausdorff Dedekind complete locally solid* tRs (L, τ) *can be embedded order densely into* $(L^{\#}, \tau^{\#})$ *having additionally* MCP *if and only if* τ *is locally solid-pseudo-order-closed.*

While 4.2 seems to be an optimal result, some further disscussion connected with 4.3 can be done. As mentionned in § 1, locally boundedly S-O-C spaces are topologically complete. However, I do not know whether $L^{\#}$ is $\tau^{\#}$-complete in 4.3 above, nor do I know whether a stronger assumption that (L, τ) is intervally complete implies $\tau^{\#}$-completeness of $L^{\#}$. The next two results show that these are really questions concerning the interval completeness.

4.4. Proposition. *The following are equivalent:*

(i) *Any* DC *Hausdorff locally pseudo*-s-o-c (L, τ) *has* $\tau^{\#}$*-complete* $L^{\#}$.

(ii) *Any* DC *Hausdorff locally pseudo*-s-o-c (L, τ) *has intervally* $\tau^{\#}$*-complete* $L^{\#}$.

(iii) *Any* DC *Hausdorff locally solid* (L, τ) *having* MCP *is intervally* τ*-complete (and hence* τ*-complete* [A&B], 7.7).

Proof. (i) ⇒ (iii): MCP ⇒ locally pseudo-s-o-c $\overset{(i)}{\Rightarrow} L^{\#}$ is $\tau^{\#}$-complete ⇒ L is intervally τ-complete (as L may be identified with a solid subspace of $L^{\#}$).

(iii) ⇒ (ii): Given (L, τ) as in (ii), it has $(L^{\#}, \tau^{\#})$ having MCP by 4.3 above. Hence by (iii) applied to $(L^{\#}, \tau^{\#})$, the latter is intervally $\tau^{\#}$-complete.

(ii) ⇒ (i) follows by 1.8.

4.5. Proposition. *The following are equivalent for Hausdorff* (L, τ):

(i) *Any* DC *intervally* τ*-complete* (L, τ) *has* $\tau^{\#}$*-complete* $L^{\#}$.

(ii) *Any* DC *intervally* τ*-complete* (L, τ) *has intervally* $\tau^{\#}$*-complete* $L^{\#}$.

(iii) *Any* DC τ*-complete* (L, τ) *has intervally* $\tau^{\#}$*-complete* $L^{\#}$.

Proof. (iii) ⇒ (ii): Let a DC intervally complete (L, τ) be given. Then the topological Riesz completion $(\hat{L}, \hat{\tau})$ is an enlargement of (L, τ) ([A&B], 7.3+7.8). Hence $(L^{\hat{}\#}, \tau^{\hat{}\#})$ enlarges (L, τ) as well. It follows by a maximality argument (cf. 3.3) that $(L^{\hat{}\#}, \tau^{\hat{}\#}) \hookrightarrow (L^{\#}, \tau^{\#})$ continuously. On the other hand, by 3.4 (ii) $(L^{\#}, \tau^{\#}) \hookrightarrow (L^{\hat{}\#}, \tau^{\hat{}\#})$ continuously. Consequently $(L^{\#}, \tau^{\#}) \cong (L^{\hat{}\#}, \tau^{\hat{}\#})$ and the latter space is intervally complete by (iii). Thus $(L^{\#}, \tau^{\#})$ is intervally complete, which shows (ii).

(ii) ⇒ (i): $(L^{\#\hat{}}, \tau^{\#\hat{}})$ enlarges $(L^{\#}, \tau^{\#})$ ([A&B], loc. cit.) and so, by a maximality argument, $L^{\#\hat{}} \hookrightarrow L^{\#}$. The inverse inclusion being obvious, $L^{\#\hat{}} \cong L^{\#}$ i.e., $L^{\#}$ is $\tau^{\#}$-complete.

(i) ⇒ (iii) is trivial.

4.6. Definition. A locally solid tRs (L, τ) is said to be *enlarged* if $L \cong L^{\#}$ (i.e., they are isomorphic).

Note that it follows from this definition that $L^{\#}$ is a vector space. As it is solid in $C^{\infty}(\Omega_L)$ (see 0.3), $L^{\#}$ and L are DC. Furthermore, $\tau^{\#} = \tau^{\infty} \cap L^{\#} = \tau^{\infty} \cap L = \tau$. Consequently,

4.7. (L, τ) *is enlarged iff* L *is* DC *and* $(L, \tau) \cong (L^{\#}, \tau^{\#})$.

Any (L, τ) having the BOC property is Hausdorff and by 2.5 is enlarged. Examples of enlarged spaces without BOC are not difficult to produce and can be found in [1].

4.8. Theorem. *Let* (L, τ) *be a Hausdorff enlarged space.*

(i) (L, τ) *is* τ*-complete if (and only if) it is intervally* τ*-complete.*

(ii) (L, τ) *has* MCP *if (and only if) it is locally pseudo*-s-o-c.

(iii) *Suppose* (L, τ) *is metrizable. Then* (L, τ) *is* τ*-complete if (and only if) it is locally* σ*-pseudo*-s-o-c.

(iv) (L, τ) *is locally boundedly* S-O-C *if (and only if) it is locally* s-o-c.

Proof. (i) follows by 1.8; (ii) by 4.3; (iii) by 1.9 and (iv) by 4.2.

Let L be again an arbitrary order dense Riesz subspace of C^{∞} and let $\mathscr{V}$ be a base of neighbourhoods of 0 for some topology τ on L. Without the Dedekind completeness $\mathscr{V}^{\infty}$ is perhaps not summative. On the other hand, the filter base $(\mathrm{sol}\mathscr{V})^{\#}$ associated with the min-max enlargement is summative. But $(\mathrm{sol}L)^{\#} \subset L^{\#}$(3.3), hence in order to assure that (at least) increasing Cauchy sequences do not escape from $(\mathrm{sol}L)^{\#}$, it is reasonable to expect that something more than the pseudo-s-o-c condition should be imposed on τ. Clearly, $(\mathrm{sol}L)^{\#}$ is $(\mathrm{sol}\tau)^{\#}$-complete implies that $\mathrm{sol}L$ is intervally $\mathrm{sol}\tau$-complete. Assume the latter, and moreover, that τ is metrizable. Take $0 \leqslant x_n \uparrow \subset L$, (x_n) τ-Cauchy and (x_n) in some order interval in $\mathrm{sol}L$. Then firstly, there exists $y \in L$ majorizing (x_n) (since L is full in $\mathrm{sol}L$), and secondly there exists $x \in \mathrm{sol}L$ such that $x_n \to x$ $(\mathrm{sol}\tau)$. It turns out that this condition is also sufficient.

4.9. Theorem. *Let* L *be a Riesz subspace of* C^{∞} *and let* ϱ *be a Riesz* F*-norm on* L. $(\mathrm{sol}L, \mathrm{sol}\varrho)$ *is intervally complete if and only if the following condition is satisfied:*

(*) $$0 \leqslant x_n \uparrow \leqslant y \text{ in } L,\ (x_n)\ \varrho\text{-Cauchy} \Rightarrow x_n \to x\ (\mathrm{sol}\varrho).$$

Note that this condition may be viewed as a '*σ-pseudo-Fatou property with respect to* $(\mathrm{sol}\,L, \mathrm{sol}\,\tau)$'.

4.10 COROLLARY. *Let (L, τ) be a metrizable locally solid* tRs *in* C^∞. *Then the min-max enlargement* $(\mathrm{sol}\,L)^\#$ *is* $(\mathrm{sol}\,\tau)^\#$*-complete iff the condition* $(*)$ *is satisfied.*

Proof. By 1.8 and the above, it is clear that 4.9 implies 4.10 and that $(*)$ is necessary. In order to show the sufficiency it will be first shown that $(*)$ implies that $(\mathrm{sol}\,L, \mathrm{sol}\,\varrho)$ satisfies:

$$(**)\qquad 0 \leqslant x_n\uparrow \leqslant y \text{ in } \mathrm{sol}\,L,\ (x_n)\ \mathrm{sol}\,\varrho\text{-Cauchy} \Rightarrow x_n \to x\,(\mathrm{sol}\,\varrho).$$

Indeed, as L is full in $\mathrm{sol}\,L$ it can be assumed that $y \in L$. Moreover, it can also be assumed that

$$\sum_{n=1}^{\infty} \mathrm{sol}\,\varrho(x_n - x_{n-1}) < \infty, \quad x_0 = 0.$$

In view of the definition of $\mathrm{sol}\,\varrho$, it is clear that $v_n \geqslant x_n - x_{n-1}$, $v_n \in L$, $v_n \leqslant y$ (replacing v_n by $v_n \wedge y$ if needed) may be found such that

$$\sum_{n=1}^{\infty} \varrho(v_n) < \infty.$$

Consider $u_n := \sum_{i=1}^{n} v_i$; (u_n) satisfies assumptions of $(*)$; hence $\mathrm{sol}\,\varrho - \sum_{i=1}^{\infty} v_i$ exists and is equal to $(o) - \sum_{i=1}^{\infty} v_i$ in $\mathrm{sol}\,L$. Furthermore, $x = \sup x_n \in \mathrm{sol}\,L$ ($\mathrm{sol}\,L$ is Dedekind complete as an order ideal in C^∞) and

$$x - x_n = (o) - \sum_{i=n+1}^{\infty} x_i - x_{i-1} \leqslant \sum_{i=n+1}^{\infty} v_i = (o) - \Big(\sum_{i=1}^{\infty} v_i - \sum_{i=1}^{n} v_i\Big).$$

Hence $\mathrm{sol}\,\varrho(x - x_n) \leqslant \mathrm{sol}\,\varrho\big(\sum_{i=1}^{\infty} v_i - \sum_{i=1}^{n} v_i\big) \to 0$ with $n \to \infty$. This shows $(**)$. Now $(**)$ implies interval completeness of $(\mathrm{sol}\,L, \mathrm{sol}\,\varrho)$ by a result of Veksler ([6], see 5.4 below).

After this excursion into the framework of § 2, let us come back to the general (isomorphic) point of view. According to the well established terminology *F-lattice* is a complete metrizable locally solid tRs, and if it is not complete then it is termed *F-normed lattice* (recall Banach lattice, normed lattice, etc.).

4.11. THEOREM. *An F-normed lattice (L, ϱ) can be embedded order densely into an F-lattice $(L^{\delta\#}, \varrho^{\delta\#})$ if and only if the following condition is satisfied:*

$$(*)\qquad 0 \leqslant x_n\uparrow \leqslant y \ \text{in } L,\ (x_n)\ \varrho\text{-Cauchy} \Rightarrow x_n \to x\,(\varrho^\delta).$$

Recall that ϱ^δ replaces $\mathrm{sol}\,\varrho$, as L^δ does for $\mathrm{sol}\,L$.

4.12. COROLLARY. *L^δ is intervally ϱ^δ-complete iff $(*)$ holds.*

Indeed, L^δ is solid in L^u whence in $L^{\delta\#}$.

The next corollary is also a direct consequence of 1.9.

4.13. COROLLARY. *An F-normed* DC *lattice (L, ϱ) can be embedded order densely into an F-lattice $(L^\#, \varrho^\#)$ if and only if ϱ is σ-pseudo-order-continuous ($\Leftrightarrow$ σ-pseudo-s-o-c): $0 \leqslant x_n\uparrow x$ in L, (x_n) ϱ-Cauchy $\Rightarrow x_n \to x(\varrho)$.*

Indeed, in this case $L^\delta = L$, $\varrho^\delta = \varrho$, and hence $(*)$ becomes:

$$0 \leqslant x_n\uparrow \leqslant y \ \text{ in } L,\ (x_n)\ \varrho\text{-Cauchy} \Rightarrow x_n \to x(\varrho).$$

But by the Dedekind completeness, $0 \leqslant x_n\uparrow \leqslant y$ iff $x_n\uparrow x$, thus $(*)$ becomes the σ-pseudo-order-continuity.

It is visible that $c_0^\# = l_\infty$. This shows that the order-continuity of the topology τ is not preserved under $\#$, in general.

4.14. THEOREM. *Suppose (L, τ) is a Hausdorff locally solid* tRs *with an order-continuous topology. Then $\tau^\#$ is order-continuous on $L^\#$ if and only if $(L^\#, \tau^\#) \cong (\hat{L}, \hat{\tau})$.*

Proof. The 'if' part is trivial since $\hat{\tau}$ is order-continuous on L ([A&B], 10.6). Suppose $\tau^\#$ is order-continuous on $L^\#$. Then (L, τ) is isomorphic to a dense subspace in $(L^\#, \tau^\#)$ which is complete. Hence $(\hat{L}, \hat{\tau}) \cong (L^\#, \tau^\#)$.

§ 5. Concluding remarks. (1) The three main theorems mentioned in the introduction of this paper seem to be new even in the setting of normed Riesz spaces.

(2) The extension procedure '$\#$' associating $(L^\#, \tau^\#)$ with a DC (L, τ) has been first considered, in normed spaces, by Abramovič [1]. He has shown that an intervally complete normed DC Riesz space (L, τ) has a complete (i.e., Banach) maximal enlargement $(L^\#, \tau^\#)$. Independently this procedure has been discovered about ten years later by Szeptycki [5], who deals with F-norms but places himself in the framework of solid subspaces of L^0. Then L^0 plays the role of the universal completion L^u. The existence of the topology of convergence in measure in L^0 simplifies the situation considerably. In particular, it permits the identification of the sup x without introducing M as in [4], or C^∞ in this paper. Szeptycki shows that a solid F-space (L, ϱ) injected continuously in L^0 has the complete enlargement $(L^\#, \varrho^\#)$. When locally s-o-c topologies are concerned,

see Wnuk [7] and compare [A&B], 11.10. Wnuk has shown, in particular, that $(L^{\#}, \tau^{\#})$ is complete provided (L, τ) is locally s-o-c, and 4.14.

(3) Recall

5.1. THEOREM. *Let (L, ϱ) be an F-normed lattice. The following are equivalent:*

(i) *L is ϱ-complete.*

(ii) *σ-MCP.*

(iii) *σ-POC property.*

(iv) *Riesz–Fischer property: $0 \leqslant x_n$ and $\sum_{n=1}^{\infty} \varrho(x_n) < \infty$ then $(o)-\sum_{n=1}^{\infty} x_n$ exists in L.*

(v) *$0 \leqslant x_n$ and $\sum_{n=1}^{\infty} \varrho(x_n) < \infty$ then $(o)-\sum_{n=1}^{\infty} x_n$ exists in L and $\varrho\left(\sum_{n=1}^{\infty} x_n\right) \leqslant \sum_{n=1}^{\infty} \varrho(x_n)$.*

The equivalence of the first three conditions is referred to, at least in the Russian literature, as Amemiya's theorem. It is formulated there for normed Riesz spaces—according to the general setting in which these authors work. But the proof as given e.g. in [3], is valid without any change for the F-normed case. The Riesz–Fischer property is intensively used by Luxemburg and Zaanen and their school (see, e.g. [4]). The equivalence of (i), (iv) and (v) is usually credited to Halperin, Luxemburg and Zaanen, although the main trick of using multipliers 'k' and a subseries converging with 'k^{-3}' is the same in both proofs. Again, Luxemburg and Zaanen deal with norms only—by the general setting of their work. However, this is not the case of the monograph [A&B], yet the result ([A&B], 16.2) is recorded in its original form, which may be somewhat misleading. Anyway, the proof as given there works for F-norms after one trivial change.

(4) When the interval completeness is concerned, only the condition (ii) above may have a meaningful analogue. This is precisely the condition (**) as used in 4.10. Here is the result of Veksler [6], which is recorded for the sake of completeness (again he originally deals with homogeneous norms, but the proof applies to F-norms):

5.2. PROPOSITION. *An F-normed lattice (L, ϱ) is intervally complete iff (**) is satisfied:*

$$0 \leqslant x_n \uparrow \leqslant y,\ (x_n)\ \varrho\text{-Cauchy} \Rightarrow x_n \to x(\varrho).$$

Proof. Let (x_n) be Cauchy. It may be assumed that $0 \leqslant x_n \leqslant y$, and that $\sum \varrho(x_{n+1}-x_n) < \infty$. Define

$$u_n = x_1 + \sum_{k=1}^{n-1} (x_{k+1}-x_k)_+, \qquad v_n = \sum_{k=1}^{n-1} (x_{k+1}-x_k)_-.$$

Then $\varrho(u_{n+p}-u_n) \leqslant \sum_{k=n}^{\infty} \varrho(x_{k+1}-x_k)$; hence (u_n) is Cauchy. Similarly, (v_n) is Cauchy. For each n, k, p $\varrho\big((u_{k+p}-v_n)\wedge y-(u_k-v_n)\wedge y\big) \leqslant \varrho\big((u_{k+p}-v_n)-(u_k-v_n)\big) = \varrho(u_{k+p}-u_k)$, hence the sequence

$$\big((u_k-v_n)\wedge y\big), \qquad k = 1, 2, \ldots,$$

is order-bounded, increasing and Cauchy. By assumption, there exists

$$w_n = \lim (u_k - v_n)\wedge y \qquad (k \to \infty).$$

Moreover, $0 \leqslant w_n \downarrow$, $\varrho(w_{n+p}-w_n) = \lim_k \varrho\big((u_k-v_{n+p})\wedge y-(u_k-v_n)\wedge y\big) \leqslant \lim_k \varrho\big((u_k-v_{n+p})-(u_k-v_n)\big) = \varrho(v_{n+p}-v_n)$. Hence (w_n) is Cauchy. By assumption, there exists $x = \lim w_n$. But $x = \lim x_n$. Indeed, $x_n = u_n - v_n = (u_n-v_n)\wedge y$; whence $\varrho(x_n-w_n) = \lim_k \varrho\big((u_n-v_n)\wedge y-(u_k-v_n)\wedge y\big) \leqslant \lim_k \varrho(u_n-u_k) \leqslant \sum_{m=n}^{\infty} \varrho(u_{m+1}-u_m) \leqslant \sum_{m=n}^{\infty} \varrho(x_{m+1}-x_m) \to 0$ with n. Thus (w_n) and (x_n) are equivalent Cauchy sequences and $x_n \to x$.

(5) In connection with 4.10 and 4.11 it may be interesting to note that the following result holds.

5.3. PROPOSITION. *Let (L, ϱ) be an F-lattice; then $(L^\delta, \varrho^\delta)$ is such.*

Proof. It may be assumed that $(L^\delta, \varrho^\delta) = (\mathrm{sol}\,L, \mathrm{sol}\,\varrho)$ in L^u. It is sufficient to check the Riesz–Fischer property. Given $(x_n) \subset (\mathrm{sol}\,L)_+$ so that $\sum \mathrm{sol}\,\varrho(x_n) < \infty$, it is possible to find (u_n) such that $x_n \leqslant u_n \in L$ and $\sum \varrho(u_n) < \infty$. Thus $(o)-\sum u_n$ exists in L and majorizes $\sum_{n=1}^{m} x_n$ for all $m \in N$. As $\mathrm{sol}\,L$ is DC, $(o)-\sum u_n$ exists therein.

This permits to embed order densely any locally pseudo-s-o-c F-normed lattice (L, ϱ) into a Dedekind complete F-lattice.

Indeed, by ([A&B], 17.4) and 1.3, $(\hat{L}, \hat{\varrho})$ contains (L, ϱ) order densely, whence by ([A&B], 23.21) $\hat{L} \subset L^u$, and $(L^{\wedge\delta}, \hat{\varrho}^\delta)$ is a complete enlargement of (L, ϱ). Another one is $(L^{\wedge\delta\#}, \varrho^{\wedge\delta\#})$.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
BRANCH IN POZNAŃ
ul. Mielżyńskiego 27/29
61-725 POZNAŃ

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Quasi-complements in F-spaces

by

L. DREWNOWSKI (POZNAŃ)

Dedicated to Professor Jan Mikusiński on the occasion of his 70th birthday

Abstract. Let E be an F-space (a complete metrizable topological vector space). Two closed subspaces X and Y in E are called *quasi-complements* (to each other) if $X \cap Y = \{0\}$ and $X+Y$ is dense in E; if, in addition, $X+Y \neq E$, then they are *proper quasi-complements*. The paper presents some extensions to F-spaces of the results about quasi-complements in Banach spaces obtained by various authors since the introduction of this notion by F. J. Murray in 1945. In particular, the following results are established:

(1) Every non-minimal infinite-codimensional closed subspace in a separable F-space has a proper quasi-complement (an analogue of the Murray–Mackey theorem).

(2) If X and Y are proper quasi-complements in a separable F-space E, then there exist quasi-complements $Y_1 \subset Y$ to X such that $\dim(Y/Y_1) = \infty$ (an extension of a theorem of R. C. James).

(3) If X and Y are proper quasi-complements in a locally convex F-space E, then there exist quasi-complements $Y_2 \supset Y$ to X such that $\dim(Y_2/Y) = \infty$ (a generalization of a result of A. N. Pličko in case of Banach spaces; more restrictive versions were established by R. C. James and W. B. Johnson). Some extensions of results of V. G. Vinokurov and of V. I. Gurarii and M. I. Kadec are also included.

1. Introduction. We shall only consider Hausdorff topological vector spaces (TVS's), in particular, locally convex spaces (LCS's). By an *F-space* we mean a complete metrizable TVS, and a *Fréchet space* is a locally convex F-space.

Let E be a TVS and X, Y two closed subspaces of E that are *transversal*, i.e., $X \cap Y = 0$ $(=\{0\})$. If $X+Y = E$, then X, Y are called *complements* (to each other) in E, while if $X+Y$ is known only to be dense in E, then they are called *quasi-complements*. (The last notion appeared for the first time in [23] in the case of normed spaces.) Quasi-complements which are not complements are said to be *proper quasi-complements*.

We shall be mainly interested in the existence and properties of proper quasi-complements. In view of this, the following remarks are in place.