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The generalization of Cellina's Fixed Point Theorem

by

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Abstract. Let $L^1(T, Z)$ be the Banach space of integrable functions from a compact space T into a Banach space Z. A set $K \subset L^1(T, Z)$ is called *decomposable* if, for every $u, v \in K$ and measurable $A \subset T$, $u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K$. In this note we prove that each compact mapping from a closed and decomposable subset $K \subset L^1(T, Z)$ into itself has a fixed point.

§1. **Introduction.** In paper [2] Cellina proved that the set K_P of all functions integrable on a closed interval $[a, b]$ whose values belong to a fixed closed subset P of a Euclidean space R^m has a fixed point property; this means that each compact mapping from K_P into itself has a fixed point. The set K_P can be nonconvex; thus the result of Cellina is interesting when confronted with Schauder's Fixed Point Theorem, where the assumption of convexity is essential (see [3], [8]).

In this note we generalize the above result to an arbitrary closed and decomposable subset K of the space of integrable functions. The decomposability of a set K means that for each $u, v \in K$ and A measurable $u \cdot \chi_A + v \cdot \chi_{[a,b] \setminus A} \in K$, where χ_A stands for the characteristic function of A.

Obviously, the set K_P in the theorem of Cellina is decomposable.

This generalization is quite easy to obtain if we apply a certain theorem on continuous selections proved by the author in [5]. The theorem is an abstract version of Antosiewicz and Cellina's Selection Theorem [1] and can also be applied to the problem of the existence of solutions for the functional-differential inclusion $\dot{x}(t) \in F(t, x(\cdot))$ (see [6]). The required facts about the selections are given in §3. We formulate the main results in §2 and prove it in §4.

§2. **The main result.** Let T be a compact topological space with a σ-field $\mathfrak{M}$ of measurable subsets of T given by a nonnegative, regular Borel measure dt and let Z be a separable Banach space with norm $|\cdot|$. By $L^1(T, Z)$ we denote the Banach space of functions $u\colon T \to Z$, integrable in the Bochner sense, with norm $\|u\| = \int_T |u(t)|\, dt$.

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We call a set $K \subset L^1(T, Z)$ *decomposable* if $u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K$ for every $u, v \in K$ and $A \in \mathfrak{M}$. The family of all nonempty closed and decomposable subsets of $L^1(T, Z)$ we denote by $d(L^1)$.

From this moment let K be a fixed set from $d(L^1)$. The main result is the following:

THEOREM. *Let φ: $K \to K$ be a compact mapping. Then φ has a fixed point.*

COROLLARY. *Let Ω be an abstract space with a σ-field Σ and let φ: $\Omega \times K \to K$ be a function measurable in the first variable and compact in the second. Then there exists a Σ-measurable function s: $\Omega \to K$ such that, for each $\omega \in \Omega$, $\varphi(\omega, s(\omega)) = s(\omega)$ holds. This function s is a Σ-measurable selection of the map P from Ω into closed subsets of K given by $P(\omega) = \{s \in K: \varphi(\omega, s) = s\}$ which is Σ-measurable* (see [4], [7]).

§3. Selection Theorem. Let S and X be topological spaces. Denote by $\mathrm{cl}(X)$ the family of all nonempty and closed subsets of X and let P: $S \to \mathrm{cl}(X)$ be the multivalued map. The function p: $S \to X$ is a selection of P if, for each $s \in S$, we have $p(s) \in P(s)$.

The map P: $S \to \mathrm{cl}(X)$ is called *lower semicontinuous* (*l.s.c.*), if the set $P^- U = \{s \in S: P(s) \cap U \neq \emptyset\}$ is open for each open $U \subset X$.

The following selection theorem was proved in [5]:

SELECTION THEOREM. *Assume that S is a compact topological space and the map L: $S \to d(L^1)$ is l.s.c. Then L admits a continuous selection.*

We apply this theorem to the maps L_ε defined on the set

$$S = \mathrm{cl}\,\mathrm{co}\,\varphi(K), \tag{1}$$

for each $\varepsilon > 0$ by the formulas

$$L_\varepsilon(s) = \mathrm{cl}\{u \in K: |u(t) - s(t)| < \operatorname*{ess\,inf}_{u \in K} |u(t) - s(t)| + \varepsilon \text{ almost everywhere in } T\}, \tag{2}$$

where ess inf stands for the essential infimum and φ and K are as in the Theorem.

The fact that the sets $L_\varepsilon(s)$ are nonempty follows from the observation that for each $s \in S$ there exists an element $u_s \in K$ such that $|u_s(t) - s(t)| = \operatorname*{ess\,inf}_{u \in K} |u(t) - s(t)|$ a.e. in T (see [5], Prop. 2.1). The lower semicontinuity and the decomposability of L_ε given by (2) can easily be deduced from Proposition 2.3 in [5] if we observe that the map ψ defined by $\psi(s) = \operatorname*{ess\,inf}_{u \in K} |u(t) - s(t)|$ is a Lipschitz function in L^1-norm. For this purpose fix s_1 and s_2 from S and let $u_1 \in K$ be such an element that $|u_1(t) - s_1(t)| = \psi(s_1)(t)$ a.e. in T.

Then the Lipschitz condition follows from the inequalities

$$\psi(s_2)(t) - \psi(s_1)(t) \leqslant |u_1(t) - s_2(t)| - |u_1(t) - s_1(t)| \leqslant |s_1(t) - s_2(t)|$$

a.e. in T.

§4. Proof of the Theorem. Let S be defined by (1). Obviously S is a convex and compact subset of $L^1(T, Z)$. Consider the map L_ε given by (2) and let l_ε: $S \to K$ be a continuous selection of L_ε. From the definition of L_ε it follows that for every $s \in \varphi(K)$ the inequality

$$\|l_\varepsilon(s) - s\| \leqslant \varepsilon \cdot \|\chi_T\| \tag{3}$$

holds.

Consider the continuous maps $\varphi \circ l_\varepsilon$: $S \to \varphi(K)$. The Schauder Fixed Point Theorem implies that for each $\varepsilon > 0$ there exist points s_ε such that

$$\varphi[l_\varepsilon(s_\varepsilon)] = s_\varepsilon. \tag{4}$$

Those points belongs to $\varphi(K)$ and from (3) it follows that for each $\varepsilon > 0$ we have

$$\|l_\varepsilon(s_\varepsilon) - s_\varepsilon\| \leqslant \varepsilon \cdot \|\chi_T\|. \tag{5}$$

Obviously the net $\{s_\varepsilon\}$ is totally bounded and we may assume that it converges. Let $s_0 = \lim_{\varepsilon \to 0} s_\varepsilon$. Then also $\lim_{\varepsilon \to 0} l_\varepsilon(s_\varepsilon) = s_0$ because of (5). Taking the limits in (4) we notice that s_0 is the fixed point of φ, which completes the proof.

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