STUDIA MATHEMATICA, T. LXXVIII. (1984)

On the positive projection constant

by

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Abstract. The positive projection constant of l_n^p, $p \neq 2$, is proportional to $\min\{n^{1/p}, n^{1/p'}\}$, and for l_n^2 it is proportional to a $(n/\log 2n)^{1/2}$. These results are extended to symmetric spaces.

Introduction. We study the positive projection constant and the positive distance for finite dimensional Banach lattices. A Banach lattice is called λ-*injective* if it is λ-complemented in any Banach lattice in which it is embedded as a sublattice. The infimum of all such numbers λ is the positive projection constant. Cartwright [1] proved that the finite dimensional Banach lattices with positive projection constant 1 are the spaces $(\sum_{j=1}^{k} \oplus l_{n_j}^1)_\infty$. While this is a purely isometric result, Lindenstrauss and Tzafriri [2] gave the answer for the isomorphic problem: There is a function $f(\lambda)$ so that every lattice with positive projection constant λ is $f(\lambda)$ order isomorphic to one of the spaces $(\sum_{j=1}^{k} \oplus l_{n_j}^1)_\infty$.

Our main concern is to find out how small $f(\lambda)$ can be choosen. It turns out that for the classical spaces l_n^p, $p \neq 2, f(\lambda)$ can be chosen to be a constant times λ. Surprisingly, this is not sufficient for $p = 2$. Here we need a constant times $\lambda\sqrt{\log 2\lambda}$.

We also show that the function $c\lambda(\log 2\lambda)^2$ is sufficient for the class of symmetric lattices. On the other hand this result cannot be improved significantly as the example l_n^2 shows.

I would like to thank Hermann König, Kiel, for several helpful discussions on the subject of this paper.

1. Preliminaries. We consider here a finite dimensional Banach lattice E. $\{e_i\}_{i=1}^n$ denote its normalized atoms. We call such a basis 1-*symmetric* if

$$\left\|\sum_1^n a_i e_i\right\| = \left\|\sum_1^n \varepsilon_i a_i e_{\pi(i)}\right\|$$

for all $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$, and all permutations π of $\{1, \ldots, n\}$.

The dual basis is denoted by $\{e_i^*\}_{i=1}^n$. To simplify our notation we put

$$\lambda(k)=\Big\|\sum_{i=1}^{k} e_i\Big\| \quad \text{and} \quad \lambda^*(k)=\Big\|\sum_{i=1}^{k} e_i^*\Big\|.$$

The positive projection constant of E is defined by

$$\gamma_\infty^+(E)=\sup_{E\subset F}\inf\{\|P\| \mid P \text{ is a positive projection from } F \text{ onto } E\},$$

where F ranges over all lattices which contain E as a sublattice.

By the positive distance between E and F we mean

$$d^+(E,F)=\inf\{\|I\|\,\|I^{-1}\| \mid I \text{ is an order isomorphism}\}.$$

By $(\sum_{j=1}^{N}\oplus l_{k_j}^1)_\infty$ we understand the space of double sequences with norm

$$\Big\|\sum_{j=1}^{N}\sum_{i=1}^{k_j} a_{ij}u_{ij}\Big\| = \sup_{1\le j\le N}\sum_{i=1}^{k_j}|a_{ij}|.$$

If $k_j=n$ for all j we write $l_N^\infty(l_n^1)$. Of course, $l^\infty(l_n^1)$ is the infinite dimensional analogue.

It is known ([1], [2]) that

$$\gamma_\infty^+\Big(\big(\sum_{j=1}^{N}\oplus l_{k_j}^1\big)_\infty\Big)=1,$$

E can be embedded into $l^\infty(l_n^1)$ as follows. Suppose x_j^*, $j\in N$ are positive functionals that are dense in the set of positive extreme points of the dual unit sphere.

$$I(e_i)=\sum_j\langle x_j^*, e_i\rangle u_{ij}. \tag{1.1}$$

We shall refer to it below as "a standard embedding" for the sake of brevity.

If the number of extreme points is finite E embeds into $l_N^\infty(l_N^1)$ for some $N\in N$.

2. Positive projection constant, positive factorization and positive distance. The first lemma collects some well-known facts.

LEMMA 2.1. *Suppose E is an n-dimensional Banach lattice. Then the following are equivalent.*

(i) *For every lattice F which contains E as a sublattice there is a projection P from F onto E with $\|P\|\le\lambda$.*

(ii) *For every lattice F which contains E as a sublattice there is a positive projection P from F onto E with $\|P\|\le\lambda$.*

(iii) *There are positive operators $R\in L(E, l^\infty(l_n^1))$, $S\in L(l^\infty(l_n^1), E)$ with $SR=\mathrm{id}_E$ and $\|S\|\,\|R\|\le\lambda$.*

(iv) *For every lattice F with sublattice G and positive operator $T\in L(G,E)$ there is a positive operator $\tilde T\in L(F,E)$ with $\tilde T|_G=T$ and $\|\tilde T\|\le\lambda\|T\|$.*

Proof. (iii) follows easily from (ii): We choose as R a positive embedding and for S a positive projection with $\|S\|\le\lambda$. Now we show that (iv) follows from (iii). Instead of T we consider the map $RT\in L(G, l^\infty(l_n^1))$. By [3] there is an extension $\widetilde{RT}\in L(F, l^\infty(l_n^1))$ with $\|\widetilde{RT}\|=\|RT\|$. Thus $S\circ\widetilde{RT}$ gives the extension of T.

(ii) follows from (iv) by extending the identity $\mathrm{id}_E\in L(E,E)$. Obviously $\widetilde{\mathrm{id}}_E\in L(F,E)$ is a positive projection.

Trivially (i) follows from (ii). So it remains to prove that (i) implies (ii). By (ii) and (iii) it suffices to consider the case where $F=l^\infty(l_n^1)$ and E is embedded as in (1.1). Without loss of generality we may assume that the dual unit ball of E has finitely many extreme points, i.e. E embeds in $l_N^\infty(l_n^1)$ for some $N\in N$.

We show that for every projection P from $l_N^\infty(l_n^1)$ onto E there is a positive projection P^+ from $l_N^\infty(l_n^1)$ onto E with $\|P\|\ge\|P^+\|$.

The projection can be written in the form

$$P(y)=\sum_{i=1}^{n}\langle f_i, y\rangle I(e_i),\qquad y\in l_N^\infty(l_n^1),$$

where $\{e_i\}_{i=1}^n$ is the 1-unconditional basis given by the atoms and I the standard embedding (1.1). By dualization we get

$$\begin{aligned}\|P\| &= \max_{\|y\|=1}\Big\|\sum_{i=1}^{n}\langle f_i, y\rangle I(e_i)\Big\|\\ &= \max_{1\le j\le N}\max_{\varepsilon_i=\pm1}\sum_{l=1}^{N}\max_{1\le k\le n}\Big|\sum_{i=1}^{n}\varepsilon_i\langle x_j^*, e_i\rangle\langle f_i, u_{kl}\rangle\Big|\\ &\ge \max_{1\le j\le N}\sum_{l=1}^{N}\max_{1\le k\le n}2^{-n}\sum_{\varepsilon}\Big|\sum_{i=1}^{n}\varepsilon_i\langle x_j^*, e_i\rangle\langle f_i, u_{kl}\rangle\Big|.\end{aligned}$$

On the other hand we have the elementary estimate for $a_i\in R$

$$\max_{1\le i\le n}|a_i|\le 2^{-n}\sum_{\varepsilon}\Big|\sum_{i=1}^{n}\varepsilon_i a_i\Big|.$$

Thus

$$\begin{aligned}\|P\| &\ge \max_{1\le j\le N}\sum_{l=1}^{N}\max_{1\le k\le n}\max_{1\le i\le n}|\langle x_j^*, e_i\rangle\langle f_i, u_{kl}\rangle|\\ &\ge \max_{1\le j\le N}\sum_{l=1}^{N}\max_{1\le i\le n}|\langle x_j^*, e_i\rangle\langle f_i, u_{il}\rangle|.\end{aligned} \tag{2.1}$$

Now we choose as P^+ the positive projection given by

$$\langle \tilde{f}_i, u_{kl}\rangle = \begin{cases} \left(\sum_{l=1}^{N} |\langle x_l^*, e_i\rangle \langle f_i, u_{il}\rangle|\right)^{-1} |\langle f_i, u_{il}\rangle| & \text{if } i = k,\\ 0 & \text{if } i \neq k.\end{cases}$$

By the same computation as above one gets that $\|P^+\|$ is bounded by the expression on the right-hand side of (2.1). ■

The following lemma is an immediate consequence of Lemma 2.1.

LEMMA 2.2. (i) $\gamma_\infty^+(E) \leqslant \gamma_\infty^+(F) d^+(E, F)$, (ii) $\gamma_\infty^+(E) \leqslant \gamma_\infty(E)$.

The following lemma will be used several times.

LEMMA 2.3. *Let $\{e_i\}_{i=1}^n$ be a 1-symmetric basis of E. Then*

$$\gamma_\infty^+(E) = \inf \max_{\|x^*\|=1} \sum_{l=1}^{N} \max_{1 \leqslant i \leqslant n} |\langle x^*, e_i\rangle \langle x_l, e_i^*\rangle|, \tag{2.2}$$

where the infimum is taken over all sequences $\sum_{l=1}^{N} \|x_l\| = n$.

Proof. The proof uses the same notation as that of Lemma 2.1. We also assume that the dual unit ball has finitely many extreme points. We prove first that $\gamma_\infty^+(E)$ is greater than the right-hand expression.

For a positive projection P we get, by (2.1) and the fact that $\langle f_i, u_{kl}\rangle \geqslant 0$, that

$$\|P\| \geqslant \max_{1 \leqslant j \leqslant N} \sum_{l=1}^{N} \max_{1 \leqslant i \leqslant n} \langle x_j^*, e_i\rangle \langle f_i, u_{il}\rangle. \tag{2.3}$$

Now we define $\langle x_l, e_i^*\rangle := \langle f_i, u_{il}\rangle$. Because of $1 = \langle f_i, I(e_i)\rangle$ and (1.1)

$$n = \sum_{j=1}^{N} \sum_{i=1}^{n} \langle x_j^*, e_i\rangle \langle x_j, e_i^*\rangle \leqslant \sum_{j=1}^{N} \|x_j^*\| \|x_j\| = \sum_{j=1}^{N} \|x_j\|.$$

This and (2.3) give that $\gamma_\infty^+(E)$ is greater than the right-hand expression of (2.2). Now we prove the opposite inequality. Suppose x_l, $l = 1, \ldots, N$, is a sequence with $\sum_{l=1}^{N} \|x_l\| = n$ so that

$$\max_{\|x^*\|=1} \sum_{l=1}^{N} \max_{1 \leqslant i \leqslant n} |\langle x^*, e_i\rangle \langle x_l, e_i^*\rangle| \tag{2.4}$$

is $1+\varepsilon$-close to the infimum. Since $\{e_i\}_{i=1}^n$ is a 1-symmetric basis, we may assume that with every x_l the vector

$$\sum_{i=1}^{n} \langle x_l, e_i^*\rangle e_{\pi(i)}, \quad \pi \text{ a permutation of } \{1, \ldots, n\} \tag{2.5}$$

is also in the sequence. Without loss of generality, we may assume that the extreme points $x_1^*, \ldots, x_N^*$ used in (1.1) for a standard embedding satisfy

$$n = \sum_{l=1}^{N} \|x_l\| = \sum_{l=1}^{N} \|x_l\| \|x_l^*\| = \sum_{l=1}^{N} \langle x_l, x_l^*\rangle. \tag{2.6}$$

Now (2.5) and (2.6) assure that the functionals f_i defined by the formulae:

$$\begin{aligned} \langle f_i, u_{kl}\rangle &= 0 \quad \text{if } i \neq k,\\ \langle f_i, u_{il}\rangle &= |\langle x_l, e_i^*\rangle|, \end{aligned} \qquad l = 1, \ldots, N,$$

represent a projection with norm less that (2.4). The computations are as above. ■

LEMMA 2.4. *Suppose $\{e_i\}_{i=1}^n$ is a 1-symmetric basis of E and k_i, $i = 1, \ldots, r$, are natural numbers so that $\sum_{i=1}^{n} k_i = n$. Then we have for some $c > 0$*

(i) $$d^+\left(\left(\sum_{i=1}^{r} \oplus l_{k_i}^1\right)_\infty, E\right) \leqslant \left\|\sum_{i=1}^{r} \lambda^*(k_i) e_i\right\|,$$

(ii) $$\inf_{\gamma_\infty^+(F)=1} d^+(F, E) \leqslant \min_{\sum_{i=1}^{r} k_i = n} \left\|\sum_{i=1}^{r} \lambda^*(k_i) e_i\right\| \leqslant c \inf_{\gamma_\infty^+(F)=1} d^+(F, E).$$

Proof. Let u_{ij}, $i = 1, \ldots, r$, $j = 1, \ldots, k_i$, denote the natural basis in $\left(\sum_{i=1}^{r} \oplus l_{k_j}^1\right)_\infty$ and let e_{ij}, $i = 1, \ldots, r$, $j = 1, \ldots, k_i$, an enumeration of the 1-symmetric basis in E. Then, by considering the diagonal map I with $I(u_{ij}) = \lambda^*(k_i) e_{ij}$, we get (i). The left-hand inequality in (ii) follows from (i) and the fact [1], [3] that

$$\gamma_\infty^+\left(\left(\sum_{l=1}^{r} \oplus l_{k_l}^1\right)_\infty\right) = 1.$$

To prove the right-hand inequality we observe first that since I and I^{-1} are positive and $\{e_i\}_{i=1}^n$ is a 1-symmetric basis we may assume that I is diagonal, i.e. $I(u_{ij}) = a_{ij} e_{ij}$ for scalars a_{ij}. Then we observe that we may also assume that $a_{ij} = a_{il}$ for $1 \leqslant j$, $l \leqslant k_i$. We finish the proof with an elementary computation. ■

Since $\{e_i\}_{i=1}^n$ is 1-symmetric one might conjecture that

$$\min_{\Sigma k_l = n} \left\|\sum_{l=1}^{r} \lambda^*(k_l) e_l\right\| \quad \text{and} \quad \min_{1 \leqslant k \leqslant n} \lambda^*(k) \lambda([n/k])$$

are proportional. This is not true. The space $l_n^{2,\infty}$ with norm

$$\|x\| = \max_{\pi} \max_{1 \leqslant k \leqslant n} \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |x(\pi(i))|$$

serves as an counterexample. One gets that

$$\min_{\sum k_i = n} \Big\| \sum_{i=1}^{r} \lambda^*(k_i) e_i \Big\| \leqslant c\sqrt{n/\log 2n}$$

if one chooses $k_i \sim n/i \log n$, $i = 1, \ldots, [n/\log n]$. But the second term is proportional to $\sqrt{n}$.

3. The positive projection constant of l_n^p. We now estimate the positive projection constants of l_n^p, $1 \leqslant p \leqslant \infty$, $n \in N$. It turns out that for $p \neq 2$ $\gamma_\infty^+(l_n^p)$ and

$$\inf_{\gamma_\infty^+(F)=1} d^+(F, l_n^p)$$

are proportional. This is not true for l_n^2. By Lemma 2.4 we have

$$\inf_{\gamma_\infty^+(F)=1} d^+(F, l_n^2) \sim \sqrt{n} \quad \text{while} \quad \gamma_\infty^+(l_n^2) \sim \sqrt{n/\log 2n}.$$

THEOREM 3.1. *There are constants $a_p, b_2 > 0$ such that for all $n \in N$*

(i) $$a_p \min\{n^{1/p}, n^{1/p'}\} \leqslant \gamma_\infty^+(l_n^p) \leqslant \min\{n^{1/p}, n^{1/p'}\} \quad \text{if } p \neq 2,$$

(ii) $$a_2 \sqrt{n/\log 2n} \leqslant \gamma_\infty^+(l_n^2) \leqslant b_2 \sqrt{n/\log 2n},$$

b_2 is less than 10.

The main part of Theorem 3.1 is easily derived from the following proposition.

PROPOSITION 3.2. *Let $\{e_i\}_{i=1}^n$ be a 1-symmetric basis of E and $\mathrm{id} \in L(l_n^2, E)$ the natural identity $\mathrm{id}((a_i)_{i=1}^n) = \sum_1^n a_i e_i$. Then we have*

$$\gamma_\infty^+(E) \geqslant c\sqrt{n} \left(\Big\| \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \cdot e_k^* \Big\| \, \|\mathrm{id}\| \right)^{-1}$$

for some universal $c > 0$.

Proof of Theorem 3.1. By Lemma 2.1 it is clear that in order to compute $\gamma_\infty^+(E)$ it is enough to compute

$$\inf\{\|P\| \mid P \text{ projects } l^\infty(l_n^1) \text{ onto } E\}.$$

The left-hand inequalities are immediate consequences of Proposition 3.2. The right-hand inequality of (i) follows by considering the identity maps between l_n^p and l_n^1 and between l_n^p and l_n^∞.

So it is left to prove the right-hand inequality of (ii). We may assume that $n = 2^r$, $r \in N$. We apply Lemma 2.3 and choose the sequence $x_{k,j} \in l_n^2$

$$x_{k,j} = 2^{k/2} \frac{1}{1+\log_2 n} \sum_{i=j2^k+1}^{(j+1)2^k} e_i, \quad k = 0, \ldots, \log_2 n, \quad j = 0, \ldots, 2^{-k}n-1.$$

We get

$$\gamma_\infty^+(l_n^2) \leqslant \max_{\|x\|_2=1} \sum_{k,j} \max_{1 \leqslant i \leqslant n} |x(i) x_{k,j}(i)|$$
$$= \max_{\|x\|_2=1} \sum_{k=0}^{\log n} \frac{1}{\log_2 2n} 2^{k/2} \sum_j \max_{j2^k < i \leqslant (j+1)2^k} |x(i)|$$

at this point we introduce the norm

$$\|x\|_{2,1} = \sup_{\pi} \sum_{k=1}^{n} \left| \frac{1}{\sqrt{k}} x(\pi(k)) \right| \leqslant \sqrt{\log_2 2n}\, \|x\|_2$$

and get

$$\gamma_\infty^+(l_n^2) \leqslant \frac{1}{\sqrt{\log_2 2n}} \max_{\|x\|_{2,1}=1} \sum_{k=0}^{\log n} 2^{k/2} \sum_j \max_{j2^k < i \leqslant (j+1)2^k} |x(i)|.$$

Since it is enough to consider the maximum over the extreme points of the unit ball for the norm $\|\ \|_{2,1}$ we get

$$\gamma_\infty^+(l_n^2) \leqslant \frac{1}{\sqrt{\log 2n}} \max_{l=1,\ldots,n} \frac{1}{\sqrt{l}} \Big\{ \sum_{2^k \leqslant n/l} 2^{k/2} \cdot l + \sum_{2^k > n/l} 2^{-k/2} n \Big\}$$
$$\leqslant 2\sqrt{2}(1+\sqrt{2}) \sqrt{n/\log 2n}. \quad \blacksquare$$

In order to prove Proposition 3.2 we require the following lemma.

LEMMA 3.3. *There is an universal $c > 0$ such that for all $x \in \mathbf{R}^n$*

$$\frac{c}{\sqrt{n}} \|x\|_2 \leqslant \frac{1}{n!} \sum_{\pi} \max_{1 \leqslant k \leqslant n} \left| \frac{1}{\sqrt{k}} x_{\pi(k)} \right|,$$

where π denotes a permutation of the set $\{1, \ldots, n\}$.

Proof of Proposition 3.2. We apply Lemma 2.3. Choose a sequence x_l, $l = 1, \ldots, n$, such that

$$(1+\varepsilon)\gamma_\infty^+(E) \geqslant \max_{\|x^*\|=1} \sum_{l=1}^{N} \max_{1 \leqslant i \leqslant n} |\langle x^*, e_i \rangle \langle x_l, e_i^* \rangle|$$
$$\geqslant \max_{\|x^*\|=1} \sum_{l=1}^{N} \frac{1}{n!} \sum_{\pi} \max_{1 \leqslant i \leqslant n} |\langle x^*, e_{\pi(i)} \rangle \langle x_l, e_i^* \rangle|$$
$$\geqslant \Big\| \sum \frac{1}{\sqrt{k}} e_k^* \Big\|^{-1} \sum_{l=1}^{N} \frac{1}{n!} \sum_{\pi} \max_{1 \leqslant i \leqslant n} \left| \frac{\langle x_l, e_i^* \rangle}{\sqrt{\pi(i)}} \right|.$$

By Lemma 3.3 we get

$$(1+\varepsilon)\gamma_\infty^+(E) \geqslant \Big\|\sum_i \frac{1}{\sqrt{i}}\, e_i^*\Big\|^{-1} \sum_{l=1}^{N} \frac{c}{\sqrt{n}} \Big(\sum_i |\langle x_l, e_i^*\rangle|^2\Big)^{1/2}$$
$$\geqslant \frac{c}{\sqrt{n}} \Big\|\sum_i \frac{1}{\sqrt{i}}\, e_i^*\Big\|^{-1} \|\mathrm{id}\|^{-1} \sum_{l=1}^{N} \|x_l\|$$
$$= c\sqrt{n}\, \Big\|\sum_i \frac{1}{\sqrt{i}}\, e_i^*\Big\|^{-1} \|\mathrm{id}\|^{-1},$$

where $\mathrm{id} \in L(l_n^2, E)$ is the natural identity. ■

LEMMA 3.4 ([4]). *Let $n-j \geqslant k$ and $1 \leqslant j,\ k \leqslant n$. Then*

$$\binom{n-j}{k}\binom{n}{k}^{-1} \leqslant \Big(1-\frac{k-1}{n}\Big)^{j}.$$

LEMMA 3.5. *For all $s \in N$, $r \in \boldsymbol{R}$ with $2s \leqslant r$ we have*

$$\Big(1-\frac{1}{r}\Big)^{s} \leqslant 1-\frac{1}{10}\frac{s}{r}.$$

LEMMA 3.6. *Let A and B be finite sets with $|A| = a$, $|B| = b$ and $A \cap B = \emptyset$. Suppose $t \in N$.*

(i) *If $2a(t-1) \leqslant a+b$, then we have*

$$\frac{1}{10}\frac{at}{a+b}\, t!\binom{a+b}{t} \leqslant \operatorname{card}\{(x_1, \ldots, x_t) \subset A \cup B \mid \exists i\colon x_i \in A\}.$$

(ii) *If $2a(t-1) \geqslant a+b$, then we have*

$$\tfrac{1}{3}\, t!\binom{a+b}{t} \leqslant \operatorname{card}\{(x_1, \ldots, x_t) \subset A \cup B \mid \exists i\colon x_i \in A\}.$$

Proof.

$$\operatorname{card}\{(x_1, \ldots, x_t) \mid \exists i\colon x_i \in A\} = t!\binom{a+b}{t} - \operatorname{card}\{(x_1, \ldots, x_t) \mid \forall i\colon x_i \in B\}$$
$$= t!\binom{a+b}{t} - t!\binom{b}{t}$$
$$= t!\binom{a+b}{t}\Big(1-\binom{b}{t}\binom{a+b}{t}^{-1}\Big).$$

By Lemma 3.4 we get

$$\geqslant t!\binom{a+b}{t}\Big(1-\Big(1-\frac{t-1}{a+b}\Big)^{a}\Big).$$

Case (i) follows from Lemma 3.5 and case (ii) by using the inequality $1-r \leqslant e^{-r}$. ■

LEMMA 3.7. *Assume that $n \geqslant j_0 \geqslant j_1 \geqslant \ldots \geqslant j_k \geqslant 1$, $j_l \in N$, and $2^{l+3} j_l \leqslant n$, $l = 0, \ldots, k$. Let P_k denote the set of permutations π of the set $\{1, \ldots, n\}$ such that*

(3.1) $$\{\pi(2^k), \ldots, \pi(2^{k+1}-1)\} \cap \{1, \ldots, j_k\} \neq \emptyset$$

and

(3.2) *for all $l = 0, \ldots, k-1$ we have*

$$\{\pi(2^l), \ldots, \pi(2^{l+1}-1)\} \subset \{j_l+1, \ldots, n\}.$$

Then we have

(3.3) $$|P_0| \geqslant \frac{1}{10}\, n!\, \frac{j_0}{n},$$

(3.4) $$|P_k| \geqslant \frac{1}{10}\, n!\, \frac{2^k j_k}{n}\Big(1-\frac{1}{n}\sum_{l=0}^{k-1} 2^{l+1} j_l\Big) \quad \text{if} \quad k \neq 0.$$

Before we start the proof we would like to point out that condition (3.2) makes sense since $2^{l+1} \leqslant n-j_l$. This follows from the inequality $2^{l+3} j_l \leqslant n$.

Proof. We have to consider in how many ways we can permute $\{1, \ldots, n\}$ so that (3.1) and (3.2) hold. We use a counting argument. We start with $l = 0$, i.e. the set $\{1\}$. According to (3.2) we have $n-j_0$ choices. For $l = 1$, i.e. the set $\{2, 3\}$, we have $(n-j_1-1)(n-j_1-2)$ choices. In general, we have for $l = 0, \ldots, k-1$

(3.5) $$\Big(n-j_l-\sum_{i=0}^{l-1} 2^i\Big) \ldots \Big(n-j_l-\sum_{i=0}^{l} 2^i+1\Big) = \prod_{r=2^l-1}^{2^{l+1}-2} (n-j_l-r).$$

Now we consider in how many ways we can insert the set $\{2^k, \ldots, 2^{k+1}-1\}$. There are still $n-2^k+1$ places left. We apply Lemma 3.6 (i) with $a = j_k$, $t = 2^k$, $a+b = n-2^k+1$. Since $2^{k+3} j_k \leqslant n$ the assumptions of Lemma 3.6 (i) are fulfilled. There are at least

(3.6) $$\frac{1}{10} j_k 2^k \frac{1}{n-2^k+1}\, 2^k!\binom{n-2^k+1}{2^k} \quad \text{choices.}$$

We have now $n-2^{k+1}+1$ elements left. There are no restrictions on them. Thus we obtain

(3.7) $$(n-2^{k+1}+1)!$$

choices. Collecting (3.5), (3.6) and (3.7) we get altogether

$$(n-2^{k+1}+1)!\frac{1}{10}j_k 2^k \frac{1}{n-2^k+1} 2^k! \binom{n-2^k+1}{2^k} \prod_{l=0}^{k-1} \prod_{r=2^l-1}^{2^{l+2}-2} (n-j_l-r)$$

$$= \frac{1}{10}j_k 2^k \frac{1}{n-2^k+1}(n-2^k+1)! \prod_{l=0}^{k-1} \prod_{r=2^l-1}^{2^{l+1}-2} (n-r)\left(1-\frac{j_l}{n-r}\right) \quad \text{choices.}$$

Now we apply the elementary inequality $(1-\alpha)(1-\beta) \geqslant 1-\alpha\beta$ if $0 < \alpha, \beta < 1$ to obtain (3.4) ∎

Proof of Lemma 3.3. We choose j_k to be the greatest number such that for $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant 0$ with $\sum_{i=1}^{n} x_i^2 = 1$ we have

$$x_i \geqslant \sqrt{2^k/2n} \quad \text{for all} \quad i \leqslant j_k,\ k = 0, \ldots, r = [\log_2 n]+1. \tag{3.8}$$

Let P_k be as defined in Lemma 3.7. Clearly we have

$$\max_{1 \leqslant i \leqslant n} \frac{x_{\pi(i)}}{\sqrt{i}} \geqslant \frac{1}{\sqrt{2n}} \qquad (\pi \in P_k,\ k = 0, \ldots, r)$$

because of (3.1).

Thus it remains to show that for some universal $c > 0$ we have

$$\Big|\bigcup_{k=0}^{r} P_k\Big| \geqslant cn!. \tag{3.9}$$

We consider two cases. First, suppose there is some $k = N$, with $2^{k+3}j_k \geqslant n$. Then we apply Lemma 3.6 with $t = 2^k$, $a+b = n$, $a = j_k$. We consider the set of all permutations with $\{\pi(2^k), \ldots, \pi(2^{k+1}-1)\} \cap \{1, \ldots, j_k\} \neq \emptyset$.

By Lemma 3.6 we get (3.9).

Now we assume that for all k we have $2^{k+3}j_k \leqslant n$. Thus we may apply Lemma 3.7. Since $P_k \cap P_{k'} = \emptyset$ for $k \neq k'$ we get

$$\Big|\bigcup_{k=0}^{r} P_k\Big| = \sum_{k=0}^{r} |P_k| \tag{3.10}$$

$$\geqslant cn! \sum_{k=0}^{r'} \frac{1}{n} 2^k j_k \left(1 - \frac{1}{n}\sum_{l=0}^{k-1} 2^{l+1} j_l\right) \quad \text{for all } r' \leqslant r.$$

Because of (3.8) and $2^{l+3}j_l \leqslant n$ we get

$$\sum_{k=0}^{r} \frac{1}{n} 2^k j_k \geqslant \sum_{i=1}^{j_0} x_i^2 \geqslant \tfrac{1}{2} \quad \text{and} \quad \frac{1}{n} 2^{l+1} j_l \leqslant \tfrac{1}{4} \quad \text{for all } l = 0, \ldots, r.$$

These two inequalities and (3.10) give (3.9).

4. Positive projection constant and positive distance of symmetric spaces. It is clear from the previous paragraph that the positive projection constant and the positive distance cannot be the same, (up to a constant). The n-dimensional Hilbert spaces are counterexamples.

In this paragraph we prove that both these expressions are, nevertheless, very close to each other. More precisely, they are, up to some logarithmic factor, proportional to

$$\varkappa(E) = \min_{1 \leqslant k \leqslant n} \lambda^*(k)\lambda([n/k]).$$

In fact, we have

THEOREM 4.1. *Let $\{e_i\}_{i=1}^n$ be a 1-symmetric basis of E. Then*

$$c(\log 2\varkappa(E))^{-2}\varkappa(E) \leqslant \gamma_\infty^+(E) \leqslant \min_{\gamma_\infty^+(F)=1} d^+(E, F) \leqslant \varkappa(E).$$

For the proof we require the following simple lemmas.

LEMMA 4.2. *Let π be a permutation of $\{1, \ldots, n\}$ and*

$$\alpha(k, l) = \begin{cases} \dfrac{k \cdot l}{n} & \text{if} \quad k \cdot l \leqslant n, \\ 1 & \text{if} \quad k \cdot l > n. \end{cases}$$

Then there are positive numbers c_1, c_2 such that

$$c_1\alpha(k, l) \leqslant \frac{1}{n!}\sum_{\pi} \max_{1 \leqslant i \leqslant k} \Big\langle e_i, \sum_{j=1}^{l} e_{\pi(j)}^* \Big\rangle \leqslant c_2\alpha(k, l).$$

The left-hand inequality is a consequence of Lemma 3.6.

LEMMA 4.3 ([4]). *Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ be symmetric bases of E and F with $\lambda_E(k) = \lambda_F(k)$, $k = 1, \ldots, n$. Then we have for all $a_i \in \boldsymbol{R}$, $i = 1, \ldots, n$,*

$$\Big\|\sum_{i=1}^{n} a_i e_i\Big\| \leqslant c(\log\min\{\lambda_E(n), \lambda_E^*(n)\})\Big\|\sum_{i=1}^{n} a_i f_i\Big\|,$$

where $c > 0$ is a absolute number.

LEMMA 4.4. $\min_{1 \leqslant k \leqslant n} \lambda(k)\lambda^*([n/k]) \geqslant \tfrac{1}{2}\min\{\sqrt{\lambda(n)}, \sqrt{\lambda^*(n)}\}$.

Proof of Theorem 4.1. The second inequality follows from Lemma 2.2 while the third follows from Lemma 2.4 (ii). Throughout the proof of the first inequality we assume that the extreme points of the dual unit ball are the points $\lambda^*(k)^{-1}\sum_{l=1}^{k} \varepsilon_l e_{\pi(l)}^*$, π a permutation, $\varepsilon_l = \pm 1$, $k = 1, \ldots, n$. Indeed, by Lemma 4.3 and 4.4 we may assume this if we admit a factor $\log 2\varkappa(E)$.

Now we apply Lemma 2.3.

For every $\varepsilon > 0$ there is a sequence x_l, $l = 1, \ldots, N$, with $n = \sum_{l=1}^{N} \|x_l\|$

and

$$\varepsilon+\gamma_\infty^+(E) \geq \max_{\|x^*\|=1} \sum_{l=1}^N \max_{1\le i\le n} |\langle x^*, e_i\rangle \langle x_l, e_i^*\rangle|.$$

We now group the vectors x_l. Put

$$M_k = \{l \mid x_l \text{ attains its norm on a functional with exactly } k \text{ non-zero coordinates}\}.$$

Clearly we may assume that each x_l, $l\in M_k$, has exactly k positive coordinates. Thus we have

$$n = \sum_{l=1}^N \|x_l\| = \sum_{k=1}^n \sum_{l\in M_k} \lambda^*(k)^{-1} \sum_{i=1}^n |\langle x_l, e_i^*\rangle| = \sum_{k=1}^n \sum_{l\in M_k} \lambda^*(k)^{-1}\|x_l\|_1 \tag{4.1}$$

and, since $\{e_i\}_{i=1}^n$ is 1-symmetric,

$$\varepsilon+\gamma_\infty^+(E) \geq \max_{\|x^*\|=1} \frac{1}{n!}\sum_\pi \sum_{l=1}^N \max_{1\le i\le n} \left|\langle x^*, e_i\rangle \left\langle \sum_{m=1}^n x_l(m)e_{\pi(m)}, e_i^*\right\rangle\right|.$$

Because of our assumption on the extreme points of the dual unit ball we get

$$\varepsilon+\gamma_\infty^+(E) \geq \max_{1\le j\le n} \lambda^*(j)^{-1}\frac{1}{n!}\sum_\pi \sum_{l=1}^N \max_{1\le i\le n} \left|\left\langle \sum_{r=1}^j e_r^*, e_i\right\rangle \left\langle \sum_{m=1}^n x_l(m)e_{\pi(m)}, e_i^*\right\rangle\right|$$

$$= \max_{1\le j\le n} \lambda^*(j)^{-1}\frac{1}{n!}\sum_\pi \sum_{l=1}^N \max_{1\le i\le j} \left|\left\langle \sum_{m=1}^j x_l(m)e_{\pi(m)}, e_i^*\right\rangle\right|.$$

For every x_l, $l\in M_k$, there are exactly k non-zero, positive coordinates $\{i_1, \dots, i_k\}$. There is a set $G = G(x_l)$ of permutations such that for $\pi_1, \pi_2 \in G$ we have $\{\pi_1(i_1), \dots, \pi_1(i_k)\} \neq \{\pi_2(i_1), \dots, \pi_2(i_k)\}$. Of course $|G| = \binom{n}{k}$. For every $\pi\in G$ there is a set D_π with $|D_\pi| = k!(n-k)!$ which leave $\{\pi(i_1), \dots, \pi(i_k)\}$ invariant

$$\varepsilon+\gamma_\infty^+(E) \geq \max_{1\le j\le n} \lambda^*(j)^{-1} \sum_{k=1}^n \sum_{l\in M_k} \binom{n}{k}^{-1} \times$$

$$\times \sum_{\pi\in G} \max_{1\le i\le j} \left|\left\langle (k!(n-k)!)^{-1} \sum_{\eta\in D_\pi} \sum_{s=1}^k x_l(i_s) e_{\eta\pi(i_s)}, e_i^*\right\rangle\right|$$

$$= \max_{1\le j\le n} \lambda^*(j)^{-1} \sum_{k=1}^n \sum_{l\in M_k} \binom{n}{k}^{-1} \sum_{\pi\in G} \max_{1\le i\le j} \frac{1}{k}\|x_l\|_1 \left|\left\langle \sum_{s=1}^k e_{\pi(i_s)}, e_i^*\right\rangle\right|.$$

By Lemma 4.2 we get

$$\gamma_\infty^+(E) \geq c \max_{1\le j\le n} \lambda^*(j)^{-1} \sum_{k=1}^n \sum_{l\in M_k} \alpha(j,k)\frac{\|x_l\|_1}{k}.$$

Putting

$$b_k = \frac{1}{n}\sum_{l\in M_k} \lambda^*(k)^{-1}\|x_l\|_1 \tag{4.2}$$

we get

$$\gamma_\infty^+(E) \geq c \max_{1\le j\le n} \lambda^*(j)^{-1} n \sum_{k=1}^n b_k \lambda(k)^{-1}\alpha(j,k).$$

Because of (4.1), (4.2) this means that

$$\gamma_\infty^+(E) \geq c \inf \max_{1\le j\le n} \sum_{k\le \frac{n}{j}} \lambda(j)\lambda^*(k) b_k + \sum_{k>\frac{n}{j}} \frac{n}{kj}\lambda(j)\lambda^*(k) b_k, \tag{4.3}$$

where the infimum is taken over all $\{b_k\}_{k=1}^n$ with $\sum_{k=1}^n b_k = 1$. Now we consider two cases. Firstly $\lambda^*(n) \leq \lambda(n)$. Without loss of generality we may assume that there are natural numbers $k_0, k_1, \dots, k_r$ with $r \leq \log\lambda^*(n)$, $k_0 = 0$, $k_r = n$ and $\lambda^*(k_j) = 2^j$, $j = 1, \dots, r$. By (4.3) we get for some constant $c > 0$ and a sequence $b_1, \dots, b_n$,

$$c\gamma_\infty^+(E) \geq \max_{1\le j\le n} \sum_{k_l\le\frac{n}{j}} \sum_{k_{l-1}<k\le k_l} b_k\lambda^*(k)\lambda(j) + \sum_{k_l>\frac{n}{j}} \sum_{k_{l-1}<k\le k_l} b_k \frac{n}{kj}\lambda^*(k)\lambda^*(j).$$

For at least one l we have

$$2\sum_{k_{l-1}<k\le k_l} b_k \geq (\log 2\lambda^*(n))^{-1}$$

since $r \leq \log\lambda^*(n)$ and $\sum_{k=1}^n b_k = 1$. Therefore

$$c\gamma_\infty^+(E) \geq (\log 2\lambda^*(n))^{-1} \max_{\frac{n}{k}\le j\le n} \lambda(j)\lambda^*(k)\frac{n}{kj}.$$

Since $\lambda(j)/j$ is decreasing

$$c\gamma_\infty^+(E) \geq (\log 2\lambda^*(n))^{-1}\lambda^*(k)\lambda([n/k]).$$

The case $\lambda(n) \leq \lambda^*(n)$ is treated in the same way. ■

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Received August 30, 1982 (1796)

STUDIA MATHEMATICA, T. LXXVIII. (1984)

A counterexample for the strong maximal operator

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Abstract. The following conjecture is stated:

Let $f \in L(\log^+ L)^{n-1}(\boldsymbol{R}^n)$; then f^* is integrable over every set of finite measure if and only if $f \in L(\log^+ L)^n(\boldsymbol{R}^n)$. ($f^*$ denotes the strong maximal function.)

We give a counterexample in $\boldsymbol{R}^2$.

Introduction and statement of results. For a function $f \in L_{\text{loc}}(\boldsymbol{R}^n)$, Hardy and Littlewood defined the maximal function Mf at each $x \in \boldsymbol{R}^n$ by

$$Mf(x) = \sup_{r>0} \frac{1}{|Q(x, r)|} \int_{Q(x,r)} |f(y)|\, dy,$$

where $Q(x, r)$ stands for the open cubic interval of center x and side length $2r$.

A covering theorem due to Vitali leads immediately to the weak type (1,1) of the maximal operator, i.e.

$$\left|\frac{x}{Mf(x)} > \lambda\right| \leqslant \frac{C}{\lambda} \|f\|_1; \tag{1}$$

C a constant independent of f and λ.

Using Whitney's covering theorem, the converse inequality can be proved, see [4], p. 57. Let $f \in L_1(\boldsymbol{R}^n)$; then

$$\frac{\tilde{C}}{\lambda} \int_{(Mf>\lambda)} |f| \leqslant |x/Mf(x) > \lambda|. \tag{2}$$

The strong maximal operator $f \to f^*$ is defined by

$$f^*(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(y)|\, dy,$$

where the supremum is taken over the set of all intervals I (cells with sides parallel to the axes) containing the point x.

In 1971, N. A. Fava proved the following (see [1])

$$|x/f^*(x) > 4\lambda| \leqslant C \int \frac{|f|}{\lambda} \left(\log^+ \frac{|f|}{\lambda}\right)^{n-1} dy. \tag{3}$$