(iv) *there exist* $k \in P^q$, $p \in N$ *and measurable functions* F_n *such that* $f_n = F_n^{(k)}$, $F_n \to 0$ *almost everywhere in* R^q *and*

$$|F_n(x)| < C\exp(p|x|^r)$$

for some $C > 0$ *and almost all* $x \in R^q$.

References

[1] P. Antosik, J. Mikusiński, R. Sikorski, *Theory of distributions. The sequential approach*, PWN–Elsevier, Warszawa–Amsterdam 1973.
[2] I. M. Gelfand, G. E. Shilov, *Generalized functions*, vol. 2, Academic Press, N.Y.–London 1968.
[3] M. Hasumi, *Note on the n-dimensional tempered ultradistributions*, Tôhoku Math. J. 13 (1961), 94–104.
[4] A. Kamiński, S. Pilipović, *On some spaces of distributions*, this volume, 489–498.
[5] L. Kitchens, C. Swartz, *Convergence in the dual of certain* $K\{M_p\}$*-spaces*, Colloq. Math. 30 (1974), 149–155.
[6] G. Sampson, Z. Zieleźny, *Hypoelliptic convolution equations in* $\mathscr{K}'_p$, $p > 1$, Trans. Amer. Math. Soc. 223 (1976), 133–154.
[7] L. Schwartz, *Théorie des distributions*, Paris 1966.
[8] S. Sznajder, *On some classes of distributions*, Colloq. Math. 23 (1971), 133–139.
[9] Z. Zieleźny, *On the space of convolution operators in* $\mathscr{K}'_1$, Studia Math. 31 (1968), 111–124.

INSTITUTE OF MATHEMATICS
OF THE POLISH ACADEMY OF SCIENCES
BRANCH IN KATOWICE

Received November 16, 1982 (1837)

STUDIA MATHEMATICA, T. LXXVII. (1984)

Sobolev's and local derivatives

by

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Dedicated to my teacher
Professor Jan Mikusiński
on his 70th birthday

Abstract. The paper deals with local derivatives of functions of several variables having values in a fixed Banach space. It is shown that the local derivative and Sobolev's derivative are equivalent.

Local derivatives of functions of one real variable with values in a Banach space were considered by J. Mikusiński in [2], and earlier, in [4], local derivatives of functions of q real variables with values in a Hilbert space were introduced. In [6] the author gave a list of properties, theorems and also some comments concerning local derivatives. The functions considered in this paper are defined in a q-dimensional Euclidean space R^q; their values are elements of a Banach space $\mathscr{X}$.

By a local derivative of a function f of a real variable we mean the local limit of the expression

$$\frac{1}{h}[f(x+h)-f(x)]$$

as $h \to 0$. In other words, g is a local derivative of f if

$$\lim_{h\to 0}\int_a^b\left|\frac{1}{h}[f(x+h)-f(x)]-g(x)\right|dx = 0 \tag{1}$$

holds for every bounded interval (a, b). (We assume that the integrand in (1) is locally integrable).

Let f and g be locally integrable functions on an open set $\mathcal{O}$. If, for each real valued infinitely derivable function φ with bounded support in $\mathcal{O}$, the following equality holds:

$$\int g\varphi\,dx = -\int f\varphi'\,dx,$$

then g is called *weak derivative of* f or *Sobolev's derivative* f' of f (cf. [3], p. 172).

We shall show that the Sobolev derivative f' and the local derivative of f are equivalent. We generalize this result to any number of variables and to any order of the derivative. This is the subject of this note.

This result to be somewhat unexpected because weak convergence is essentially more general than local convergence. For example, $\frac{1}{h}\sin\frac{x}{h}$ converges weakly to 0, whereas local limit does not exist.

1. The points of the q-dimensional Euclidean space R^q are denoted by $x, y, z, \ldots$; $x = (\xi_1, \ldots, \xi_q)$, $y = (\eta_1, \ldots, \eta_q)$. The set of all non-negative integer points of R^q will be denoted by P^q. We adopt the notation: $x+y = (\xi_1+\eta_1, \ldots, \xi_q+\eta_q)$, $\lambda x = (\lambda\xi_1, \ldots, \lambda\xi_q)$, $xy = (\xi_1\eta_1, \ldots, \xi_q\eta_q)$, $x^m = \xi_1^{\mu_1}\ldots\xi_q^{\mu_q}$, where $m = (\mu_1, \ldots, \mu_q) \in P^q$ and λ is a real number. The letter e_i denotes the point whose ith coordinate is 1 and all the remaining ones are 0. It will also be convenient to use the notation: $e = (1, \ldots, 1)$.

Let $a = (a_1, \ldots, a_q)$ and $b = (\beta_1, \ldots, \beta_q)$. The set of points $x \in R^q$ such that $a_j < \xi_j < \beta_j$ $(j = 1, \ldots, q)$ will be called a *q-dimensional open interval* and denoted by $a < x < b$ or (a, b). Infinite values for a_j and β_j are admitted. If a_j and β_j are finite, then the set of all points $x \in R^q$ whose coordinates satisfy inequality $a_j \leqslant \xi_j \leqslant \beta_j$ $(j = 1, \ldots, q)$ will be called a *q-dimensional closed interval* and denoted by $a \leqslant x \leqslant b$ or $[a, b]$.

All integrals considered in this paper are meant as Bochner integrals (see [2]).

The symbol

$$\int_{x_0}^{x} f(t)\,dt^m$$

will stand for the iterated integral of order $m \in P^q$ of a locally integrable function f (see [1], p. 69).

We adopt the definition of the difference operator of order m ($m \in P^q$, $h = (\chi_1, \ldots, \chi_q) \in R^q$)

$$\Delta^{(m,h)}f = \Delta_1^{(\mu_1,\chi_1)}\ldots\Delta_q^{(\mu_q,\chi_q)}f,$$

where the symbols on the right-hand side mean the iteration of difference operators:

$$\Delta_i^{(\mu_i,\chi_i)}f(x) = \sum_{j=0}^{\mu_i}(-1)^{\mu_i-j}\binom{\mu_i}{j}f(x+je_i\chi_i),$$

and

$$\Delta_i^{(0,\chi_i)}f = f.$$

By the *m-th local derivative* of a function f in R^q we mean the local limit of the expression

$$\frac{1}{h^m}\Delta^{(m,h)}f(x)$$

as $h \to 0$. In other words, g is the mth local derivative of f if

$$\lim_{h\to 0}\int_I\left|\frac{1}{h^m}\Delta^{(m,h)}f(x)-g(x)\right|dx = 0 \tag{2}$$

holds for every bounded interval I in R^q. In order for this definition to be sensible, we always assume that the integrand in (2) is a locally integrable function of x. The mth local derivative of f will be denoted by $D_{\mathrm{loc}}^m f$.

If a locally integrable function f has a local derivative, then its local derivative is locally integrable because the limit of a locally convergent sequence of locally integrable functions is locally integrable. If g is a local derivative of f, then every function equivalent to g is also a local derivative of f. No other function has this property. If a measurable function f, of a real variable, has a local derivative $D_{\mathrm{loc}}^1 f$, then both f and $D_{\mathrm{loc}}^1 f$ are locally integrable (see [2]).

The local derivative is also a distributional derivative because each locally convergent sequence of distributions is distributionally convergent.

2. A function F will be called the mth local primitive of f, if f is the mth local derivative of F.

The following are the main theorems in this section.

THEOREM 1. *If f is a locally integrable function, $m \geqslant e$ ($m \in P^q$), then for every x_0 the indefinite integral*

$$F(x) = \int_{x_0}^{x} f(t)\,dt^m$$

is the m-th local primitive for f.

THEOREM 1′. *If f is a locally integrable function and $0 \leqslant m \leqslant e$ ($m \in P^q$), then for almost every x_0 the indefinite integral*

$$F(x) = \int_{x_0}^{x} f(t)\,dt^m$$

is the m-th local primitive for f.

Before giving the proof of Theorem 1 we shall need three lemmas.

LEMMA 1 (see [4]). *If f is a locally integrable function in R^q, then*

$$\Delta^{(e,h)}\int_{x_0}^{x} f(t)\,dt = \int_{x}^{x+h} f(t)\,dt.$$

LEMMA 2 (see [4]). *If f is a locally integrable function in R^q, $m \in P^q$, then*

$$\Delta^{(m,h)} \int\limits_{x}^{x+h} f(t)\,dt = \int\limits_{x}^{x+h} \Delta^{(m,h)} f(t)\,dt.$$

LEMMA 3. *If f is a locally integrable function in R^q and $m \geqslant e$, then the equality*

$$\Delta^{(m,h)} \int\limits_{x_0}^{x} f(t)\,dt^m = \int\limits_{0}^{\chi_1} d\tau_{11} \ldots \int\limits_{0}^{\chi_1} d\tau_{1\mu_1} \ldots \int\limits_{0}^{\chi_q} d\tau_{q1} \ldots \int\limits_{0}^{\chi_q} f(x+\tau)\,d\tau_{q\mu_q}, \tag{3}$$

where $\tau = (\tau_{11}+\ldots+\tau_{1\mu_1}, \ldots, \tau_{q1}+\ldots+\tau_{q\mu_q})$, $m = (\mu_1, \ldots, \mu_q)$, $h = (\chi_1, \ldots, \chi_q)$, holds.

Proof. It is easy to verify that

$$\Delta_i^{(e_i,\chi_i)} \int\limits_{x_0}^{x} G(t)\,dt^{e_i} = \int\limits_{\xi_i}^{\xi_i+\chi_i} G(t)\,dt^{e_i} = \int\limits_{0}^{\chi_i} G(t+e_i x)\,dt^{e_i} \tag{4}$$

for an arbitrary continuous function G. Using the notation from [1] (p. 69) of the iterated integral of order $m \in P^q$, we have

$$\int\limits_{x_0}^{x} f(t)\,dt^{m+e_i} = \int\limits_{\xi_{0i}}^{\xi_i} d\tau_{i\mu_i+1} \int\limits_{x_0}^{\bar{x}} f(t)\,dt^m, \tag{5}$$

$\bar{x} = x - e_i x + e_i \tau_{i\mu_i+1}$.

By Lemma 1, we have

$$\Delta^{(e,h)} \int\limits_{x_0}^{x} f(t)\,dt = \int\limits_{x}^{x+h} f(t)\,dt = \int\limits_{0}^{h} f(x+t)\,dt.$$

We now proceed by induction with respect to m. Assume that the assertion is true for all m satisfying $e \leqslant m \leqslant k$, $k \in P^q$. We have to prove that the assertion is true for $k = m+e_i$ with arbitrary i, $1 \leqslant i \leqslant q$. Let us denote the right-hand side of (3) by $F(x)$. Then, by the induction assumption, we have

$$\Delta^{(m,h)} \int\limits_{x_0}^{\bar{x}} f(t)\,dt^m = F(\bar{x}),$$

where $\bar{x} = x - e_i x + e_i \tau_{i\mu_i+1}$. Hence, by Lemma 2 and equalities (4), (5), we have

$$\Delta^{(m+e_i,h)} \int\limits_{x_0}^{x} f(t)\,dt^{m+e_i} = \Delta^{(m,h)} \Delta_i^{(e_i,\chi_i)} \int\limits_{\xi_{0i}}^{\xi_i} d\tau_{i\mu_i+1} \int\limits_{x_0}^{\bar{x}} f(t)\,dt^m$$

$$= \Delta^{(m,h)} \int\limits_{\xi_i}^{\xi_i+\chi_i} d\tau_{i\mu_i+1} \int\limits_{x_0}^{\bar{x}} f(t)\,dt^m = \int\limits_{\xi_i}^{\xi_i+\chi_i} d\tau_{i\mu_i+1} \Delta^{(m,h)} \int\limits_{x_0}^{\bar{x}} f(t)\,dt^m$$

$$= \int\limits_{\xi_i}^{\xi_i+\chi_i} F(\bar{x})\,d\tau_{i\mu_i+1} = \int\limits_{0}^{\chi_i} F(x+e_i\tau_{i\mu_i+1})\,d\tau_{i\mu_i+1}$$

$$= \int\limits_{0}^{\chi_i} d\tau_{i\mu_i+1} \int\limits_{0}^{\chi_1} d\tau_{11} \ldots \int\limits_{0}^{\chi_1} d\tau_{1\mu_1} \ldots \int\limits_{0}^{\chi_q} d\tau_{q1} \ldots \int\limits_{0}^{\chi_q} f(x+\tau+e_i\tau_{i\mu_i+1})\,d\tau_{q\mu_q},$$

where $\bar{x} = x - e_i x + e_i \tau_{i\mu_i+1}$, $\tau = (\tau_{11}+\ldots+\tau_{1\mu_1}, \ldots, \tau_{q1}+\ldots+\tau_{q\mu_q})$. Our assertion is thus proved.

The proof of Theorem 1, which has not yet been given, can now be deduced by using Lemma 3.

Proof of Theorem 1. Using the definition of the mth local derivative, Lemma 3 and the Fubini Theorem, we have

$$\int\limits_{a}^{b} \left| \frac{1}{h^m} \Delta^{(m,h)} F(x) - f(x) \right| dx = \int\limits_{a}^{b} \left| \frac{1}{h^m} \Delta^{(m,h)} \int\limits_{x_0}^{x} f(t)\,dt^m - f(x) \right| dx \tag{6}$$

$$= \int\limits_{a}^{b} \left| \frac{1}{h^m} \int\limits_{0}^{\chi_1} d\tau_{11} \ldots \int\limits_{0}^{\chi_1} d\tau_{1\mu_1} \ldots \int\limits_{0}^{\chi_q} d\tau_{q1} \ldots \int\limits_{0}^{\chi_q} f(x+\tau)\,d\tau_{q\mu_q} - f(x) \right| dx$$

$$= \int\limits_{a}^{b} \left| \frac{1}{h^m} \int\limits_{0}^{\chi_1} d\tau_{11} \ldots \int\limits_{0}^{\chi_1} d\tau_{1\mu_1} \ldots \int\limits_{0}^{\chi_q} d\tau_{q1} \ldots \int\limits_{0}^{\chi_q} [f(x+\tau) - f(x)]\,d\tau_{q\mu_q} \right| dx$$

$$\leqslant \frac{1}{h^m} \int\limits_{0}^{\chi_1} d\tau_{11} \ldots \int\limits_{0}^{\chi_1} d\tau_{1\mu_1} \ldots \int\limits_{0}^{\chi_q} d\tau_{q1} \ldots \int\limits_{0}^{\chi_q} d\tau_{q\mu_q} \int\limits_{a}^{b} |f(x+\tau) - f(x)|\,dx,$$

where $\tau = (\tau_{11}+\ldots+\tau_{1\mu_1}, \ldots, \tau_{q1}+\ldots+\tau_{q\mu_q})$.

Expression (6) tends to 0. Indeed, by Corollary 1.2 (see [2], p. 166), for given $\varepsilon > 0$, there exists an index h_0 such that

$$\int\limits_{a}^{b} |f(x+\tau) - f(x)|\,dx < \varepsilon \quad \text{for} \quad 0 < \tau < h_0.$$

Hence

$$\int_a^b \left| \frac{1}{h^m} \Delta^{(m,h)} F(x) - f(x) \right| dx$$

$$\leqslant \frac{1}{h^m} \int_0^{\chi_1} d\tau_{11} \ldots \int_0^{\chi_1} d\tau_{1\mu_1} \ldots \int_0^{\chi_q} d\tau_{q1} \ldots \int_0^{\chi_q} \varepsilon\, d\tau_{q\mu_q} = \frac{1}{h^m} \chi_1^{\mu_1} \ldots \chi_q^{\mu_q} \varepsilon = \varepsilon$$

for $h \leqslant h_0$. This proves that the mth local derivative of F is f. Consequently F is the mth local primitive of f.

The proof of Theorem 1′ is similar to the proof of Theorem 1. We only need to apply the following

LEMMA 3′. *If f is a locally integrable function in R^q, $0 \leqslant m \leqslant e$, then*

$$\Delta^{(m,h)} \int_{x_0}^{x} f(t)\, dt^m = \int_0^{\chi_1} d\tau_1^{\mu_1} \ldots \int_0^{\chi_q} f(x+mt)\, d\tau_q^{\mu_q} \text{ a.e.,}$$

where $t = (\tau_1, \ldots, \tau_q)$, $m = (\mu_1, \ldots, \mu_q)$, $h = (\chi_1, \ldots, \chi_q)$.

Proof. Without loss of generality we may assume that the first i $(1 \leqslant i \leqslant q)$ variables of a point m are 1 and all the remaining $(q-i)$-variables are 0. Let $m_1 = (1, \ldots, 1) \in P^i$ and let m_2 denote a point in P^{q-i} whose all coordinates are 0. Then we have $m = (m_1, m_2) \in P^q$. Let $x = (\xi_1, \ldots, \xi_q) \in R^q$. By x_1 we denote the point $(\xi_1, \ldots, \xi_i)$ in R^i and by x_2 the point $(\xi_{i+1}, \ldots, \xi_q)$ in R^{q-i}. We note that $x = (x_1, x_2)$, $t = (t_1, t_2)$ and $h = (h_1, h_2)$. Then

$$\Delta^{(m,h)} \int_{x_0}^{x} f(t)\, dt^m = \Delta^{(m_1,h_1)} \int_{x_{01}}^{x_1} f(t_1, x_2)\, dt_1.$$

By the Fubini Theorem, it follows that the function

$$F(x_1, x_2) = \int_{x_{01}}^{x_1} f(t_1, x_2)\, dt_1$$

is determined for almost all points x_2 in R^{q-i}. Let x_2 be an arbitrary fixed point such that the function $F(x_1, x_2)$ is determined. Applying Lemma 1 and substituting $x_1 + u_1 = t_1$, we get

$$\Delta^{(m_1,h_1)} \int_{x_{01}}^{x_1} f(t_1, x_2)\, dt_1 = \int_{x_1}^{x_1+h_1} f(t_1, x_2)\, dt_1 = \int_0^{h_1} f(x_1+u_1, x_2)\, du_1.$$

Under the sign of integral, we can replace u_1 by t_1 and write the integral in the form

$$\int_0^{h_1} f(x_1+t_1, x_2)\, dt_1 = \int_0^{\chi_1} d\tau_1^{\mu_1} \ldots \int_0^{\chi_q} f(x+mt)\, d\tau_q^{\mu_q},$$

where $t = (\tau_1, \ldots, \tau_q)$, $m = (\mu_1, \ldots, \mu_q)$, $\mu_j = 1$ for $j = 1, \ldots, i$ and $\mu_j = 0$ for $j = i+1, \ldots, q$. This means that

$$\Delta^{(m,h)} \int_{x_0}^{x} f(t)\, dt^m = \int_0^{\chi_1} d\tau_1^{\mu_1} \ldots \int_0^{\chi_q} f(x+mt)\, d\tau_q^{\mu_q} \text{ a.e.}$$

for $0 \leqslant m \leqslant e$. Thus the proof of Lemma 3′ is finished.

THEOREM 2. *If a locally integrable function f has a local derivative $D^e_{\mathrm{loc}} f$, then*

$$\frac{1}{h^e} \Delta^{(e,h)} f(x) \to D^e_{\mathrm{loc}} f(x)$$

almost everywhere, as $h \to 0$.

In order to prove Theorem 2 we shall use the theorems given below.

THEOREM ON LOCAL CONVERGENCE (see [2], Th. 1, p. 173). *If f is a locally integrable function in R^q, then for almost every point $x_0 \in R^q$ we have*

$$f(x_0 + ht) \to f(x_0) \ \textit{loc.}, \quad \textit{as} \quad h \to 0;$$

in other words, for each bounded interval $I \subset R^q$, we have

$$\lim_{h \to 0} \int_I |f(x_0+ht) - f(x_0)|\, dt = 0 \quad \textit{for a.e. } x_0 \in R^q.$$

THEOREM 3 (see [6], Th. 5). *If a function f has a locally integrable local derivative $D^e_{\mathrm{loc}} f$, then*

$$\Delta^{(e,h)} f(x) = \int_x^{x+h} D^e_{\mathrm{loc}} f(t)\, dt$$

for each fixed $h \in R^q$ and almost all $x \in R^q$.

Proof of Theorem 2. By Theorem 3 it follows that

$$\Delta^{(e,h)} f(x) = \int_x^{x+h} D^e_{\mathrm{loc}} f(u)\, du$$

for almost all $x \in R^q$ and for each $h \in R^q$. Substituting $x + ht = u$, we get

$$\Delta^{(e,h)} f(x) = h^e \int_0^e D^e_{\mathrm{loc}} f(x+ht)\, dt \tag{7}$$

for almost all $x \in R^q$ and for each $h \in R^q$. According to the Theorem on Local Convergence, for almost every point $x \in R^q$ we have

$$D^e_{\mathrm{loc}} f(x+ht) \to D^e_{\mathrm{loc}} f(x) \text{ loc.}, \quad \text{as} \quad h \to 0,$$

which implies

$$\int_0^e D^e_{\mathrm{loc}} f(x+ht)\,dt \to D^e_{\mathrm{loc}} f(x) \text{ a.e.,} \quad \text{as} \quad h\to 0.$$

Hence and from (7) it follows that

$$\frac{1}{h^e}\Delta^{(e,h)} f(x) \to D^e_{\mathrm{loc}} f(x) \text{ a.e.,} \quad \text{as} \quad h\to 0.$$

Thus the proof of Theorem 2 is finished.

3. The following theorem tells us that the local derivative $D^m_{\mathrm{loc}} f$ and the Sobolev derivative $f^{(m)}$ coincide with each other, when $f^{(m)}$ exists.

THEOREM 4. *The distributional derivative $f^{(m)}$ of f is equal to the m-th local derivative $D^m_{\mathrm{loc}} f$ of f if $f^{(m)}$ and f are locally integrable functions.*

Proof. Consider the function

$$F(x) = \int_{x_0}^{x} f^{(m)}(t)\,dt^m.$$

By Theorems 1 and 1′, it follows that, for almost every x_0, F is the mth local primitive for $f^{(m)}$. Hence

$$D^m_{\mathrm{loc}} F(x) = F^{(m)}(x) = f^{(m)}(x). \tag{8}$$

Thus the difference $F-f$ has its mth distributional derivative equal to 0. By Theorem 4.5.4 (see [1], p. 106), we have

$$F(x) = f(x) + \sum_{j=0}^{\mu_1-1} \xi_1^j f_{1j}(x) + \ldots + \sum_{j=0}^{\mu_q-1} \xi_q^j f_{qj}(x), \tag{9}$$

where f_{ij} are constant in ξ_i. (If $\mu_i = 0$ for some i in formula (9), then the corresponding sum is taken to be 0.) It is easy to see that

$$\varphi(x) = \sum_{j=0}^{\mu_1-1} \xi_1^j f_{1j}(x) + \ldots + \sum_{j=0}^{\mu_q-1} \xi_q^j f_{qj}(x)$$

is a locally integrable function. Since

$$\Delta_i^{(\mu_i,\varkappa_i)} \sum_{j=0}^{\mu_i-1} \xi_i^j f_{ij}(x) = 0 \quad (i = 1, \ldots, q),$$

we have

$$\Delta^{(m,h)}\varphi(x) = \Delta^{(m-\mu_1 e_1, h-e_1\varkappa_1)} \Delta_1^{(\mu_1,\varkappa_1)} \sum_{j=0}^{\mu_1-1} \xi_1^j f_{1j}(x) + \ldots$$
$$\ldots + \Delta^{(m-\mu_q e_q, h-e_q\varkappa_q)} \Delta_q^{(\mu_q,\varkappa_q)} \sum_{j=0}^{\mu_q-1} \xi_q^j f_{qj}(x) = 0$$

and $D^m_{\mathrm{loc}}\varphi(x) = 0$. Since the mth local derivative of F exists, we obtain

$$D^m_{\mathrm{loc}} F = D^m_{\mathrm{loc}} f,$$

in view of (9). Hence and by equality (8) it follows that

$$f^{(m)} = D^m_{\mathrm{loc}} f.$$

Thus the proof of Theorem 4 is finished.

Remark. 1. Locally integrable functions $f(x)$ and $g(x)$ are equal iff they are equal as distributions which turns out on the same iff $f(x) = g(x)$ almost everywhere.

Remark 2. If f is a locally integrable function and its distributional derivative $f^{(m)}$ is a locally integrable function, then $f^{(m)}$ is a Sobolev derivative of order m.

References

[1] P. Antosik, J. Mikusiński, R. Sikorski, *Theory of distributions. The sequential approach*, Amsterdam–Warszawa 1973.

[2] J. Mikusiński, *The Bochner Integral*, Birkhäuser, 1978.

[3] [S. M. Nikolskii] С. М. Никольский, *Приближение функций многих переменных и теоремы вложения*, Издательство Наука, Москва 1969.

[4] K. Skórnik, *Postać funkcji lokalnie całkowalnej, której m-ta pochodna lokalna znika prawie wszędzie* (*The form of locally integrable function whose m-th local derivative vanishes almost everywhere*), Zeszyty Naukowe WSP 5, Katowice 1966, 127–152.

[5] — *К теории локальных производных*, Доклады АН СССР 257, 2 (1981).

[6] — *On local derivatives*, Ann. Polon. Math. 41, 1982 (to appear).

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Received November 16, 1982 (1838)