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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
BRANCH IN POZNAŃ
ul. Mielżyńskiego 27/29
61-725 POZNAŃ

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Quasi-complements in F-spaces

by

L. DREWNOWSKI (POZNAŃ)

Dedicated to Professor Jan Mikusiński on the occasion of his 70th birthday

Abstract. Let E be an F-space (a complete metrizable topological vector space). Two closed subspaces X and Y in E are called *quasi-complements* (to each other) if $X\cap Y=\{0\}$ and $X+Y$ is dense in E; if, in addition, $X+Y\neq E$, then they are *proper quasi-complements*. The paper presents some extensions to F-spaces of the results about quasi-complements in Banach spaces obtained by various authors since the introduction of this notion by F. J. Murray in 1945. In particular, the following results are established:

(1) Every non-minimal infinite-codimensional closed subspace in a separable F-space has a proper quasi-complement (an analogue of the Murray–Mackey theorem).

(2) If X and Y are proper quasi-complements in a separable F-space E, then there exist quasi-complements $Y_1\subset Y$ to X such that $\dim(Y/Y_1)=\infty$ (an extension of a theorem of R. C. James).

(3) If X and Y are proper quasi-complements in a locally convex F-space E, then there exist quasi-complements $Y_2\supset Y$ to X such that $\dim(Y_2/Y)=\infty$ (a generalization of a result of A. N. Plička in case of Banach spaces; more restrictive versions were established by R. C. James and W. B. Johnson). Some extensions of results of V. G. Vinokurov and of V. I. Gurarii and M. I. Kadec are also included.

1. Introduction. We shall only consider Hausdorff topological vector spaces (TVS's), in particular, locally convex spaces (LCS's). By an *F-space* we mean a complete metrizable TVS, and a *Fréchet space* is a locally convex F-space.

Let E be a TVS and X, Y two closed subspaces of E that are *transversal*, i.e., $X\cap Y=0$ $(=\{0\})$. If $X+Y=E$, then X, Y are called *complements* (to each other) in E, while if $X+Y$ is known only to be dense in E, then they are called *quasi-complements*. (The last notion appeared for the first time in [23] in the case of normed spaces.) Quasi-complements which are not complements are said to be *proper quasi-complements*.

We shall be mainly interested in the existence and properties of proper quasi-complements. In view of this, the following remarks are in place.

Recall (see, e.g., [2], [3]) that a TVS is said to be *minimal* if it admits no strictly weaker Hausdorff vector topology, and *non-minimal* otherwise. In particular, we may speak of minimal and non-minimal subspaces of a given TVS. For X, Y as above, if one of them is minimal, then $X+Y$ is closed; in fact, it is even the topological direct sum of X and Y ([3], Proposition 2.3). Hence: *If one of quasi-complements is minimal, then they are complements.* This explains why we have to impose such conditions on a given subspace X which imply its non-minimality, when we wish to prove it has a proper quasi-complement.

It is not known whether any minimal non-locally convex space exists, but an LCS is minimal if and only if it is isomorphic to a product of one-dimensional spaces ([2], Postscript). From this it follows easily that: *Every minimal subspace of E has a complement* (*or quasi-complement*) *if and only if E has a separating dual space E^*.*

Some characteristic properties of non-minimal F-spaces will be recalled in the next section.

The paper presents some extensions to F-spaces of the results about quasi-complements obtained by various authors in the case of Banach spaces (or, sometimes, even normed spaces). We give first a brief review of those results. If not stated otherwise, let E denote below a separable Banach space.

(M–M) The history of quasi-complements began with the work [23] of Murray, where it was proved that if E is reflexive, then every closed subspace of E is quasi-complemented. The reflexivity assumption was soon removed by Mackey [22].

(V) The Murray–Mackey result was improved by Vinokurov ([31], [32]) as follows: If X, W are closed transversal subspaces of E, then there exist quasi-complements Y_1 and Y_2 to X in E such that $W\subset Y_1$ and $W\cap Y_2=0$. (For the existence of Y_1 see also Lohman [20].)

(G–K) Next improvement, with interesting consequences, was due to Gurarii and Kadec [7]: If X_1, X_2 are closed subspaces of E such that $X_1\subset X_2$ and $\dim X_1=\operatorname{codim} X_2=\infty$, then X_1 has a quasi-complement isomorphic to X_2, and X_2 has a quasi-complement isomorphic to X_1. (Very short and elegant proof of this has been recently provided by Weis [34], with separability of E replaced by separability of X_1 and E/X_2.)

(J–J–P) Another direction of research also goes back to Murray and Mackey and was resumed by James [8] who showed that if Y is a proper quasi-complement to X, then X has also quasi-complements Y_1 and Y_2 such that $Y_1\subset Y\subset Y_2$ and $\dim(Y/Y_1)=\dim(Y_2/Y)=\infty$. Actually James' proof of the existence of Y_2 required E to be reflexive, a condition which was disposed of by Johnson [11] and Plička [24]. Finally, Plička [25] has extended the Y_2-part of this result to *arbitrary* Banach spaces E.

For the sake of completeness, let us mention also some of other results for nonseparable Banach spaces: Every closed WCG subspace of a WCG Banach space has a quasi-complement [17]; in particular, closed subspaces in all reflexive Banach spaces have quasi-complements. If Γ is uncountable, then $c_0(\Gamma)$ has *no* quasi-complement in $l_\infty(\Gamma)$ [18]. However, every closed subspace of l_∞, in particular c_0, is quasi-complemented [27].

It seems we do not know as much about quasi-complements in Fréchet spaces. The (M–M) type theorem in [10], Corollary 10, and [33], and the (V) type theorem in [10], Theorem 12, and [20], Remark, for separable Fréchet spaces are the only results the present author has come across in the literature.

The contents and the main ideas of our paper are as follows. Section 2 contains some preliminaries, especially concerning the so-called m-independent sequences in TVS's. Such sequences (v_n) are particularly useful in the non-locally convex setting because they exist in abundance, are easy to handle with, and in many cases can replace minimal sequences usually employed when investigating quasi-complements in Banach spaces (as in [7] for instance). We use them in Section 3 to produce injective compact operators of the form $K(x)=\sum_n h_n(x)v_n$, acting in F-spaces. In turn, such operators give rise, under some circumstances, to isomorphisms of the form $J+K$, where J is an inclusion embedding. These isomorphisms are the main tool used in Section 3 in proving existence theorems of the (M–M) type (Theorem 3.3) and of the combined (G–K) and (V) types (Theorems 3.8 and 3.11). In particular we have the following analogue of the Murray–Mackey theorem: Every non-minimal closed subspace of a separable F-space has a quasi-complement (Corollary 3.5).

The rest of the paper is devoted to establishing some analogues of (J–J–P); our approach may be interesting also for those believing only in Banach spaces. Let X, Y be proper quasi-complements in an F-space E. First, in Section 4, we seek for Y_1. Let $Q\colon E\to E/X$ be the quotient map; note that $Q(Y)$ is dense in E/X and that $Q|Y$ is *not* an isomorphism onto $Q(Y)$. The problem of finding a required Y_1 is of course identical with the problem of finding a closed subspace Y_1 of infinite codimension in Y such that $Q(Y_1)$ is dense in $Q(Y)$.

Equivalently, transporting the topology of $Q(Y)$ back to Y and denoting it by ϱ, our problem is this: We are given an F-space Y and a strictly weaker metrizable vector topology ϱ on Y, and we wish to find a closed infinite-codimensional subspace Y_1 of Y which is dense in (Y,ϱ).

It turns out that if Y is separable, then a result asserting that this is indeed possible was already essentially established by Kalton [12], Theorem 1, because the Mackey topology appearing there can be obviously replaced by any strictly weaker metrizable vector topology. Not satisfied with this, we prove in fact a stronger result (Theorem 4.1) providing us with uncountably many desirable Y_1's (Corollary 4.3) when E/X is separable.

Next we look for Y_2, but restrict ourselves to E a Fréchet space (the reason for doing so is given in Remark 5.10(c)). This time let R denote the quotient map $E\to E/Y$. Our departing observation is that it suffices to construct a closed infinite-dimensional subspace $\mathscr{Y}_2$ of E/Y transversal to $R(X)$, for then $Y_2 = R^{-1}(\mathscr{Y}_2)$ will be as required. Motivated by this and inspired by the works of Pličko ([24], [25]), we prove in Theorem 5.6 of Section 5 that if E is a Fréchet space and X is its non-closed subspace that is a continuous linear image of another Fréchet space, then it is possible to find a closed infinite-dimensional subspace Z in E transversal to X. From this the required analogue of the "Y_2-part" of (J–J–P) follows.

2. m-independent and other sequences. Let (v_n) be a sequence in a TVS E. Following [15] and [5], we say it is *topologically linearly independent* if, for any sequence of scalars (t_n),

$$\sum_{n=1}^{\infty} t_n v_n = 0 \Rightarrow (t_n) = 0.$$

If this holds when $(t_n) \in m = l_\infty$, then (v_n) is said to be (topologically linearly) *m-independent*. The latter notion is strictly broader than the former, see [15], Theorem 1. In our work we could equally well use either of them, but we prefer to use m-independent sequences, for two reasons. One is that the sequences of coefficients (t_n) of the type $\big(h_n(z)\big)$ appearing below are bounded, and the other is the following extremely simple existence result from [5], Proposition:

For every linearly independent sequence (x_n) in E there exist $\gamma_n > 0$ such that the sequence $(\gamma_n x_n)$ is m-independent.

Assume (v_n) is m-independent. It is obvious that this property is preserved when the topology of E is replaced by a stronger one. Also, for any *bounded* sequence (c_n) of nonzero scalars, the sequence $(c_n v_n)$ is m-independent. It follows that if E is an F-space with F-norm $\|\cdot\|$, we may always replace any given m-independent sequence, without changing its linear span, by a new one, (v_n) say, so as to have $\sum_n \|v_n\| < \infty$. These simple facts will be used below without further reference.

We denote by $\langle (v_n)\rangle$ (or by $\langle v_n\colon n \in M\rangle$ if the index set $M \subset N$ is different from N) the subspace in E consisting of those $x \in E$ which have an expansion of the form $x = \sum_n t_n v_n$ with $(t_n) \in m$. Evidently, m-independence implies that the coefficients t_n of the expansion are uniquely determined by x. Also, $\operatorname{lin}(v_n) \subset \langle (v_n)\rangle \subset [(v_n)]$, where $\operatorname{lin}(v_n)$ and $[(v_n)]$ are the linear span and the closed linear span in E of (v_n), respectively. An m-independent sequence (v_n) with $[(v_n)] = E$ is called an *m-quasi-basis* for E. Thus, by the above result, every separable TVS has an m-quasi-basis.

Now, let X be a closed subspace of E and $Q\colon E\to E/X$ the quotient map. We shall say that a sequence $(v_n) \subset E$ is *m-independent of X* if the sequence (Qv_n) is m-independent in E/X. Clearly, then (v_n) is m-independent in the usual sense. Before formulating, in the proposition below, some obvious properties of such sequences, we introduce two further definitions.

We say that a sequence (z_n) is a *union* of two given sequences (x_n) and (y_n) if (z_n) can be partitioned into two disjoint subsequences one of which coincides with (x_n), and the other with (y_n). We shall also use (implicitly, in Section 3) some clear extensions of this definition when we are given more than two sequences, or when some of them are finite.

We say that a subspace W of the TVS $E = (E,\tau)$ is *dominated* (resp., *strictly dominated*) by an F-space if there exists a vector topology ν on W such that (W,ν) is an F-space and $\tau\cap W \leqslant \nu$ (resp. $\tau\cap W < \nu$); $\tau\cap W$ is the topology on W induced by τ. Evidently, by the closed graph theorem, such a topology ν if exists, is unique. These properties of W may be equivalently expressed by saying that W is a continuous linear image of an F-space (resp. by a map that is not relatively open). Domination or strict domination by other classes of TVS's, or by single TVS's, are defined similarly.

2.1. Proposition. *Let X be a closed subspace of a* TVS E. *If a sequence $(v_n) \subset E$ is m-independent of X, then:*

(a) $X\cap\langle (v_n)\rangle = 0$;

(b) *$(c_n v_n)$ is m-independent of X for every $(c_n) \in m$ with all $c_n \neq 0$;*

(c) *If (u_n) is an m-independent sequence in X, then any sequence (w_n) that is a union of (u_n) and (v_n) is m-independent and $X\cap\langle (w_n)\rangle = \langle (u_n)\rangle$.*

Let W be a subspace of E transversal to X; then:

(d) *Every countable-dimensional subspace of W has a Hamel basis (w_n) which is m-independent of X. Moreover, if W is dominated by an F-space, then (w_n) can be chosen so that $\langle (w_n)\rangle \subset W$;*

(e) *If E/X is separable and metrizable and $\overline{X+W} = E$, then W contains a sequence (w_n) which is m-independent of X and such that $X+\operatorname{lin}(w_n) = E$.*

In the remaining part of this section, let E, F be F-spaces, with F-norms denoted by $\|\cdot\|$.

If (h_n) is an equicontinuous sequence of linear functionals defined on a closed subspace Z of E, and (v_n) is a sequence in F such that $\sum_n \|v_n\| < \infty$, then we may define a continuous (linear) operator $K\colon Z \to F$ by

$$Kz = \sum_{n=1}^{\infty} h_n(z) v_n,$$

and K is easily seen to be compact, i.e., it maps a neighbourhood of 0 in Z into a compact set. (See, e.g., proof of Proposition 3.3 in [3].) We shall call K the *compact operator determined by the sequences* (h_n) *and* (v_n). We note the following obvious facts: (a) $K(Z) \subset \langle (v_n) \rangle$. (b) If (h_n) is total over Z and (v_n) is m-independent, then K is injective. (c) If (h_n) is biorthogonal to a sequence $(z_n) \subset Z$, then $[(v_n)] = \overline{K(Z)}$. (d) If $E = F$ and (v_n) is m-independent of Z, then $Z \cap K(Z) = 0$.

2.2. Remark. Metrizable LCS's dominated by separable Fréchet spaces are investigated in [28]. Subspaces of Banach spaces dominated by Banach spaces have found some applications in operator theory [14].

If $X_1, \ldots, X_n$ are subspaces of the F-space E dominated by F-spaces, then so is $X_1 + \ldots + X_n$; in particular this holds when these subspaces are closed in E. This is obvious. The following fact is less evident (cf. [6]):

If L is a countable-dimensional subspace of E, then for every Banach space B whose dual B^* is weak-separable, there exists a subspace W_B in E strictly dominated by B and such that $L \subset W_B \subset \bar{L}$.*

In fact, let (v_n) be a Hamel m-quasi-basis for L such that $\sum_n \|v_n\| < \infty$, and let (h_n) be an equicontinuous sequence in B^* which is total on B. Then the compact operator $K\colon B \to E$ determined by (h_n) and (v_n) is injective and so $W_B = K(B)$ is as required. In particular, $\langle (v_n) \rangle$ is strictly dominated by $m = l_\infty$. Also, if E is separable and $\dim E = \infty$, then E has a dense subspace dominated by any given infinite-dimensional separable Banach space.

The reader will note that we have thus provided examples of non-trivial subspaces W in 3.8, 3.9 and 3.11, and X in 5.6 and 5.7.

A sequence $(z_n) \subset E$ is called *strongly regular M-basic* [13] if there exists a sequence $(h_n) \subset [(z_n)]^*$ which is biorthogonal to (z_n), equicontinuous and total on $[(z_n)]$. If (h_n) is not necessarily equicontinuous but $z = \sum_{n=1}^{\infty} h_n(z) z_n$ for all $z \in [(z_n)]$, then (z_n) is called *basic*. For a basic sequence (z_n), equicontinuity of (h_n) and regularity of (z_n) (i.e., $\inf_n \|z_n\| > 0$) are equivalent conditions.

2.3. Proposition. *If E is an F-space, then the following are equivalent:* (a) *E is non-minimal.* (b) *E contains a strongly regular M-basic sequence.* (c) *E contains a regular basic sequence.*

2.4. Proposition. *Let E be an F-space. Then every regular sequence in E which converges to 0 in a weaker metrizable vector topology on E has a strongly regular M-basic subsequence.*

These two results are due to Kalton and Shapiro [13], Theorems 3.2 and 2.1 (ii); as concerns 2.4, cf. also [3], Corollary 2.7.

3. The existence of quasi-complements. Throughout this section, E is an infinite-dimensional F-space and $\|\cdot\|$ is an F-norm defining its topology. We also assume that X, Z are closed subspaces of E, and $J\colon Z \to E$ is the identity embedding.

3.1. Lemma ([12], Lemma 2). *If $K\colon Z \to E$ is a compact operator, then the operator $T = J + K\colon Z \to E$ has closed range.*

3.2. Lemma. *Assume $Z \subset X$, and let $K\colon Z \to E$ be an injective compact operator whose range $K(Z)$ is contained in a subspace V such that $X \cap V = 0$. Then $T = J + K$ is an isomorphism from Z onto the closed subspace $Y = T(Z)$ and $X \cap Y = V \cap Y = 0$. Moreover, if $\dim Z = \infty$, then $X + Y$ is not closed, and if $K(Z)$ is dense in V, then $\overline{X+Y} = \overline{X+V}$.*

Proof. The subspace Y is closed by Lemma 3.1. Suppose $Tz = z + Kz \in X$ for some $z \in Z$. Then $Z \subset X$ implies $Kz \in X$; on the other hand $Kz \in V$, whence $Kz = 0$. Since K is one-to-one, $z = 0$. This shows $X \cap Y = 0$ and, simultaneously, injectivity of T. Thus T is an isomorphism. The equality $V \cap Y = 0$ is checked similarly.

If $\dim Z = \infty$, then K is not an isomorphism; so we may find a sequence (z_n) in Z such that $z_n \not\to 0$ while $Kz_n \to 0$. Then the sequence $z_n - Tz_n = -Kz_n$ is in $X + Y$ and converges to 0, while its projection into X, z_n, does not. Hence $X + Y$ is not closed.

Finally, assume $\overline{K(Z)} \supset V$. If $x \in X$, $v \in V$, then there is a sequence (z_n) in Z such that $Kz_n \to v$, and therefore $X + Y \ni (x - z_n) + Tz_n = x + Kz_n \to x + v$. Thus $\overline{X+Y} \supset X + V$. We also check easily that $X + Y \subset \overline{X+V}$, and this finishes the proof.

The simplest existence result for quasi-complements is contained in the following

3.3. Theorem. *If X is non-minimal and $\dim(E/X) = \infty$, then there exists a closed non-minimal subspace Y of E such that $X \cap Y = 0$ and $X + Y$ is not closed. Moreover, if E/X is separable, then Y may be chosen to be a proper quasi-complement to X in E.*

Proof. Using Proposition 2.1 we find a sequence $(v_n) \subset E$ which is m-independent of X, with $\sum_n \|v_n\| < \infty$ (and $\overline{X + \operatorname{lin}(v_n)} = E$ when E/X

is separable). By Proposition 2.3, X contains a regular basic sequence (z_n). (We may use a strongly regular M-basic sequence as well.) Let $Z = [(z_n)]$ and let $(h_n) \subset Z^*$ be the sequence biorthogonal to (z_n). The assertions of the theorem follow by applying Lemma 3.2 with K = the compact operator determined by (h_n) and (v_n), and $V = \langle(v_n)\rangle$.

3.4. Remark. The first part of the above theorem is an extension to F-spaces of a similar fact proved in [35], p. 12, for Banach spaces. (It is also remarked there that Kalton had generalized this to the case of Fréchet spaces.) The existence, in every infinite-dimensional Banach space, of closed subspaces X and Y with $X \cap Y = 0$ and $X + Y$ non-closed, was established in [21], p. 174.

3.5. Corollary. *Every non-minimal infinite-codimensional closed subspace of a separable F-space has a proper quasi-complement (which can be chosen so as to have a basis).*

3.6. Corollary. *Every separable non-minimal F-space has a pair of proper quasi-complements (which can be chosen so as to be isomorphic and have bases).*

Proofs. These two corollaries follow easily from Theorem 3.3 and its proof. For instance, to get 3.6 we take a regular basic sequence (x_n) and apply the proof of 3.3 with $X = Z = [(x_{2n})]$.

Our next theorem is an extension of the "Y_1 part" of (V).

3.7. Theorem. *Let $\dim X = \infty$, and let W be a closed subspace of E transversal to X. If E/W is separable and has no infinite-dimensional minimal subspace, then X has a quasi-complement Y in E such that $W \subset Y$.*

Proof (cf. [20]). Let $Q\colon E \to E/W$ be the quotient map. By Corollary 3.5, the subspace $\overline{Q(X)}$ has a quasi-complement $\mathscr{Y}$ in E/W. (This is obvious when codim $\overline{Q(X)} < \infty$.) Then $Y = Q^{-1}(\mathscr{Y})$ is as required.

We now proceed to somewhat more elaborated results which will constitute joint extensions of (G–K) and (V). We shall need the following definition.

A pair (U, V) of closed subspaces of a TVS F such that $U \subset V$ will be said to *have property* $(*)$ if there exists an equicontinuous sequence $(f_n) \subset V^*$ which is total over U. If $U = V$, then we simply say that U has property $(*)$.

Note that (U, V) has property $(*)$ if and only if there exists a continuous seminorm p on V that is a norm on U such that the normed space (U, p) admits a total sequence $(h_n) \subset (U, p)^*$.

In fact, if (f_n) is as in the definition, then $p(v) = \sup_n |f_n(v)|$ is the required seminorm. For the converse we may assume $|h_n| \leqslant p$ on U, and use the Hahn–Banach theorem to get a sequence (f_n) as required in the definition.

Using this observation and a standard construction (see, e.g., [29], p. 154) we easily see that if (U, V) has property $(*)$, then there exists a biorthogonal system (u_n), (g_n) with $(u_n) \subset U$, $(g_n) \subset V^*$, (g_n) equicontinuous on V and total over U.

3.8. Theorem. *Suppose $Z \subset X$, both Z and E/X are infinite-dimensional, Z has property $(*)$ and E/X is separable. In addition, let W be a subspace of E transversal to X and dominated by an F-space.*

Then X has a proper quasi-complement Y which is isomorphic to Z and transversal to W.

Proof. Applying Proposition 2.1(d), (e) to the subspace X of $\overline{X+W}$ (the latter playing the role of E in 2.1) and next to the subspace $\overline{X+W}$ of E, and then using 2.1 (c), we find an m-independent sequence (v_n) in E such that for $V = \langle(v_n)\rangle$ and for some decomposition N_1, N_2 of N and $V_i = \langle v_n\colon n \in N_i\rangle$, $i = 1, 2$, we have

$$X \cap V = 0; \quad V_1 \subset W, \quad \overline{X+V_1} = \overline{X+W};$$
$$\overline{X+W} \cap V_2 = 0, \quad \overline{X+W+V_2} = E.$$

Then evidently $\overline{X+V} = E$, and we may also assume that $\sum_n \|v_n\| < \infty$.

Since Z has property $(*)$, there is a biorthogonal system (z_n), (h_n) with $(h_n) \subset Z^*$ equicontinuous and total on Z.

For $i = 1, 2$, let $K_i\colon Z \to V_i \subset E$ be the compact operator determined by the sequences $(h_n\colon n \in N_i)$ and $(v_n\colon n \in N_i)$. Then $K = K_1 + K_2\colon Z \to V_1 + V_2 \subset V$ is a compact injective operator whose range is dense in V. Now it suffices to define Y as in Lemma 3.2, and it will remain only to check that $W \cap Y = 0$. Suppose $Tz = z + K_1 z + K_2 z \in W$ for some $z \in Z$. Then $K_2 z \in W + Z + V_1 \subset \overline{X+W}$; but $K_2(Z) \subset V_2$ by definition, so $K_2 z = 0$. Now $z + K_1 z \in W$, whence $z \in W$; since $X \cap W = 0$, we have $z = 0$. This concludes the proof.

3.9. Corollary. *If X is non-minimal and E/X is separable, then for every closed subspace $W \subset E$ transversal to X there is a proper quasi-complement Y to X in E transversal to W.*

3.10. Lemma. *Let $X \subset Z$, and let $K_1, K_2\colon Z \to E$ be compact operators satisfying the following conditions:*

(1a) $K_1(Z) \subset Z$, (1b) $K_1|X$ *is one-to-one*, (1c) $X \cap K_1(X) = 0$;
(2a) $\ker K_2 = X$, (2b) $Z \cap K_2(Z) = 0$.

Then $T = J + K_1 + K_2$ is an isomorphism from Z onto $Y = T(Z)$ and $X \cap Y = 0$. Moreover, if $\dim X = \infty$, then $X + Y$ is not closed.

Proof. Since $K = K_1 + K_2$ is compact, Y is closed by Lemma 3.1. Suppose $Tz = z + K_1 z + K_2 z \in X$ for some $z \in Z$. Then $K_2 z \in Z$ by (1a), whence $K_2 z = 0$ by (2b). This and (2a) imply $z \in X$. Thus $z + K_1 z \in X$

and $z \in X$, so $K_1 z \in X$ and next $K_1 z = 0$ by (1c). Finally, $z = 0$ by (1b). This proves that $X \cap Y = 0$ and also that T is injective; so T is an isomorphism.

If $\dim X = \infty$, then we apply Lemma 3.2 to see that the subspace $X + T(X)$ is not closed (i.e., is not the topological direct sum of X and $T(X)$), whence neither is $X + Y$.

3.11. THEOREM. *Assume $X \subset Z$, E/X is separable, and both X and E/Z are infinite-dimensional.*

If (X, Z) and Z/X have property $(*)$, then X has a proper quasi-complement Y isomorphic to Z.

Moreover, if W is a subspace of Z dominated by an F-space and transversal to X, then Y may be chosen so that $W \cap Y = 0$.

Proof. First consider the case when $\dim(Z/X) = \infty$. Let N_1, N_2 be a decomposition of N such that both N_1 and N_2 are infinite. From the assumption that (X, Z) and Z/X have property $(*)$ it follows that there exists an equicontinuous sequence $(h_n) \subset Z^*$ and a sequence $(z_n) \subset Z$ such that

$$(z_n)_{n\in N_1} \subset X,\ (h_n)_{n\in N_1} \text{ is total on } X \text{ and biorthogonal to } (z_n)_{n\in N_1},$$

$$(h_n)_{n\in N_2} \text{ is biorthogonal to } (z_n)_{n\in N_2}, \text{ and } \bigcap_{n\in N_2} \ker h_n = X.$$

Next, applying Proposition 2.1 to $X \subset Z$ and $Z \subset E$ (note that Z/X and E/Z are separable) and then using 2.1 (c), we find an m-independent sequence (v_n) in E with $\sum_n \|v_n\| < \infty$ and such that, denoting $V_i = \langle v_n \colon n \in N_i\rangle$ for $i = 1, 2$, we have

$$X \cap V_1 = 0, \quad \overline{X + V_1} = Z; \quad Z \cap V_2 = 0, \quad \overline{Z + V_2} = E.$$

Now for $i = 1, 2$ let $K_i \colon Z \to V_i \subset E$ be the compact operator determined by $(h_n \colon n \in N_i)$ and $(v_n \colon n \in N_i)$. It is clear that the hypotheses of Lemma 3.10 are fulfilled, whence $T = J + K_1 + K_2$ is an isomorphism between Z and $Y = T(Z)$, $X \cap Y = 0$ and $X + Y$ is not closed.

If $n \in N_1$, then $v_n = -z_n + Tz_n \in X + Z$, whence

$$\overline{X + Y} \supset \overline{X + \operatorname{lin}(v_n \colon n \in N_1)} = Z.$$

If $n \in N_2$, then $v_n = (-z_n - K_1 z_n) + Tz_n \in Z + Y$, whence

$$\overline{Z + Y} \supset \overline{Z + \operatorname{lin}(v_n \colon n \in N_2)} = E.$$

It follows that $\overline{X + Y} = E$.

Now assume we are also given W as in the "moreover" part. Then (v_n) may be chosen so that, in addition to the conditions stated above, it will satisfy the following: There is a decomposition N_{11}, N_{12} of N_1 such that if $V_{1j} = \langle v_n \colon n \in N_{1j}\rangle$ for $j = 1, 2$, then

$$V_{11} \subset W \quad \text{and} \quad \overline{X + W} \cap V_{12} = 0.$$

Then $K_1 = K_{11} + K_{12}$, where $K_{1j} \colon Z \to V_{1j} \subset E$ is the compact operator determined by $(h_n \colon n \in N_{1j})$ and $(v_n \colon n \in N_{1j})$, $j = 1, 2$.

Suppose $Tz = z + K_{11}z + K_{12}z + K_2 z \in W \subset Z$ for some $z \in Z$. Then $K_2 z \in Z$, whence $K_2 z = 0$ and so $z \in X$. Now $z \in X$ and $z + K_{11}z + K_{12}z \in W$, so $K_{12}z \in X + W$, whence $K_{12}z = 0$. Finally, we have $z + K_{11}z \in W$, whence $z \in W$; but also $z \in X$, and therefore $z = 0$.

At last, consider the case $\dim(Z/X) = k < \infty$. By Theorem 3.8, X has a proper quasi-complement Y_1 isomorphic to X and such that $W \cap Y_1 = 0$. By Proposition 5.1, $X + Y_1 + W$ has infinite codimension in E, so we may find a subspace $V \subset E$ with $(X + Y_1 + W) \cap V = 0$ and $\dim V = k$. Then $Y = Y_1 + V$ is as required.

3.12. Remark. It is not incidental that the proper quasi-complements X, Y constructed in the above theorems have isomorphic non-minimal closed subspaces $X_1 \subset X$, $Y_1 \subset Y$ (e.g., $X_1 = Z$, $Y_1 = Y$ in Theorem 3.8). By [2], Theorem 4.1 (b), *all* proper quasi-complements in F-spaces share this property.

4. Diminution of quasi-complements.

In this section we obtain in Corollary 4.3 an analogue of the "Y_1 part" of (J–J-P) in the case of F-spaces. We follow the idea explained in the Introduction.

4.1. THEOREM. *Let $E = (E, \tau)$ be an F-space, and let ϱ be a strictly weaker metrizable vector topology on E such that (E, ϱ) is separable. Then there exists a strongly regular M-basic sequence (w_n) in (E, τ) with $w_n \to 0(\varrho)$ such that for any infinite subset A of N the subspace $\operatorname{lin}(w_n \colon n \in A)$ is dense in (E, ϱ).*

Proof. Let us agree that limits, series, closures, etc., unless specified otherwise, are to be understood in the sense of τ. Also, let $\|\cdot\|$ and $|\cdot|$ be F-norms defining τ and ϱ, respectively.

We start by choosing a sequence (y_n) so that ϱ-$[(y_n)] = E$ and $\sum_n \|y_n\| < \infty$. Then it is easy to define a strictly increasing sequence (r_k) in N such that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \|r_k^{-n} y_n\| < \infty. \tag{1}$$

Define a function $z \colon (0, 1] \to E$ by

$$z(t) = \sum_{n=1}^{\infty} t^n y_n.$$

If $0 < t_j \leqslant 1$ and $t_j \to 0$, then

$$y_n = \lim_{j\to\infty} t_j^{-n}\Big[z(t_j) - \sum_{i=0}^{n-1} t_j^i y_i\Big] \quad \forall n \in \boldsymbol{N}, \tag{2}$$

where $y_0 = 0$. For $k \in \boldsymbol{N}$ let $v_k = z(r_k^{-1})$; then (1) implies

$$\sum_{k=1}^{\infty} \|v_k\| < \infty.$$

Next, as $\varrho < \tau$, using Proposition 2.4 we find a strongly regular M-basic sequence (z_k) in (E, τ) such that

$$|r_k^k z_k| \to 0 \quad \text{as} \quad k \to \infty.$$

Then

$$|r_k^n z_k| \to 0 \quad \text{as} \quad k\to\infty \quad \forall n \in \boldsymbol{N}. \tag{3}$$

Let $Z = [(z_k)]$, and let $(h_k) \subset Z^*$ be the sequence biorthogonal to (z_k). Define $K\colon Z \to E$ to be the compact operator determined by (h_k) and (v_k), and let $J\colon Z\to E$ be the identity embedding. Then $T = J+K\colon Z\to E$ has evidently a finite-dimensional kernel, hence there is m such that the restriction $T|[(z_k)_{k\geqslant m}]$ is one-to-one; by Lemma 3.1 it has closed range. Without loss of generality we may assume $m = 1$, i.e., T is an isomorphism between Z and $W = T(Z)$. It follows that $w_k = Tz_k = z_k + v_k$ $(k \in \boldsymbol{N})$ is a strongly regular M-basis of W.

Let (k_j) be a strictly increasing sequence in $\boldsymbol{N}$, and set $F = \varrho\text{-}[(w_{k_j})]$. Then, using (2) with $t_j = r_{k_j}^{-1}$ and (3), we have for every $n \in \boldsymbol{N}$

$$\lim_{j\to\infty} r_{k_j}^n\Big[w_{k_j} - \sum_{i=0}^{n-1} r_{k_j}^{-i} y_i\Big] = \lim_{j\to\infty} r_{k_j}^n z_{k_j} + \lim_{j\to\infty} r_{k_j}^n\Big[v_{k_j} - \sum_{i=0}^{n-1} r_{k_j}^{-i} y_i\Big] = y_n,$$

where the limits are taken in the ϱ-sense. From this we have clearly $y_1 \in F$ and, by induction, $y_n \in F$ for all $n \in \boldsymbol{N}$. Hence $F = E$, which concludes the proof.

4.2. Remark. The above proof combines some of the ideas used in a known construction of the so-called hypercomplete sequences ([29], p. 59) with the proof of Theorem 1 in [12]. (Note that it is, in fact, unessential, whether we assume separability of (E, ϱ) (as above) or separability of (E, τ) (as in [12]): the former case can be easily reduced to the latter one.)

Theorem 4.1 suggests the following question: Is it possible to find a separable F-space F containing a dense subspace E dominated by an F-space and such that every infinite-dimensional subspace of E is dense in F? We can make only these two remarks:

(1) Such an F-space F, if exists, is q-minimal ([2], [3]). For, otherwise, it would have a non-minimal Hausdorff quotient and this together with Proposition 2.3 would imply that F contains a strongly regular semi-basic (or minimal) sequence (u_n) (see [2] and [13] for explanations). By [2], Theorem 2.8, we can assume $u_n \in E$ for all $n \in \boldsymbol{N}$. But then if $(f_n) \subset [(u_n)]^*$ is the sequence biorthogonal to (u_n), it is obvious that $E\cap \ker f_n \supset \operatorname{lin}(u_k)_{k\neq n}$ are infinite-dimensional subspaces of E which are not dense in F.

(2) Our question would have (trivially) an affirmative answer if a separable F-space existed without nontrivial closed subspaces. Unfortunately, this is apparently an open problem (cf. [26], p. 114).

4.3. Corollary. *Let X, Y be proper quasi-complements in an F-space E such that E/X is separable. Then there exists a closed subspace W of Y with a strongly regular M-basis (w_n) such that $[(w_n)_{n\in A}]$ is a quasi-complement to X for every infinite subset A of $\boldsymbol{N}$.*

4.4. Remark. Note that Y need not be separable. For instance, if $E = l_\infty \oplus c_0$ and $X = l_\infty$, then from Theorem 3.8 (with $Z = X$, $W = \{0\}$) we get a quasi-complement Y to X isomorphic to X. However, from separability of E/X it follows trivially that any quasi-complement Y to X contains a separable quasi-complement to X.

5. Enlargement of quasi-complements.

In this section we extend to arbitrary Fréchet spaces the "Y_2-part" of (J–J–P).

In general, i.e., for F-spaces, we have only this result.

5.1. Proposition. *If X, Y are proper quasi-complements in an F-space E, then $X+Y$ has uncountable codimension in E. Hence there exists an increasing sequence (Y_n) of quasi-complements to X such that $Y \subset Y_n$ and $\dim(Y_n/Y) = n$ $(n \in \boldsymbol{N})$.*

Proof. The first assertion follows from a more general fact that if E is an ultrabarrelled TVS (= barrelled TVS in [1]) and W is its subspace strictly dominated by an F-space, then the codimension of W in E is uncountable. (Otherwise W would be ultrabarrelled by [1], p. 90, thus contradicting [1], § 8 (4).) The second assertion is an obvious consequence of the first one.

5.2. Remark. We were unable to verify our conjecture that the codimension of $X+Y$ in E is at least $2^{\aleph_0}$. (It is so when E is a Fréchet space, see Corollary 5.9 below.)

From now on, we shall deal exclusively with LCS's (except for some remarks). As explained in the Introduction, we shall first look for conditions under which, given a subspace X of an LCS E, there exists a closed infinite-dimensional subspace Z of E such that $Z\cap X = 0$. Our approach to this question was highly inspired by the works of Pličko [24] and [25].

5.3. Lemma. *Let L be a finite-dimensional subspace and C a closed convex subset in an* LCS E. *If $L\cap C=\emptyset$ and either C is bounded or $L\cap(C-x)=\{0\}$ for some $x\in C$, then there exists a closed hyperplane H in E such that*

$$L\subset H \quad \text{and} \quad H\cap C=\emptyset.$$

Proof. The assertion will follow from the geometrical form of the Hahn–Banach theorem (see, e.g., [9], 7.3.1) if we show that there exists a neighbourhood U of 0 in E such that

$$L\cap(C+U)=\emptyset.$$

Suppose it is not so. Then there exist nets $(y_\alpha)_{\alpha\in A}$ in L and $(x_\alpha)_{\alpha\in A}$ in C such that

$$y_\alpha - x_\alpha \to 0.$$

Now consider two cases.

Case 1. (y_α) has a bounded subnet $(y_{\alpha_\beta})_{\beta\in B}$. Then since $\dim L<\infty$, by passing to a further subnet we may assume that (y_{α_β}) converges to some $y\in L$. Then also $x_{\alpha_\beta}\to y$ and so $y\in C$ because C is closed. Thus $y\in L\cap C$, which is impossible. Note that if C is bounded, then Case 1 does actually occur, as easily seen.

Case 2. (y_α) has no bounded subnet. Then, if $\|\cdot\|$ is any norm on L, we must have $c_\alpha=\|y_\alpha\|^{-1}\to 0$; without loss of generality it may be assumed that $c_\alpha<1$ for all $\alpha\in A$. Now, using the fact that $\dim L<\infty$ again, by passing to a suitable subnet we may assume that $(c_\alpha y_\alpha)$ converges to some $y\in L$, where of course $\|y\|=1$. Then for each $x\in C$ we have $c_\alpha y_\alpha - c_\alpha x - c_\alpha(x_\alpha - x)\to 0$, $c_\alpha y_\alpha\to y$ and $c_\alpha x\to 0$, whence $c_\alpha(x_\alpha-x)\to y$. But $0<c_\alpha<1$ and $0, x_\alpha - x\in C-x$ imply $c_\alpha(x_\alpha-x)\in C-x$, and therefore $y\in C-x$. Thus $0\neq y\in L\cap(C-x)$. Therefore, the alternative hypothesis that $L\cap(C-x)=\{0\}$ for some $x\in C$ is violated.

5.4. Proposition. *Let W be an infinite-dimensional subspace of an* LCS E, *and (C_n) a sequence of closed convex subsets of E such that, for each $n\in \boldsymbol{N}$, $W\cap C_n=\emptyset$ and either C_n is bounded or $W\cap(C_n-x_n)=\{0\}$ for some $x_n\in C_n$.*

Then there exists a closed subspace Z in E such that

$$\dim(Z\cap W)=\infty \quad \text{and} \quad Z\cap C_n=\emptyset \quad \forall\, n\in\boldsymbol{N}.$$

Proof (cf. proof of Theorem 1 in [24]). We shall construct by induction a linearly independent sequence (w_n) in W and a sequence (H_n) of closed hyperplanes in E such that for every $n\in \boldsymbol{N}$

(1) $$H_n\cap C_n=\emptyset$$

and

(2) $$H_i\supset [w_j]_{j=1}^n \quad \text{for} \quad 1\leqslant i\leqslant n.$$

We start by choosing any $w_1\in W\setminus\{0\}$ and applying Lemma 5.3 to $L=[w_1]$ and $C=C_1$. This gives us a closed hyperplane H_1 satisfying (1) and (2) for $n=1$.

Assume we have already defined w_i and H_i for $i=1,\ldots,k$ $(k\geqslant 1)$ so that (1) and (2) hold for $n\leqslant k$. Then we apply Lemma 5.3 to find a closed hyperplane $H_{k+1}\supset[w_j]_{j=1}^k$ such that $H_{k+1}\cap C_{k+1}=\emptyset$, and next we choose any w_{k+1} in $W\cap H_1\cap\ldots\cap H_{k+1}\setminus[w_j]_{j=1}^k$. Evidently, (1) and (2) are then fulfilled for all $n\leqslant k+1$. This concludes the inductive construction.

It is now obvious that the assertion of the proposition is satisfied with $Z=\bigcap_{n=1}^{\infty} H_n$.

5.5. Corollary. *Let X and W be subspaces of an* LCS E *with* $\dim W=\infty$ *and $X\cap W=0$. Suppose that there is a sequence (C_n) of closed convex subsets of E such that*

$$X\setminus\{0\}=\bigcup_{n=1}^{\infty} C_n.$$

Then there exists a closed subspace Z of E such that

$$\dim(Z\cap W)=\infty \quad \text{and} \quad X\cap Z=0.$$

5.6. Theorem. *Let E be an* LCS *and X a subspace of E. Assume that one of the following conditions is satisfied.*

(a) *E is barrelled and X is strictly dominated by a separable Fréchet space.*

(b) *E is separable, metrizable and barrelled (in particular, a separable Fréchet space) and X is strictly dominated by a Fréchet space.*

(c) *E is a Fréchet space and X is strictly dominated by a Fréchet space.*

Then there exists a closed infinite-dimensional subspace Z of E such that $Z\cap X=0$.

The theorem admits an equivalent reformulation; restricted to the case (c), it says the following:

Let T be a continuous linear mapping from a Fréchet space F into a Fréchet space E. If T is not relatively open (which is the case when $T(F)$ is not barrelled), then there exists a closed infinite-dimensional subspace Z in E such that $Z\cap T(F)=0$.

Proof. Let τ denote the topology of E and ξ the (unique) topology on X such that $\tau\cap X<\xi$ and (X,ξ) is a Fréchet space. Choose a base (V_m) of absolutely convex open neighbourhoods of 0 in (X,ξ) such that $V_1\supset V_2\supset\ldots$ For every $m\in\boldsymbol{N}$ let $\overline{V_m}$ denote the closure of V_m in E.

Claim. *There is m such that $X_m=\lim \overline{V_m}$ is not a barrelled subspace of E.*

Otherwise, for every m, $\overline{V_m}$ is a neighbourhood of 0 in $(X_m, \tau \cap X_m)$, whence $\overline{V_m} \cap X$ (= the closure of V_m in $(X, \tau \cap X)$) is a neighbourhood of 0 in $(X, \tau \cap X)$. Thus the identity map $(X, \xi) \to (X, \tau \cap X)$ is nearly open, and since (X, ξ) is a Fréchet space, it must be open and so $\xi = \tau \cap X$, contrary to the assumption.

Without loss of generality we may assume that our claim holds for $m = 1$, i.e., $X_1 = \lim \overline{V_1}$ is not barrelled. Then, by a result in [16] or [30], X_1 is of uncountable codimension in E. Let W be any infinite-dimensional subspace in E transversal to X_1.

We shall now consider separately cases (a), (b) and (c) of the theorem.

(a): We first observe that for every $x \in X \setminus \{0\}$ there exists $m(x) \in N$ such that $0 \notin x + \overline{V_{m(x)}}$. Since $X \setminus \{0\}$ has the Lindelöf property under ξ, there is a sequence (x_n) in $X \setminus \{0\}$ such that

$$X \setminus \{0\} = \bigcup_{n=1}^{\infty} (x_n + V_{m(x_n)}).$$

Denote $C_n = x_n + \overline{V_{m(x_n)}}$. Since $W \cap X_1 = 0$, $X_1 \supset \overline{V_1} \supset C_n - x_n$ and $X_1 \supset x_n + \overline{V_1} \supset C_n$ for every $n \in N$, we can apply Proposition 5.4 to find a closed subspace Z of E such that $\dim(Z \cap W) = \infty$ and $Z \cap C_n = \emptyset$ $\forall n \in N$. The latter equalities imply $Z \cap (X \setminus \{0\}) = \emptyset$.

(b): Since E is separable, metrizable and locally convex, we can find a sequence (A_n) of closed convex subsets of E such tht $E \setminus \{0\} = \bigcup_n A_n$. Then the countable family of sets $(k\overline{V_1}) \cap A_n \subset X_1 \setminus \{0\}$, where $k, n \in N$, covers $X_1 \setminus \{0\}$. From Corollary 5.5 we have a closed subspace Z in E with $\dim Z = \infty$ and $Z \cap X_1 = 0$, hence also $Z \cap X = 0$.

(c): Since $\xi > \tau \cap X$, there is a sequence (x_n) in X such that $x_n \to 0$ (τ) but $x_n \not\to 0$ (ξ). Then define $E_1 = \tau\text{-}[(x_n)]$ and $Y = E_1 \cap X$. Applying (b), we find a closed infinite-dimensional subspace Z of $(E_1, \tau \cap E_1)$ such that $Z \cap Y = 0$. Since Z is closed in E and evidently $Z \cap X = 0$, Z is as required.

5.7. Corollary. *If the assumptions of Theorem 5.6 are fulfilled and, in addition, Y is a closed subspace of E such that $X \cap Y = 0$, then there exists a closed subspace Z of E satisfying*

$$Y \subset Z, \quad \dim(Z/Y) = \infty \quad \textit{and} \quad X \cap Z = 0.$$

Proof. Let Q: $E \to E/Y$ be the quotient map, and R: $(X, \xi) \to (E, \tau)$ the identity map. We apply Theorem 5.6 in its equivalent formulation to the map QR: $(X, \xi) \to (E, \tau)/Y$, and find a closed infinite-dimensional subspace $\mathscr{Z}$ of E/Y such that $\mathscr{Z} \cap Q(X) = 0$. Then $Z = Q^{-1}(\mathscr{Z})$ has the required properties.

In a similar way we obtain the promised extension of (J–J–P):

5.8. Corollary. *Let X, Y be proper quasi-complements in a Fréchet space E. Then there exists a quasi-complement Z to X such that $Y \subset Z$ and* $\dim(Z/Y) = \infty$.

5.9. Corollary. *Let X, Y be two closed subspaces of a Fréchet space E such that $X+Y$ is not closed. Then there exists a closed infinite-dimensional subspace Z in E such that $(X+Y) \cap Z = 0$. (Hence $\dim E/(X+Y) \geqslant 2^{\aleph_0}$.)*

5.10. Remarks. (a) Theorem 5.6 remains valid if $E = (E, \tau)$ is a TVS with separating dual E^*, and the respective assumptions on E in (a), (b) and (c) are replaced by the following ones: (a′) E is ultrabarrelled; (b′) E is separable, metrizable and ultrabarrelled; (c′) E is an F-space. (The assumptions on X remain unchanged.) Indeed, E is easily seen to be a barrelled LCS under its Mackey topology $\mu = \tau(E, E^*) \leqslant \tau$, and (E, μ) is metrizable or separable if such is (E, τ). In cases (a′) and (b′) we may therefore apply the corresponding parts of Theorem 5.6 to (E, μ) and get a μ-closed infinite-dimensional subspace Z in E with $Z \cap X = 0$; then Z is also τ-closed. The assertion in case (c′) follows from (b′) in exactly the same way as (c) from (b) in 5.6.

(b) Another extension of Theorem 5.6 (b), (c) is obtained by replacing the original assumption about X by "X is strictly dominated by a B_r-complete LCS". (We deduce (c) from (b) as before.) Of course, this extension of 5.6 may be combined with that indicated in the previous remark.

(c) The author does not know whether 5.8 is valid for general (even separable) F-spaces, and the same concerns of course validity of 5.6 when E is an F-space and X is its subspace strictly dominated by an F-space, even if the latter is a separable Fréchet (or Banach) space. Note that if 5.6 were true under such assumptions, then this and Remark 2.2 would imply the existence of many non-trivial closed subspaces in every infinite-dimensional F-space, thus solving negatively the problem mentioned at the end of Remark 4.2.

(d) The author has shown in [4] that every infinite-dimensional Fréchet space E contains a dense barrelled subspace X with $\operatorname{codim} X \geqslant 2^{\aleph_0}$ such that no infinite-dimensional closed subspace of E is transversal to X. (A similar, but less precise result is stated in [24], Theorem 2, without proof.) Thus some assumption on X in Theorem 5.6, stronger than just the requirement that $\operatorname{codim} X$ is infinite, are necessary.

(e) From Theorem 5.6 (c) and Remark 2.2 it follows that if E is a Fréchet space (or, using Remark (a) above, an F-space with a separating dual), then for every countable-dimensional subspace L in E there exists a closed subspace Z in E such that $\dim Z = \infty$ and $Z \cap L = 0$.

(f) We have the following curious result: Let E be a Banach space, (x_n) a basic sequence in E, and Z a closed subspace of E such that $Z \cap \operatorname{lin}(x_n)$

$= 0$. (See (e) above for such Z's with $\dim Z = \infty$.) Then there exists a subsequence (x_{k_n}) such that $Z \cap [(x_{k_n})] = 0$.

In fact, if $Q: E \to E/Z$ is the quotient map, then (Qx_n) is a linearly independent sequence in E/Z, and therefore contains a topologically linearly independent subsequence (Qx_{k_n}), see [19]. Assuming, as we may, that (x_n) is normalized, we have $[(x_{k_n})] = \langle (x_{k_n}) \rangle$, and the subsequence (x_{k_n}) is as required in view of 2.1 (a).

Added in proof. (a) Answering a question asked by the present author, R. Pol has shown that if Z is an *analytic* linear subspace in a separable F-space E, then $\operatorname{codim} Z$ is either $< \aleph_0$ or equals $2^{\aleph_0}$. From this it follows immediately that the conjecture stated in Remark 5.2 holds true for *separable* F-spaces. (The case of general F-spaces is still open.)

(b) A version of Theorem 5.6 (c), with X assumed to be a nonbarrelled subspace of the Fréchet space E, has been recently obtained by M. Valdivia in *A property of Fréchet spaces*, preprint. It is easy to see that Valdivia's result and our Theorem 5.6 (c) are, in fact, equivalent.

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INSTITUTE OF MATHEMATICS
A. MICKIEWICZ UNIVERSITY, POZNAŃ, POLAND

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