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A counterexample for the strong maximal operator

by

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Abstract. The following conjecture is stated:

Let $f \in L(\log^+ L)^{n-1}(\boldsymbol{R}^n)$; then f^* is integrable over every set of finite measure if and only if $f \in L(\log^+ L)^n(\boldsymbol{R}^n)$. ($f^*$ denotes the strong maximal function.)

We give a counterexample in $\boldsymbol{R}^2$.

Introduction and statement of results. For a function $f \in L_{\text{loc}}(\boldsymbol{R}^n)$, Hardy and Littlewood defined the maximal function Mf at each $x \in \boldsymbol{R}^n$ by

$$Mf(x) = \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|\,dy,$$

where $Q(x,r)$ stands for the open cubic interval of center x and side length $2r$.

A covering theorem due to Vitali leads immediately to the weak type (1,1) of the maximal operator, i.e.

$$\left|\frac{x}{Mf(x)} > \lambda\right| \leqslant \frac{C}{\lambda} \|f\|_1; \tag{1}$$

C a constant independent of f and λ.

Using Whitney's covering theorem, the converse inequality can be proved, see [4], p. 57. Let $f \in L_1(\boldsymbol{R}^n)$; then

$$\frac{\tilde{C}}{\lambda} \int_{(Mf>\lambda)} |f| \leqslant |x/Mf(x) > \lambda|. \tag{2}$$

The strong maximal operator $f \to f^*$ is defined by

$$f^*(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(y)|\,dy,$$

where the supremum is taken over the set of all intervals I (cells with sides parallel to the axes) containing the point x.

In 1971, N. A. Fava proved the following (see [1])

$$|x/f^*(x) > 4\lambda| \leqslant C \int \frac{|f|}{\lambda} \left(\log^+ \frac{|f|}{\lambda}\right)^{n-1} dy. \tag{3}$$

Now, for the Hardy–Littlewood maximal operator, inequalities (1) and (2) lead to the theorem:

Let $f \in L^1(\boldsymbol{R}^n)$. The following conditions are equivalent:

(i) $\int_{(Mf>1)} Mf < +\infty$;

(ii) $f \in L\log^+ L$.

De Guzmán proposed the following problem (see [4], p. 64): Are these conditions equivalent?

Let $f = L^1(\boldsymbol{R}^n)$. Then

(i) $\int_{(f^*>1)} f^* < +\infty$;

(ii) $f \in L(\log^+ L)^2 (R^2)$.

Using inequality (3), it is easy to prove that if $f \in L(\log^+ L)^2$, then $\int_{(f^*>1)} f^* < +\infty$.

The purpose of this paper is to show that the conditions are not equivalent, more explicitly, that there exists a function f such that f does not belong to $L(\log^+ L)^2$ and such that

$$\int_{(f^*>1)} f^* < +\infty.$$

Given the function $g: \boldsymbol{R}^2 \to \boldsymbol{R}$ defined by

$$g(x, y) = \frac{\chi_B(x, y)}{(y-\frac{1}{4})\,|\log(y-\frac{1}{4})|^3},$$

where $B = \{(x, y)/\mathrm{Max}(|x-\frac{1}{2}|;\ |y-\frac{1}{2}|) \leqslant \frac{1}{4}\}$.

Clearly, $g \in L(\log^+ L)$ and $g \notin L(\log^+ L)^2$. We have proved in [3] that g^* is locally integrable but since $V_{1/4}^{3/4}[g(\cdot, y)] = 0$ for all $y \in \boldsymbol{R}$, where $V_{1/4}^{3/4}[g(\cdot, y)]$ denotes the variation of $g(\cdot, y)$ in $[\frac{1}{4}, \frac{3}{4}]$, another result in [3] proves that g^* is not integrable over the set $\{(x, y)/g^*(x, y) > 1\}$; hence, $\{(x, y)/g^*(x, y) > 1\}$ is not bounded and inequality (3) yields that g^* is not integrable over every set of finite measure.

Now, given the rotation in an angle $\frac{1}{4}\pi$ centered at $(\frac{1}{2}, \frac{1}{2})$,

$$R(x, y) = \left(\tfrac{1}{2}\sqrt{2}(x+y)+\tfrac{1}{2},\ \tfrac{1}{2}\sqrt{2}(y-x)+\tfrac{1}{2}\right),$$

we prove that $(g \circ \boldsymbol{R})^*$ is integrable over every set of finite measure and that the set $\{(x, y)/(g \circ \boldsymbol{R})^* > 1\}$ is bounded. Comparing these results with the two results we have stated before about g^*, the geometric nature of the strong maximal operator can immediately be observed.

Given $f \in L^1(\boldsymbol{R}^n)$, the following operators can be defined:

$$M_i f(x) = \sup_{a<x_i<b} \frac{1}{b-a} \int_a^b |f(x_1, \ldots, x_{i-1}, u, x_{i+1}, x_n|\, du \qquad (i = 1, 2, \ldots, n).$$

The operation $M_n \ldots M_1 f$ is well defined for any function f in $L(\log^+ L)^{n-1}$ (see [1]). Fava, Gatto and Gutiérrez proved the following: Let $f \in L(\log^+ L)^{n-1}$. Then $M_n \ldots M_1 f$ is integrable over every set of finite measure if and only if $f \in L(\log^+ L)^n$ (see [2]).

This work shows then the difference in $\boldsymbol{R}^2$ between the strong maximal function f^* and $M_2 M_1 f$.

Proof of the results.

THEOREM. *There exists a function $f \in L\log^+ L$ such that $f \notin L(\log^+ L)^2$ and such that f^* is integrable over the set $\{(x, y)/f^*(x, y) > 1\}$.*

Proof. Let $B = \{(x, y)\colon \mathrm{Max}(|x-\frac{1}{2}|;\ |y-\frac{1}{2}|) \leqslant \frac{1}{4}\}$.

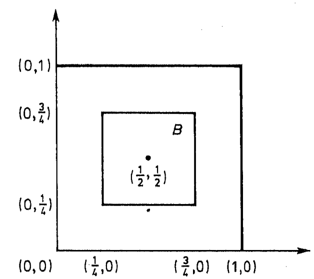

Fig. 1

We consider the rotation in an angle $\pi/4$ centered at $(\frac{1}{2}, \frac{1}{2})$,

$$R(x, y) = \left(\frac{\sqrt{2}(x+y)+1}{2},\ \frac{\sqrt{2}(y-x)+1}{2}\right).$$

Let

$$g(x, y) = \frac{1}{(y-\frac{1}{4})}\ \frac{\chi_B(x, y)}{|\log(y-\frac{1}{4})|^3};$$

and let $f\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ be defined by

$$f(x, y) = g\circ R(x, y) = \frac{\chi_B\left(\frac{\sqrt{2}(x+y)+1}{2}, \frac{\sqrt{2}(y-x)+1}{2}\right)}{\frac{2\sqrt{2}(y-x)+1}{4}\left|\operatorname{Log}\frac{2\sqrt{2}(y-x)+1}{4}\right|^3}$$

$$= \frac{\chi_{R^{-1}(B)}(x, y)}{\frac{2\sqrt{2}(y-x)+1}{4}\left|\operatorname{Log}\frac{2\sqrt{2}(y-x)+1}{4}\right|^3}.$$

The function f is infinite in

$$C = \{(x, y)/\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2}+1/\sqrt{8};\ y = x-1/\sqrt{8}\}$$

and decreasing through the lines perpendicular to C included in $R^{-1}(B)$.

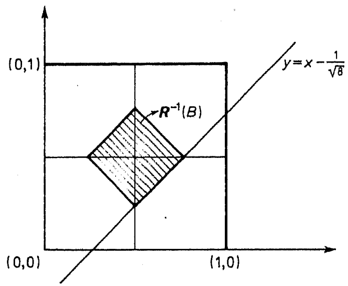

Fig. 2

We prove that $\{(x, y)/f^*(x, y) > 1\}$ is bounded. To show this, it will be sufficient to calculate the averages over the intervals I with larger side parallel to the horizontal axis and $I \subseteq Q_1$. See Fig. 3, where

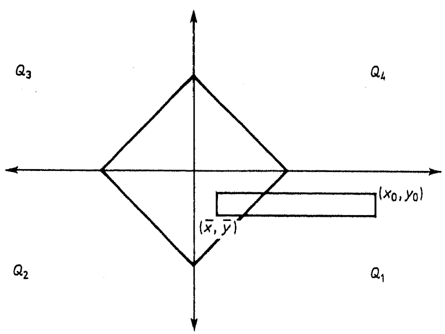

Fig. 3

$$(\bar{x}, \bar{y}), (\bar{x}, y_0) \in \left\{(x, y)/\frac{1}{2} \leqslant x \leqslant y+\frac{1}{\sqrt{8}};\ \frac{1}{2}-\frac{1}{\sqrt{8}} \leqslant y \leqslant \frac{1}{2}\right\}$$

and

$$Q_1 = \{(x, y)/x \geqslant \tfrac{1}{2};\ y \leqslant \tfrac{1}{2}\},\qquad Q_2 = \{(x, y)/x \leqslant \tfrac{1}{2};\ y \leqslant \tfrac{1}{2}\},$$
$$Q_3 = \{(x, y)/x \leqslant \tfrac{1}{2};\ y \geqslant \tfrac{1}{2}\},\qquad Q_4 = \{(x, y)/x \geqslant \tfrac{1}{2};\ y \geqslant \tfrac{1}{2}\}.$$

Consider the statement:

Given an average over an interval I which is not the one indicated in Fig. 3, more explicitly, $I \nsubseteq Q_1$ or larger side of I is not parallel to the horizontal axis as shown in Fig. 4.

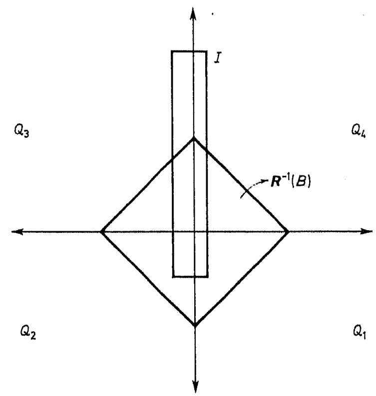

Fig. 4

We consider the interval $\tilde{I}$ which satisfies the following conditions:

(i) $|\tilde{I}| = |I|$.

(ii) Let

$$e(I) = \frac{\text{length of larger side of } I}{\text{length of smaller side of } I};$$

then $e(I) = e(\tilde{I})$.

(iii) $h_i(I\cap R^{-1}(B)\cap Q_i) = \tilde{I}\cap R^{-1}(B)\cap Q_{\Pi(i)}$ $(i = 1, 2, 3, 4)$ with $\Pi\colon \Pi_4 \to \Pi_4$ a permutation, $\Pi_4 = \{1, 2, 3, 4\}$ and where the functions h_i are compositions of S_x (a symmetry with respect to the line $y = \frac{1}{2}$), S_y (a symmetry with respect to the line $x = \frac{1}{2}$), S (a symmetry with respect to the

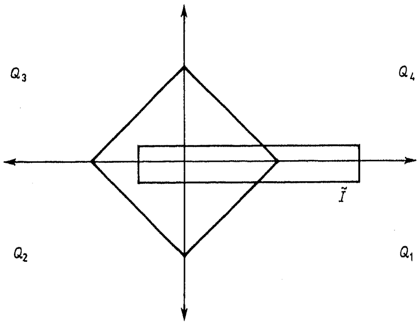

Fig. 5

line $y=-x+1$). Any of the symmetries may not be used or can be repeated.

(iv) $\operatorname{Max}_{i\in\Pi_4}|Q_i\cap I\cap R^{-1}(B)| = |Q_1\cap\tilde{I}\cap R^{-1}(B)|$.

(v) Larger side of $\tilde{I}$ is parallel to the horizontal axis.

There is only one interval $\tilde{I}$ satisfying properties (i)–(v).

Let us assume that $\int_{I\cap Q_i\cap R^{-1}(B)} f \leqslant \int_{I\cap Q_1\cap R^{-1}(B)} f$ $(i=1,2,3,4)$ which will be demonstrated afterwards. Then

$$\frac{1}{|I|}\int_{I\cap R^{-1}(B)} f = \frac{1}{|\tilde{I}|}\int_{I\cap R^{-1}(B)} f = \frac{1}{|\tilde{I}|}\sum_{i=1}^{4}\int_{I\cap Q_i\cap R^{-1}(B)} f$$
$$\leqslant \frac{4}{|\tilde{I}|}\int_{\tilde{I}\cap Q_1\cap R^{-1}(B)} f \leqslant \frac{4}{|\tilde{I}\cap Q_1|}\int_{\tilde{I}\cap Q_1\cap R^{-1}(B)} f.$$

(a) If $(\bar{x},\bar{y})\in Q_3\cap I\cap R^{-1}(B)$ (see Figs 4, 5 and 2), then $f(\bar{x},\bar{y}) \leqslant f(S_x(\bar{x},\bar{y})) \leqslant f(S_y\cdot S_x(\bar{x},\bar{y})) = f(SS_y S_x(\bar{x},\bar{y}))$, since when $a\in Q_1\cap R^{-1}(B)$, $f(a)=f(Sa)$; moreover,

$$SS_y S_x(Q_3\cap I\cap R^{-1}(B)) \subseteq Q_1\cap\tilde{I}\cap R^{-1}(B)$$

because of properties (iii), (iv).

(b) Similarly, if $(\bar{x},\bar{y})\in Q_4\cap I\cap R^{-1}(B)$, then $f(\bar{x},\bar{y}) \leqslant f(S_x(\bar{x},\bar{y})) = f(SS_x(\bar{x},\bar{y}))$, and $SS_x(Q_4\cap I\cap R^{-1}(B)) \subseteq Q_1\cap\tilde{I}\cap R^{-1}(B)$ by properties (iii), (iv).

(We use the symmetry S when the larger side of $I\cap Q_i$ is parallel to the y-axis as is Fig. 4.)

(c) If $(\bar{x},\bar{y})\in Q_2\cap I\cap R^{-1}(B)$, then $f(\bar{x},\bar{y}) \leqslant f(S_y(\bar{x},\bar{y})) = f(SS_y(\bar{x},\bar{y}))$ and $SS_y(Q_2\cap I\cap R^{-1}(B)) \subseteq Q_1\cap\tilde{I}\cap R^{-1}(B)$.

(d) If $(\bar{x},\bar{y})\in Q_1\cap I\cap R^{-1}(B)$, then $f(\bar{x},\bar{y}) = f(S(\bar{x},\bar{y}))$ and $S(Q_1\cap I\cap R^{-1}(B)) \subseteq Q_1\cap\tilde{I}\cap R^{-1}(B)$.

Using (a), we have

$$\int_{Q_3\cap I\cap R^{-1}(B)} f(\bar{x},\bar{y}) \leqslant \int_{Q_3\cap I\cap R^{-1}(B)} f(SS_y S_x(\bar{x},\bar{y})) \leqslant \int_{Q_1\cap\tilde{I}\cap R^{-1}(B)} f(\bar{x},\bar{y});$$

similarly for (b), (c) and (d).

We consider now an average over an interval $I\subseteq Q_1$ with larger side parallel to the horizontal axis. We have

$$\frac{1}{|I|}\int_{I\cap R^{-1}(B)} \frac{1}{\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\left|\operatorname{Log}\frac{2\sqrt{2}(y-x)+1}{4}\right|^3}\,dx\,dy$$
$$= \frac{1}{|I|}\int_{\bar{y}}^{y_0} dy \int_{\bar{x}}^{y+1/\sqrt{8}} \frac{1}{\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\left|\operatorname{Log}\frac{2\sqrt{2}(y-x)+1}{4}\right|^3}\,dx$$

(see Fig. 3).

Since

$$\int_{\beta}^{\alpha} \frac{dx}{(ax+b)\,|\operatorname{Log}(ax+b)|^3} = \frac{1}{2a\,[\operatorname{Log}(ax+b)]^2}\Big|_{\beta}^{\alpha}, \qquad 0<\beta<\alpha<\frac{1-b}{a},$$

we have

$$\frac{1}{|I|}\int_{\bar{y}}^{y_0} dy \int_{\bar{x}}^{y+1/\sqrt{8}} \frac{dx}{\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\left|\operatorname{Log}\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\right|^3}$$
$$= \frac{1}{|I|}\int_{\bar{y}}^{y_0} \left|\frac{-1}{\sqrt{2}}\,\frac{1}{\left[\log\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\right]^2}\right|_{\bar{x}}^{y+1/\sqrt{8}} dy$$
$$= \frac{1}{(x_0-\bar{x})}\,\frac{1}{(y_0-\bar{y})}\,\frac{1}{\sqrt{2}}\int_{\bar{y}}^{y_0} \frac{dy}{\left[\log\left(\frac{2\sqrt{2}(y-\bar{x})+1}{4}\right)\right]^2}; \qquad |I| = (x_0-\bar{x})(y_0-\bar{y})$$

(see Fig. 3) since $(\bar{x},\bar{y}),(\bar{x},y_0)\in R^{-1}(B)$.

If $\bar{y} \leqslant y \leqslant y_0$, then $\operatorname{Max}(|\bar{x}+y|;\ |\bar{x}-y|) \leqslant 1/2\sqrt{2}$.

Consequently,

$$\frac{\sqrt{2}}{2}(y-\bar{x})+\tfrac{1}{4} \leqslant \tfrac{1}{2} < 1,$$

so that

$$h(y) = \frac{1}{[\log(\frac{1}{2}\sqrt{2}(y-\bar{x})+\frac{1}{4})]^2}$$

is increasing and bounded in $[\bar{y}, y_0]$; then

$$\sup_{\bar{y}\leqslant y\leqslant y_0} \frac{1}{[\log(\frac{1}{2}\sqrt{2}(y-\bar{x})+\frac{1}{4})]^2} = \frac{1}{[\log(\frac{1}{2}\sqrt{2}(y_0-\bar{x})+\frac{1}{4})]^2} \leqslant \frac{1}{\left[\log\left(\frac{\sqrt{2}}{2}\frac{1}{\sqrt{8}}+\frac{1}{4}\right)\right]^2} = \frac{1}{(\log 2)^2};$$

hence

$$\frac{1}{\sqrt{2}}\frac{1}{(x_0-\bar{x})}\frac{1}{(y_0-\bar{y})}\int_{\bar{y}}^{y_0} h(y)\,dy \leqslant \frac{1}{\sqrt{2}}\frac{1}{(\log 2)^2}\frac{1}{(x_0-\bar{x})} < 1$$

if

$$x_0-\bar{x} > \frac{1}{\sqrt{2}(\log 2)^2} \sim 1.$$

Consequently, $\{f^* > 1\}$ is bounded.

$f \in L\log^+ L$ and $f \notin L(\log^+ L)^2$ since $f = g\circ R$ with

$$g(x, y) = \frac{1}{(y-\frac{1}{4})}\frac{\chi_B(x, y)}{|\log(y-\frac{1}{4})|^3} \quad \text{and} \quad g \in L\log^+ L; \quad g \notin L(\log^+ L)^2.$$

Let us prove that

$$\int_{R^{-1}(B)} f^* < +\infty.$$

If $a \in Q_3 \cap R^{-1}(B)$, it is already known that

(1) $$f^*(a) \leqslant 4 \sup_{I\subseteq Q_1} \frac{1}{|I|}\int_{I\cap R^{-1}(B)} f,$$

where $S_y S_x a \in I \cap R^{-1}(B) \subseteq Q_1 \cap R^{-1}(B)$; similarly, if $a \in Q_i \cap R^{-1}(B)$, $i = 1, 2, 3, 4$.

Having in mind (1), we see that for all $a \in Q_1 \cap R^{-1}(B)$ the inequality

(2) $$4 \sup_{a\in I\subseteq Q_1} \frac{1}{|I|}\int_{I\cap R^{-1}(B)} \leqslant \frac{32}{(\log 2)^2}\cdot f_R^*(a)$$

is verified, where f_R^* is the maximal function over intervals rotated in the angle $\frac{1}{4}\pi$ (in the counterclockwise sense).

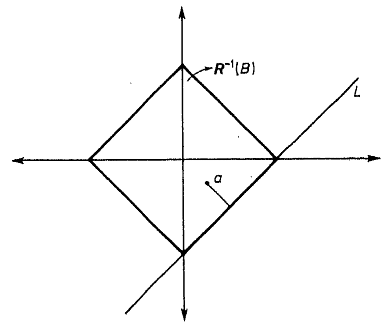

Fig. 6

Given $a \in R^{-1}(B)\cap Q_1$, we have $f_R^*(a) = g^*(Ra)$. If $Ra = (\alpha, \beta)$, then

$$g^*(Ra) = \frac{-1}{(\beta-\frac{1}{4})}\int_{1/4}^{\beta} \frac{dy}{(y-\frac{1}{4})[\log(y-\frac{1}{4})]^3} = \frac{1}{2(\beta-\frac{1}{4})}\frac{1}{[\log(\beta-\frac{1}{4})]^2} = \frac{1}{2d(a, L)}\frac{1}{(\log d(a, L))^2},$$

where L is the straight line $\{(x, y)/y = x-1/\sqrt{8}\}$ and

$$d(a, L) = d(Ra, RL) = \beta-\tfrac{1}{4} = (\tfrac{1}{2}\sqrt{2}(y-x)+\tfrac{1}{2})-\tfrac{1}{4} = \tfrac{1}{2}\sqrt{2}(y-x)+\tfrac{1}{4}$$

where $a = (x, y)$ and

$$RL = \{(x, y)/y = \tfrac{1}{4}\}.$$

Then

(3) $$f_R^*((x, y)) = \frac{2}{(2\sqrt{2}(y-x)+1)}\frac{1}{\left[\log\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\right]^2}.$$

Now, given an interval $I \subseteq Q_1$ with larger side parallel to the x-axis, we have

$$\frac{1}{|I|}\int\limits_{I\cap R^{-1}(B)} f$$

$$=\frac{1}{|I|}\int_{\bar y}^{\tilde y} dy \int_{x}^{y+1/\sqrt{8}} \frac{dx}{\left|\log\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\right|^3}\,\frac{1}{\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)}+$$

$$+\frac{1}{|I|}\int_{\tilde y}^{y_0} dy \int_{\bar x}^{x_0} \frac{dx}{\left|\log\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\right|^3}\,\frac{1}{\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)} \quad \text{(see Fig. 7)}$$

$$=\frac{1}{|I|}\int_{\bar y}^{\tilde y} \frac{-1}{\sqrt{2}}\,\frac{1}{\left[\log\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\right]^2}\Bigg|_{\bar x}^{y+1/\sqrt{8}} dy+$$

$$+\frac{1}{|I|}\int_{\tilde y}^{y_0} \frac{-1}{\sqrt{2}}\,\frac{1}{\left[\log\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\right]^2}\Bigg|_{\bar x}^{x_0} dy$$

$$\leqslant \frac{(\tilde y-\bar y)}{\sqrt{2}(x_0-\bar x)(y_0-\bar y)}\,\frac{1}{\left[\log\left(\frac{2\sqrt{2}(\tilde y-\bar x)+1}{4}\right)\right]^2}+$$

$$+\frac{1}{\sqrt{2}|I|}\int_{\tilde y}^{y_0} \frac{dy}{\left[\log\left(\frac{2\sqrt{2}(y-\bar x)+1}{4}\right)\right]^2}$$

$$\leqslant \frac{1}{\sqrt{2}(x_0-\bar x)}\left[\frac{1}{[\log(\frac{1}{2}\sqrt{2}(x_0-\bar x))]^2}+\frac{1}{\left[\log\left(\frac{2\sqrt{2}(y_0-\bar x)+1}{4}\right)\right]^2}\right];$$

since $\tilde y = x_0-1/\sqrt{8}$ (see Fig. 7).

$$(4)\qquad =\frac{1}{\sqrt{2}}\,\frac{1}{(x_0-\bar x)}\,\frac{1}{[\log(\frac{1}{2}\sqrt{2}(x_0-\bar x))]^2}+\frac{1}{[\log(\frac{1}{2}\sqrt{2}(x_0-\bar x+\delta))]^2},$$

since $y_0 = x_0-1/\sqrt{8}+\delta$ $(0\leqslant\delta\leqslant 1/\sqrt{8})$, $\delta = y_0-\tilde y$.

Now, $0\leqslant\delta\leqslant y_0-\bar y\leqslant x_0-\bar x$ since averages are taken over intervals with larger sides parallel to the x-axis. Then

$$\frac{1}{[\log(\frac{1}{2}\sqrt{2}(x_0-\bar x+\delta))]^2}\leqslant\frac{1}{(\log(\sqrt{2}(x_0-\bar x)))^2}.$$

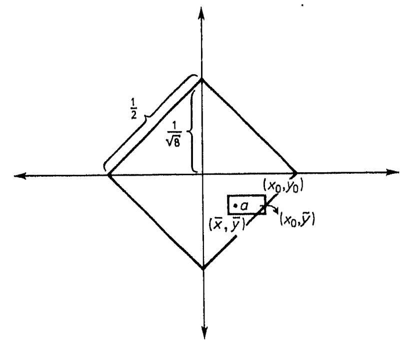

Fig. 7

Using (4), we have only to prove that

$$(5)\qquad \frac{2}{\sqrt{2}(x_0-\bar x)}\,\frac{1}{[\log(\sqrt{2}(x_0-\bar x))]^2}\leqslant C f_R^*(a)$$

$$=\frac{2C}{(2\sqrt{2}(y-x)+1)\left[\log\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\right]^2}$$

or equivalently,

$$(6)\qquad \left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\left[\log\left(\frac{2\sqrt{2}(y-x)+1}{4}\right)\right]^2$$

$$\leqslant \tfrac{1}{4}C(\sqrt{2}(x_0-\bar x))[\log(\sqrt{2}(x_0-\bar x))]^2$$

for a constant C not depending on $a=(x,y)$.

The function $x(\log x)^2$ is increasing in $[0, 1/e^2]$ and decreasing in $[1/e^2, 1]$ with maximum $(2/e)^2<1$. On the other hand, we have $\frac{1}{2}\sqrt{2}(y-x)+$ $+\frac{1}{4}\leqslant\sqrt{2}(x_0-\bar x)\leqslant\sqrt{2}\,\frac{1}{2\sqrt{2}}=\frac{1}{2}$ since $x_0-\bar x\geqslant y_0-\bar y$, so that $(x_0-\bar x)+\tilde y-y$ $\geqslant y_0-y\geqslant 0\geqslant\bar x-x$, and $y+\bar x-x\leqslant(x_0-\bar x)+\tilde y=(x_0-\bar x)+x_0-1/\sqrt{8}$. Hence $y-x+1/\sqrt{8}\leqslant 2(x_0-\bar x)$.

Consequently,

$$\tfrac{1}{4} C\, \tfrac{1}{2} (\log \tfrac{1}{2})^2 = 1 > (2/e)^2$$

is sufficient. Then $C = 8/(\log 2)^2 > 1$ and we obtain (2). Using (1) and (2), we get

$$\int_{R^{-1}(B)} f^* = \sum_{i=1}^{4} \int_{Q_i \cap R^{-1}(B)} f^* \leqslant \frac{128}{(\log 2)^2} \int_{Q_1 \cap R^{-1}(B)} f_R^* < +\infty;$$

since, clearly,

$$\int_{R^{-1}(B)} f_R^* = \int_{R^{-1}(B)} g^* R = \int_B g^* < +\infty.$$

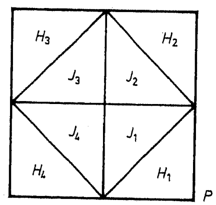

Fig. 8

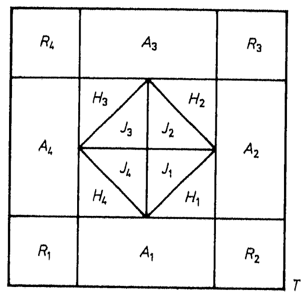

Fig. 9

Now, let $R^{-1}(B) = \bigcup_{i=1}^{4} J_i$ and let $S_i\colon H_i \to J_i$ be the corresponding symmetries. Obviously, if $a \in H_i$, then $f^*(a) \leqslant f^*(S_i a)$ $(i = 1, 2, 3, 4)$. Thus

$$\int_P f^* < +\infty, \quad \text{where} \quad P = R^{-1}(B) \cup \Big(\bigcup_{i=1}^{4} H_i\Big).$$

Let

$$S_1 = H_1 \cup J_1 \cup J_4 \cup H_4, \quad S_2 = H_2 \cup J_2 \cup J_1 \cup H_1,$$
$$S_3 = H_3 \cup J_3 \cup J_2 \cup H_2, \quad S_4 = H_4 \cup J_4 \cup J_3 \cup H_3,$$

and let $S_i\colon A_i \to S_i$ be the corresponding symmetries. Clearly,

$$f^*(a) \leqslant f^*(S_i a) \quad \text{if} \quad a \in A_i \quad (i = 1, 2, 3, 4).$$

Similar relations hold for R_i and the cubes $(H_i \cup J_i)$ $(i = 1, 2, 3, 4)$.

Then if $T = \Big(\bigcup_{i=1}^{4} A_i\Big) \cup \Big(\bigcup_{i=1}^{4} R_i\Big) \cup P$, we have $\int_T f^* < +\infty$. With a recursive proceeding, we obtain $\int_H f^* < +\infty$ for every bounded set H.

In particular, we have that $\int_{(f^*>1)} f^* < +\infty$.

Corollary. *There exists $f \notin L(\log^+ L)^2$ such that f^* is integrable over every set of finite measure.*

Let

$$f = g \circ R, \quad \text{where} \quad g(x, y) = \frac{\chi_B(x, y)}{(y - \frac{1}{4})\,|\log (y - \frac{1}{4})|^3}.$$

If $a = (a_1, a_2)$ is such that $\operatorname{Max}(|a_1 - \frac{1}{2}|;\, |a_2 - \frac{1}{2}|) \geqslant M$ with M big enough, we know that $f^*(a) \leqslant 1$. Then if $H = \{(a_1, a_2)/\operatorname{Max}(|a_1 - \frac{1}{2}|;\, |a_2 - \frac{1}{2}|) \leqslant M\}$ and

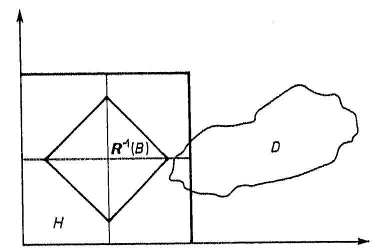

Fig. 10

D is a set of finite measure we have

$$\int_{H^c \cap D} f^*\, dx\, dy \leqslant |H^c \cap D| \leqslant |D|,$$

and since

$$\int_{H \cap D} f^*\, dx\, dy \leqslant \int_H f^*\, dx\, dy < +\infty,$$

we conclude that $\int_D f^* < +\infty$.

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The generalization of Cellina's Fixed Point Theorem

by

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Abstract. Let $L^1(T, Z)$ be the Banach space of integrable functions from a compact space T into a Banach space Z. A set $K \subset L^1(T, Z)$ is called *decomposable* if, for every $u, v \in K$ and measurable $A \subset T$, $u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K$. In this note we prove that each compact mapping from a closed and decomposable subset $K \subset L^1(T, Z)$ into itself has a fixed point.

§1. **Introduction.** In paper [2] Cellina proved that the set K_P of all functions integrable on a closed interval $[a, b]$ whose values belong to a fixed closed subset P of a Euclidean space R^m has a fixed point property; this means that each compact mapping from K_P into itself has a fixed point. The set K_P can be nonconvex; thus the result of Cellina is interesting when confronted with Schauder's Fixed Point Theorem, where the assumption of convexity is essential (see [3], [8]).

In this note we generalize the above result to an arbitrary closed and decomposable subset K of the space of integrable functions. The decomposability of a set K means that for each $u, v \in K$ and A measurable $u \cdot \chi_A + v \cdot \chi_{[a,b] \setminus A} \in K$, where χ_A stands for the characteristic function of A.

Obviously, the set K_P in the theorem of Cellina is decomposable.

This generalization is quite easy to obtain if we apply a certain theorem on continuous selections proved by the author in [5]. The theorem is an abstract version of Antosiewicz and Cellina's Selection Theorem [1] and can also be applied to the problem of the existence of solutions for the functional-differential inclusion $\dot{x}(t) \in F(t, x(\cdot))$ (see [6]). The required facts about the selections are given in §3. We formulate the main results in §2 and prove it in §4.

§2. **The main result.** Let T be a compact topological space with a σ-field $\mathfrak{M}$ of measurable subsets of T given by a nonnegative, regular Borel measure dt and let Z be a separable Banach space with norm $|\cdot|$. By $L^1(T, Z)$ we denote the Banach space of functions $u: T \to Z$, integrable in the Bochner sense, with norm $\|u\| = \int_T |u(t)|\, dt$.

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