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Connected subgroups of nuclear spaces

by

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Abstract. It is proved that closed and connected subgroups of nuclear spaces are real linear subspaces.

1. It is well known that a closed subgroup of a Lie group is a Lie subgroup. In the simplest linear case this amounts to the elementary fact that every closed and connected subgroup of \mathbf{R}^n is its linear subspace. However, considering infinite-dimensional topological linear spaces we encounter essential differences. For example, the subset of all integer-valued functions in $L^2(0, 1)$ is a closed and connected subgroup of $L^2(0, 1)$ but it fails to contain any line.

Our aim in this note is to show that in the case of nuclear spaces the situation is analogous to the finite-dimensional case, namely:

THEOREM 1. *Closed and connected subgroups of nuclear spaces are real linear subspaces.*

This theorem substantiates a conjecture of W. Wojtyński and provides one more example that nuclear spaces are closer to finite-dimensional spaces than normed spaces are.

After proving the theorem we have found that it can be derived also from the results of the first named author concerning unitary representations of groups which are quotients of nuclear spaces by its closed subgroups.

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2. All linear spaces we shall deal in the sequel are assumed to be real. We shall obtain Theorem 1 as a consequence of the following

THEOREM 2. *Let G be a subgroup of a real nuclear space X such that for each neighbourhood U of zero in X $\text{span}(G \cap U)$ is dense in X . Then G is dense in X .*

In fact, assuming that Theorem 2 holds true let G be a closed connected subgroup of a nuclear space X , let $X_1 = \text{cl}(\text{span } G)$ and let U be a neighbourhood of 0 in X .

The set $G \cap U$ generates G , hence $\text{span}(G \cap U)$ is dense in X_1 . Then, by Theorem 2, G is dense in X_1 , whence $G = \text{cl } G = X_1$.

Proof of Theorem 2. Since X is a nuclear space, let $(\|\cdot\|_i, i \in I)$ be a family of seminorms defining the topology of X such that:

1° For each $i \in I$ the seminorm $\|\cdot\|_i$ induces a Hilbert norm on the space $X/\ker\|\cdot\|_i$. The completion of $X/\ker\|\cdot\|_i$ with respect to this norm is a separable Hilbert space which will be denoted by H_i . We have the natural continuous linear operator $p_i: X \rightarrow H_i$.

2° For each $i \in I$ there is $k \in I$ such that $\|\cdot\|_k \geq \|\cdot\|_i$ and the canonical mapping $T_{ik}: H_k \rightarrow H_i$ is a Hilbert-Schmidt operator.

Let $x \in X$, $i \in I$ and $\varepsilon > 0$. In order to prove our theorem we shall find $g \in G$ such that $\|g - x\|_i < \varepsilon$. Take $k \in I$ as in 2° and $0 < r < \varepsilon/2 \|\|T_{ik}\|\|$, where $\|\|T_{ik}\|\|$ is the Hilbert-Schmidt norm of the operator T_{ik} .

Since $K = \{g \in G: \|g\|_k < r\}$ is linearly dense in X , the set $p_k(K)$ is linearly dense in H_k . Hence there are $u_1, u_2, \dots \in K$ such that the vectors $p_k(u_1), p_k(u_2), \dots$ form a complete system of vectors in H_k , i.e. they are linearly independent and total in H_k .

Let e_1, e_2, \dots be the system obtained by the orthonormalization of the vectors $p_k(u_1), p_k(u_2), \dots$. Then $p_k(u_n) = \sum_{m=1}^n \alpha_n^m e_m$, where obviously $|\alpha_n^m| < r$, $n = 1, 2, \dots$, $m = 1, \dots, n$.

The vectors $p_i(u_1), p_i(u_2), \dots$ are total in H_i and $T_{ik} p_k(u_n) = p_i(u_n)$, $n = 1, 2, \dots$. By omitting linearly dependent vectors in this sequence we obtain a complete system of vectors $p_i(u_{j_1}), p_i(u_{j_2}), \dots$ in H_i . Let f_1, f_2, \dots be the orthonormalization of this system. We have then $p_i(u_{j_n}) = \sum_{m=1}^n \beta_n^m f_m$, $n = 1, 2, \dots$. As easily verified, the matrix of the operator T_{ik} in the bases e_1, e_2, \dots and f_1, f_2, \dots has the form

$$T_{ik} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \dots \\ 0 & \gamma_{22} & \gamma_{23} & \dots \\ 0 & 0 & \gamma_{33} & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix},$$

where $\sum_{i,j=1}^{\infty} |\gamma_{ij}|^2 = \|\|T_{ik}\|\|^2$ and $\gamma_{nm} = 0$ if $m < j(n)$. Moreover,

$$|\beta_n^m| = |\gamma_{nj_n} \alpha_n^{j_n}| < r |\gamma_{nj_n}|, \quad n = 1, 2, \dots$$

Let $p_i(x) = \sum_{m=1}^{\infty} \delta_m f_m$. Take natural l such that

$$\sum_{m=l+1}^{\infty} |\delta_m|^2 < 3\varepsilon^2/4.$$

Now we can construct inductively a sequence c_l, c_{l-1}, \dots, c_1 of integers so that

$$p_i(x) - \sum_{m=1}^l c_m p_i(u_{j_m}) = \sum_{m=1}^l \varrho_m f_m + \sum_{m=l+1}^{\infty} \delta_m f_m,$$

where $|\varrho_m| < |\beta_m^m| < r |\gamma_{mj_m}|$, $m = 1, \dots, l$.

Putting $g = \sum_{m=1}^l c_m u_{j_m}$ we have $g \in G$ and

$$\|x - g\|_i^2 = \sum_{m=1}^l |\varrho_m|^2 + \sum_{m=l+1}^{\infty} |\delta_m|^2 < r^2 \sum_{m=1}^l |\gamma_{mj_m}|^2 + 3\varepsilon^2/4,$$

whence

$$\|x - g\|_i^2 < r^2 \|\|T_{ik}\|\|^2 + 3\varepsilon^2/4 < \varepsilon^2, \quad \text{q.e.d.}$$

3. The other well-known result in the theory of Lie groups is the theorem of Yamabe [2] which says that an arcwise connected subgroup of a Lie group is its Lie subgroup. Unlikely to the case of closed subgroups this theorem does not remain valid when we pass to infinite-dimensional nuclear spaces.

An easy example is the subgroup of $C^\infty(\mathbf{R})$, the nuclear space of infinitely differentiable functions on \mathbf{R} , consisting of those functions which take non-integer values only on subsets of finite measure.

This subgroup is arcwise-connected, contains constant integer-valued functions but does not contain the line of constant functions.

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