where $\tilde{\mu} = (\mu_0, \mu_1, \ldots, \mu_q)$, $\mu = (\mu_1, \ldots, \mu_q)$. This finishes the proof of the lemma.

From Theorem 2 and the lemma we obtain

COROLLARY. *If $k > q/2$, then $\mathcal{S}_k(R^q) \subset L^1(R^q)$.*

Our considerations lead to the following

THEOREM 3. *If $k > q/2$, then formula (3) is true for f belonging to $\mathcal{S}_k(R^q)$.*

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On some spaces of distributions

by

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Dedicated to Professor Jan Mikusiński on his 70th birthday

Abstract. The spaces $\mathcal{D}'_{\mathcal{A}_a}$ $(a > 0)$ of distributions are considered as a generalization of the space $\mathcal{D}'_{\mathcal{A}}$ introduced by Z. Sadlok and Z. Tyc. Some equivalent descriptions of the class $\mathcal{D}'_{\mathcal{A}_a}$ and of the convergence in $\mathcal{D}'_{\mathcal{A}_a}$ are given in terms of distributional and tempered derivatives. In particular, the spaces $\mathcal{D}'_{\mathcal{A}_a}$ turn out to be subspaces of the space $\mathcal{K}'_2$, introduced by G. Sampson and Z. Zieleźny, such that $\mathcal{K}'_2$ is an inductive limit of $\mathcal{D}'_{\mathcal{A}_p}$ $(p = 1, 2, \ldots)$. As a consequence, a characterization of the convergence in $\mathcal{K}'_2$ is obtained.

1. In connection with the theory of Hermite expansions of distributions (see [1]), Z. Sadlok and Z. Tyc introduced the class $\mathcal{D}'_{\mathcal{A}}$ of distributions such that 1° $\mathcal{D}'_{\mathcal{A}} \supset \mathcal{S}'$, 2° the Hermite coefficients $a_n = (f, h_n)$, where h_n are the Hermite functions, are uniquely defined for all distributions $f \in \mathcal{D}'_{\mathcal{A}}$ (see [6]).

Distributions of the class $\mathcal{D}'_{\mathcal{A}}$ were introduced in [6] as tempered derivatives of some order (for the definition see [1], p. 175) of functions belonging to some class $\mathcal{A}$. It is rather strange that the number 4 appears in the definition of the class $\mathcal{A}$, given in [6]. This is connected with the choice, made in [1] and [6], of constants in the definitions of the Hermite functions and tempered derivatives. However, the choice of constants is meaningless for the theory of Hermite expansions.

In this paper, we consider the general case, introducing the classes $\mathcal{A}_a$ of functions and $\mathcal{D}'_{\mathcal{A}_a}$ of distributions for arbitrary $a > 0$. For $a = \frac{1}{4}$, these classes coincide with $\mathcal{A}$ and $\mathcal{D}'_{\mathcal{A}}$, respectively.

We give several characterizations of distributions belonging to $\mathcal{D}'_{\mathcal{A}_a}$ in terms of distributional derivatives (Section 3).

Moreover, we introduce the convergence in $\mathcal{D}'_{\mathcal{A}_a}$ in some equivalent ways, by using distributional and tempered derivatives (Section 4).

It should be noted that the space $\mathcal{D}'_{\mathcal{A}_a}$ is of type $K\{M_p\}'$ in the sense of [2] and the convergence in $\mathcal{D}'_{\mathcal{A}_a}$ coincides with the weak and strong convergences in $K\{M_p\}'$ for appropriately chosen functions M_p (Section 5).

On the other hand, spaces $\mathcal{D}'_{\mathcal{A}_a}$ coincide with some subspaces $\mathcal{K}'_{2,a}$ of the space $\mathcal{K}'_2$, introduced in [7], i.e., there is an isomorphism between $\mathcal{D}'_{\mathcal{A}_a}$ and $\mathcal{K}'_{2,a}$ preserving the convergences in $\mathcal{D}'_{\mathcal{A}_a}$ and $\mathcal{K}'_{2,a}$. Since $\mathcal{K}'_2$ is an inductive limit of the spaces $\mathcal{K}'_{2,a}$ we obtain in particular characterizations of the convergence in $\mathcal{K}'_2$.

2. Functions and distributions considered in the sequel are assumed to be defined in the Euclidean space R^q and complex-valued. Notation will be as in [1]. Moreover, we adopt

$$E_\beta(x) = \exp(\beta|x|^2)$$

for $\beta \in R^1$ and $x \in R^q$. As in [1], we define the kth tempered derivatives of a distribution f:

$$D^k_a f = E_{-a}(E_a f)^{(k)}, \qquad d^k_a f = E_a(E_{-a}f)^{(k)},$$

and the kth tempered integral of a locally integrable function F:

$$S^k_a F(x) = E_{-a}(x)\int_0^x E_a(t)F(t)\,dt^k$$

for every $k \in P^q$ and $x \in R^q$, where the above integral is meant in the sense of the kth iterated integral (see [1], p. 69).

It can easily be verified that

(1) $D^0_a f = f,$ $\qquad d^0_a f = f,$

(2) $D^k_a D^m_a f = D^{k+m}_a f,$ $\qquad d^k_a d^m_a f = d^{k+m}_a f,$

(3) $D^k_a(af+bg) = aD^k_a f + bD^k_a g,$ $\qquad d^k_a(af+bg) = ad^k_a f + bd^k_a g,$

(4) $D^k_a(fg) = \sum_{0\leqslant i\leqslant k}\binom{k}{i}(D^i f)\cdot g^{(k-i)},$ $\qquad d^k_a(fg) = \sum_{0\leqslant i\leqslant k}\binom{k}{i}(d^i f)\,g^{(k-i)},$

(5) $D^{e_j}_a f(x) = f^{(e_j)}(x) + 2\xi_j f(x),$ $\qquad d^{e_j}_a f(x) = f^{(e_j)}(x) - 2\xi_j f(x),$

(6) $S^k_a(aF+bG) = aS^k_a F + bS^k G,$

(7) $|S^k_a F| \leqslant |S^k_a|F||,$

(8) $|F| \leqslant |G|$ implies $|S^k_a F| \leqslant |S^k_a|G||,$

(9) $D^k_a S^k_a F = F$

for arbitrary distributions f, g, locally integrable functions F, G, complex numbers a, b and $k, m \in P^q$ (cf. [1], pp. 175–177).

By induction, the following formulas can be derived from (5) and (6) for an arbitrary distribution f and $k \in P^q$:

$$f^{(k)}(x) = \sum_{i,j} a_{ij}x^i D^j_a f(x), \tag{10}$$

$$D^k_a f(x) = \sum_{i,j} b_{ij}x^i f^{(j)}(x), \tag{11}$$

$$d^k_a f(x) = \sum_{i,j} c_{ij}x^i f^{(j)}(x), \tag{12}$$

where the sums in (10)–(12) extend over all indices $i, j \in P^q$ such that $i, j \geqslant 0$ and $i+j \leqslant k$ and the constants a_{ij}, b_{ij}, c_{ij} are uniquely determined.

Moreover, we have

$$(D^k_a f)\cdot g = \sum_{0\leqslant i\leqslant k}(-1)^k\binom{k}{i}D^{k-i}_a(f\cdot g^{(i)}) \tag{13}$$

for arbitrary distributions f, g and $k \in P^q$.

It should be noted that the above definitions of the tempered derivatives D^k_a and d^k_a and the tempered integral S^k_a coincide for $a = \frac{1}{2}$ with that given in [1] and formulas (1)–(9) correspond to the respective ones in [1] (pp. 175–177).

Now, denote by $\mathcal{A}_a$ the set of all measurable functions F in R^q such that

$$E_{-\beta}F \in L^2(R^q)$$

for some positive number $\beta < a$ (of course, such functions F are locally integrable), and denote by $\mathcal{D}'_{\mathcal{A}_a}$ the set of all distributions f such that

$$f = D^k_a F \tag{14}$$

for some $F \in \mathcal{A}_a$ and $k \in P^q$.

In the case $a = \frac{1}{2}$, we have $\mathcal{A}_a = \mathcal{A}$ and $\mathcal{D}'_{\mathcal{A}_a} = \mathcal{D}'_{\mathcal{A}}$, where $\mathcal{A}$ and $\mathcal{D}'_{\mathcal{A}}$ are defined in [4].

It is obvious that $\mathcal{A}_a$ is a linear space. The linearity of $\mathcal{D}'_{\mathcal{A}_a}$ is less obvious and follows from the inequality

$$\int_{-\infty}^{\infty}|E_{-\beta}(\xi)S^1_a F(\xi)|^2\,d\xi \leqslant \frac{1}{2\sqrt{a}}\int_{-\infty}^{\infty}|E_{-\beta}(\xi)F(\xi)|^2\,d\xi, \tag{15}$$

where F is a function of one variable such that $FE_{-\beta} \in L^2(R^1)$ for some $0 < \beta < a$. In fact, if $F \in \mathcal{A}_a$, then $S^m_a F \in \mathcal{A}_a$ for every $m \in P^q$, by (15) and induction (cf. [1], p. 184). Thus if $f = D^k_a F$, $g = D^l_a G$ for $F, G \in \mathcal{A}_a$ and $k, l \in P^q$, then $af+bg = D^m_a(aS^{m-k}F + bS^{m-l}G) \in \mathcal{D}'_{\mathcal{A}_a}$, where $m = \max(k, l)$, in view of (3), (2), (9).

Inequality (15) for $a = \frac{1}{4}$ is stronger than that given in Lemma 1 in [6]. We are going to sketch the proof of (15). As in the proof in [1], pp. 182–183, one can show that

$$(16)\qquad \int_0^\infty |S_a^1 G|^2 \leqslant \frac{1}{2\sqrt{a}} \int_0^\infty |G|^2; \qquad \int_{-\infty}^0 |S_a^1 G|^2 \leqslant \frac{1}{2\sqrt{a}} \int_{-\infty}^0 |G|^2$$

for $G \in L^2(R^1)$. Since $E_{-\beta}(x) \leqslant E_{-\beta}(t)$ for $|t| \leqslant |x|$, we have

$$(17)\qquad \int_0^\infty |E_{-\beta} S_a^1 F|^2 \leqslant \int_0^\infty (S_a^1 |E_{-\beta} F|)^2 \leqslant \frac{1}{2\sqrt{a}} \int_0^\infty |E_{-\beta} F|^2,$$

and, analogously,

$$(18)\qquad \int_{-\infty}^0 |E_{-\beta} S_a^1 F|^2 \leqslant \frac{1}{2\sqrt{a}} \int_{-\infty}^0 |E_{-\beta} F|^2,$$

provided $FE_{-\beta} \in L^2(R^1)$. Inequalities (17) and (18) imply (15).

3. Distributions of the class $\mathscr{D}'_{\mathscr{A}_a}$ can be characterized by means of distributional derivatives of functions belonging to some classes. Namely

THEOREM 1. *The following conditions are equivalent:*

(i) $f \in \mathscr{D}'_{\mathscr{A}_a}$ *(the function F in (14) can be assumed to be measurable or continuous);*

(ii) *there exist $m \in P^q$, a positive number $\beta < a$ and a measurable (continuous) function G such that*

$$(19)\qquad f = G^{(m)} \quad \text{on} \quad R^q$$

and $E_{-\beta} G \in L^r(R^q)$ for some (each) $r \geqslant 1$;

(iii) *there exist $m \in P^q$, a positive number $\beta < a$ and a measurable (continuous) function G such that (19) holds and $E_{-\beta} G$ is bounded.*

Proof. Suppose that (i) holds, i.e., $f = D_a^k F$, where $k \in P^q$ and F is a (measurable) function such that $E_{-\beta} F \in L^2(R^q)$ for some positive number $\beta < a$. The function $\tilde{F} = S_a^1 F$, where $1 = (1, \ldots, 1) \in P^q$, is continuous and $f = D_a^{k+1} \tilde{F}$, by (2) and (9).

In view of (11) and the Leibniz formula, we have

$$f(x) = \sum_{0 \leqslant i,j \leqslant k+1} a_{ij} (x^i \tilde{F}(x))^{(j)},$$

where a_{ij} are constants. Let

$$(20)\qquad G(x) = \sum_{0 \leqslant i,j \leqslant k+1} a_{ij} \int_0^x t^i \tilde{F}(t)\, dt^{k+1-j}.$$

Of course, the function G is continuous and

$$(21)\qquad G^{(k+1)} = f.$$

By the Schwarz inequality, we get

$$|x^i \tilde{F}(x)| \leqslant |x^i E_{-a}(x)| \cdot \Big| \int_0^x |F|^2 E_{-2\beta} \Big|^{1/2} \cdot \Big| \int_0^x E_{2a+\beta} \Big|^{1/2} \leqslant A\, |x^{i+1/2}| E_\beta(x)$$

and, consequently,

$$(22)\qquad E_{-\gamma}(x) \Big| \int_0^x t^i \tilde{F}(t)\, dt^{k+1-j} \Big| \leqslant A E_{-\gamma+\beta}(x) \cdot |x^{k+i-j+3/2}| \leqslant ABE_{-\varepsilon}(x),$$

where $1/2 = (1/2, \ldots, 1/2)$, $3/2 = (3/2, \ldots, 3/2) \in P^q$,

$$(23)\qquad A = \Big(\int_{R^q} |F|^2 E_{-2\beta} \Big)^{1/2}$$

and the positive numbers B, γ, ε do not depend on $x \in R^q$ or on $i, j \leqslant k+1$ and γ, ε satisfy the inequalities $\beta + \varepsilon < \gamma < a$.

By (20) and (22), we obtain

$$(24)\qquad E_{-\gamma}(x)\, |G(x)| \leqslant ACE_{-\varepsilon}(x) \qquad (x \in R^q)$$

for some constant $C > 0$, i.e., $E_{-\gamma} G \in L^r(R^q)$, $r \geqslant 1$. But, by (21), this means that (ii) is satisfied for a continuous function G and each $r \geqslant 1$.

Suppose that (ii) holds, i.e., $G^{(m)} = f$ for some $m \in P^q$, and a measurable function G such that $E_{-\beta} G \in L^r(R^q)$ for a positive $\beta < a$ and some $r \geqslant 1$. Clearly, G is a locally integrable function.

Let ε be such a positive number that $\beta + \varepsilon < a$ and let

$$\tilde{G}(x) = \int_0^x G(t)\, dt \qquad (x \in R^q).$$

Of course, $\tilde{G}$ is continuous and

$$(25)\qquad \tilde{G}^{(m+1)} = f \quad \text{on} \quad R^q.$$

Moreover

$$(26)\qquad \sup_x |E_{-\beta-\varepsilon}(x) \tilde{G}(x)| \leqslant \sup_x \Big| \int_0^x E_{-\beta-\varepsilon}(t)\, |G(t)|\, dt \Big| \leqslant \Big(\int_{R^q} E_{-s\varepsilon} \Big)^{1/s} \Big(\int_{R^q} |G|^r E_{-r\beta} \Big)^{1/r} < \infty,$$

where $s = r/(r-1)$. Relations (25) and (26) mean that (iii) holds for a continuous function G.

Finally, suppose that condition (iii) is satisfied for some measurable G, which is, as a matter of fact, locally integrable. In view of (10) and (13), we can represent f in the form

$$f = \sum_{0\leqslant i,j\leqslant m} b_{ij} D_a^i G_j,$$

where $G_j(x) = x^j G(x)$ and b_{ij} are constants. Putting

$$F = \sum_{0\leqslant i,j\leqslant m} b_{ij} H_{ij}, \tag{27}$$

where $H_{ij} = S_a^{m+1-i} G_j$, we see that F is continuous and

$$D_a^{m+1} F = f, \tag{28}$$

owing to (3), (2) and (9).

We have, for arbitrary $\varepsilon, \gamma > 0$ such that $\beta+\varepsilon < \gamma < a$,

$$\begin{aligned} |H_{ij}(x)| &\leqslant E_{\beta-a}(x)\Big|\int_0^x E_{a-\beta}(t)\,\big|t^j G(t)\big|\,dt^{m+1-i}\Big| \\ &\leqslant \sup_t |E_{-\beta}(t)G(t)|\cdot|x^{m+1+j-i}|\cdot E_\beta(x) B E_{\gamma-\varepsilon}(x) \end{aligned} \tag{29}$$

where B is a positive constant which does not depend on $i, j \leqslant m$ or on $x \in R^q$. Hence

$$E_{-\gamma} F \in L^2(R^q), \tag{30}$$

which, together with (28), means that (14) holds for a continuous function F. The proof is thus finished.

COROLLARY 1. *If $f \in \mathscr{D}'_{\mathscr{A}_a}$, then $D_a^k f \in \mathscr{D}'_{\mathscr{A}_a}$, $d_a^k f \in \mathscr{D}'_{\mathscr{A}_a}$, $f^{(k)} \in \mathscr{D}'_{\mathscr{A}_a}$ and $\varphi f \in \mathscr{D}'_{\mathscr{A}_a}$ for each $k \in P^q$ and each smooth function φ such that $E_{-\varepsilon}\varphi^{(i)}$ is a bounded function for every $\varepsilon > 0$ and $i \in P^q$. In particular, $f \in \mathscr{D}'_{\mathscr{A}_a}$ implies $Pf \in \mathscr{D}'_{\mathscr{A}_a}$ for every polynomial P.*

Proof. Suppose that $f \in \mathscr{D}'_{\mathscr{A}_a}$. In the case of the distributions $D_a^k f$ and $f^{(k)}$, the statement of Corollary 1 is evident by the definition of the class $\mathscr{D}'_{\mathscr{A}_a}$ and Theorem 1.

If f is of the form (19) and $E_{-\beta}G$ is bounded for some positive $\beta < a$ and φ is a function defined in the corollary, then

$$\varphi f = \sum_{0\leqslant i\leqslant m} (-1)^i \binom{m}{i} [\varphi^{(i)} G]^{(m-i)}$$

and the function $E_{-\gamma}[\varphi^{(i)} G]$ is bounded for any γ such that $\beta < \gamma < a$. By Theorem 1 and the linearity of $\mathscr{D}'_{\mathscr{A}_a}$, we have $\varphi f \in \mathscr{D}'_{\mathscr{A}_a}$. In particular, $Pf \in \mathscr{D}'_{\mathscr{A}_a}$ for an arbitrary polynomial P.

Finally, we get $d_a^k f \in \mathscr{D}'_{\mathscr{A}_a}$ by applying formula (12) and the properties of the space $\mathscr{D}'_{\mathscr{A}_a}$ just proved.

4. We say that $f_n \to 0$ in $\mathscr{D}'_{\mathscr{A}_a}$ if one of the equivalent conditions given in the theorem below is satisfied.

THEOREM 2. *Let $f_n \in \mathscr{D}'_{\mathscr{A}_a}$ for $n \in N$. The following conditions are equivalent:*

(i) *there are $k \in P^q$, a positive number $\beta < a$ and measurable (continuous) functions F_n such that*

$$D_a^k F_n = f_n \quad \text{on} \quad R^q \tag{31}$$

and $E_{-\beta}F_n \to 0$ in $L^2(R^q)$;

(ii) *there are $m \in P^q$, a positive $\beta < a$ and measurable (continuous) functions G_n such that*

$$G_n^{(m)} = f_n \quad \text{on} \quad R^q \tag{32}$$

and $E_{-\beta}G_n \to 0$ in $L^r(R^q)$ for some (each) $r \geqslant 1$;

(iii) *there are $m \in P^q$, a positive $\beta < a$ and continuous functions G_n such that (32) holds and $E_{-\beta}G_n \rightrightarrows 0$ in R^q;*

(iv) *there are $m \in P^q$, a positive $\beta < a$ and measurable functions G_n such that (32) holds, the functions $E_{-\beta}G_n$ are commonly bounded (almost everywhere in R^q) and $G_n \to 0$ almost everywhere in R^q.*

Proof. Suppose that (i) holds in the weaker form, i.e., for measurable functions F_n. As in the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 1 (cf. (20)–(24)), we can deduce that there exist continuous functions G_n such that (32) holds for some $m \in P^q$ and

$$E_{-\gamma}(x)\,|G_n(x)| \leqslant C\Big(\int_{R^q} |F_n|^2 E_{-2\beta}\Big)^{1/2} E_{-\varepsilon}(x) \qquad (x \in R^q) \tag{33}$$

for arbitrary $\gamma, \varepsilon > 0$ such that $\beta+\varepsilon < \gamma < a$ and for some constant $C > 0$. Since $E_{-\beta}F_n \to 0$ in $L^2(R^q)$, inequality (33) implies that $E_{-\gamma}G_n \to 0$ in $L^r(R^q)$ for each $r \geqslant 1$, and so (ii) holds in the stronger form.

The proof of the implication (ii) $\Rightarrow$ (iii) is completely analogous to that applied in Theorem 1 and the implication (iii) $\Rightarrow$ (iv) is obvious; thus it remains to show that (iv) implies (i).

Suppose that condition (iv) is fulfilled. As in the proof of the implication (iii) $\Rightarrow$ (i) in Theorem 1 (cf. (27)–(28)), one can derive from (iv) that

$$f_n = D_a^{m+1} F_n, \tag{34}$$

where F_n are continuous function given by the formula

$$F_n = \sum_{0\leqslant i,j\leqslant m} b_{ij} H_{ij}^n, \tag{35}$$

in which b_{ij} are constants and

$$H_{ij}^n(x) = S_\alpha^{m+1-i}[x^j G_n(x)] \quad (x \in R^q).$$

Moreover,

$$(36) \quad |H_{ij}^n(x)| \leqslant E_{\beta-\alpha}(x)\Big|\int_0^x E_\alpha(t)\,|t^j|(E_{-\beta}|G_n|)(t)\,dt^{m+1-i}\Big| \leqslant BE_{\gamma-\varepsilon}(x),$$

where the numbers γ, ε ($\gamma, \varepsilon > 0$; $\beta+\varepsilon < \gamma < \alpha$) do not depend on $x \in R^q$; $i, j \leqslant m$ or $n \in N$, and $B > 0$ is a common upper bound of the functions $E_{-\beta}(t)|G_n(t)|$ (for almost all $t \in R^q$).

By (36) and the Lebesgue theorem,

$$(37) \qquad H_{ij}^n(x) \to 0 \quad \text{as} \quad n \to \infty$$

almost uniformly in R^q for all $i, j \leqslant m$. Because of (36) and (37), we can again apply the Lebesgue theorem. Consequently,

$$\int_{R^q} |H_{ij}^n|^2 E_{-2\gamma}{}^r \to 0 \quad \text{as} \quad n \to \infty$$

for all $i, j \leqslant m$, and thus

$$(38) \qquad E_{-\gamma}F_n \to 0 \quad \text{in} \quad L^2(R^q),$$

according to (35).

Relations (34) and (38) yield condition (i) for continuous functions F_n and this completes the proof.

In a similar way to that applied in the proof of Corollary 1, the following result can be obtained:

COROLLARY 2. *If $f_n \to 0$ in $\mathscr{D}'_{\mathscr{A}_\alpha}$, then $D_\alpha^k f_n \to 0$, $d_\alpha^k f_n \to 0$, $f_n^{(k)} \to 0$ and $\varphi f_n \to 0$ in $\mathscr{D}'_{\mathscr{A}_\alpha}$ for each $k \in P^q$ and each smooth function φ such that $E_{-\varepsilon}\varphi^{(i)}$ is a bounded function for every $\varepsilon > 0$ and $i \in P^q$. In particular, $f_n \to 0$ in $\mathscr{D}'_{\mathscr{A}_\alpha}$ implies $Pf_n \to 0$ in $\mathscr{D}'_{\mathscr{A}_\alpha}$ for an arbitrary polynomial P.*

5. Let $M_p(x) = \exp[\alpha(1-1/p)|x|^2]$ for $x \in R^q$. It is clear that the functions M_p are continuous on R^q and

$$1 \leqslant M_p(x) \leqslant M_{p+1}(x) < \infty \quad (p \in N,\ x \in R^q).$$

Moreover, the sequence $\{M_p\}$ fulfils conditions (M), (N), (P), formulated in [2], pp. 87 and 111, and condition (N′) from [4]. It appears that the dual of the space $K\{M_p\}$, introduced in [2], p. 86, coincides with $\mathscr{D}'_{\mathscr{A}_\alpha}$. More precisely:

THEOREM 3. *Distributions of $\mathscr{D}'_{\mathscr{A}_\alpha}$ coincide with elements of $K\{M_p\}'$ and the convergence in $\mathscr{D}'_{\mathscr{A}_\alpha}$ is equivalent to the weak (strong) convergence in $K\{M_p\}'$ with $M_p = E_{\alpha(1-1/p)}$ for $p \in N$.*

Theorem 3 is a consequence of Theorems 1 and 2 and of characterizations of elements of $K\{M_p\}'$ and of the convergence in $K\{M_p\}'$, given in [4] (cf. [2], p. 113 and [5]).

It follows from Theorem 1 that the space $\mathscr{D}'_{\mathscr{A}_\alpha}$ is a subspace of the space $\mathscr{K}'_2$ of all distributions f of the form $f = (E_p F)^{(m)}$, where $m \in P^q$, $p \in N$ and F is a measurable (continuous) bounded function. The space $\mathscr{K}'_2$ and, more generally, the space $\mathscr{K}'_p$ with $p > 1$ were introduced in [7].

It can be shown (cf. Theorems 1 and 3) that elements of $\mathscr{D}'_{\mathscr{A}_\alpha}$ are linear continuous functionals on the space $\mathscr{K}_{2,\alpha}$ of all smooth functions such that

$$p_{k,\beta}(\varphi) = \sup_{x \in R^q}\big(E_\beta(x)\,|\varphi^{(k)}(x)|\big) < \infty$$

for any positive $\beta < \alpha$ and $k \in P^q$ endowed with the pseudonorms $p_{k,\beta}$, and the convergences in $\mathscr{D}'_{\mathscr{A}_\alpha}$ and in $\mathscr{K}'_{2,\alpha}$ are equivalent. It is worth noting that $\mathscr{K}'_2$ is an inductive limit of the spaces $\mathscr{K}'_{2,p}$ ($p = 1, 2, \ldots$).

As a consequence, we obtain the following characterization of the convergence in $\mathscr{K}'_2$:

THEOREM 4. *Let $f_n \in \mathscr{K}'_2$ for $n \in N$. The following conditions are equivalent:*

(i) *$f_n \to 0$ weakly (strongly) in $\mathscr{K}'_2$,*

(ii) *there exist $m \in P^q$, $p \in N$ and continuous functions G_n such that (32) holds and $E_{-p}G_n \to 0$ in $L^r(R^q)$ for some (each) r, $1 \leqslant r \leqslant \infty$;*

(iii) *there exist $m \in P^q$, $p \in N$ and measurable functions G_n such that (32) holds, the functions $E_{-p}G_n$ are commonly bounded (almost everywhere in R^q) and $G_n \to 0$ almost everywhere in R^q.*

Remark. Note that similar characterizations to those given in Theorem 4 can be formulated for the convergence in $\mathscr{K}'_s$ for an arbitrary $s \geqslant 1$ (see [3] and [7]), in view of Corollary in [4].

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Remarks on $K\{M_p\}'$-spaces

by

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To my teacher
Professor Jan Mikusiński
on his 70th birthday

Abstract. The characterization of elements of the dual of the space $K\{M_p\}$ given by I. M. Gelfand and G. E. Shilov, and also the characterization of the convergence in $K\{M_p\}'$ given by L. Kitchens and C. Swartz are simplified under an additional condition on the sequence $\{M_p\}$. In particular, a simple description of the convergence in various spaces of distributions is obtained.

1. The space $K\{M_p\}$, introduced in [2] by means of a non-decreasing sequence of extended real-valued functions M_p, embraces various spaces of test functions considered in the theory of distributions. On the other hand, the space $K\{M_p\}'$ (the dual of $K\{M_p\}$) embraces various types of spaces of distributions of finite order.

In [2] (p. 113) we find a representation of elements of $K\{M_p\}'$ under conditions (M), (N), (P), imposed on the sequence $\{M_p\}$. This representation can be written in the form of a finite sum of derivatives (in a generalized sense) of functions which become bounded after dividing by a function of the sequence $\{M_p\}$. In terms of such representations, the convergence in $K\{M_p\}'$ is characterized in [5] under the same conditions on $\{M_p\}$.

However, in all known particular cases of the space $K\{M_p\}'$, e.g., in the spaces $\mathscr{D}_K'$, $\mathscr{S}'$ (see [7]), $\mathscr{K}_p'$ (see [6]), H_r' (see [8]), $\mathscr{D}_{\mathscr{A}_a}'$ (see [4]), elements can be described in a simpler way by using one derivative of finite order. Similarly, the convergence in $\mathscr{S}'$ (see [1], p. 169), in $\mathscr{D}_{\mathscr{A}_a}'$ and in $\mathscr{K}_2'$ (see [4]) can be expressed by means of single distributional derivatives. Therefore the natural question arises when elements of $K\{M_p\}'$ and the convergence in $K\{M_p\}'$ can be characterized in that simplified way.

In this note we give an additional condition, constituting a modification of (N) (denoted by (N′)), which guarantees such characterizations. Note that the system of conditions (M), (N), (N′), (P) is a little