STUDIA MATHEMATICA, T. LXXVIII. (1984)

On commutative algebras of unbounded operators

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Abstract. This paper deals with a kind of algebras of unbounded linear operators in a Hilbert space, called extended C^* and W^*-algebras. It is shown that any commutative EC^*-algebra is a subalgebra of the algebra of spectral integrals for a certain spectral measure. It is also shown that for any spectral measure the algebra of all spectral integrals for this measure is an EC^*-algebra. The conditions for this algebra to be an EW^*-algebra are investigated. Some implications and relations to known results are also discussed.

1. Introduction.

1. There are some possible ways of investigating algebras of unbounded linear operators in a Hilbert space. Such algebras are usually explored under the assumption that all operators have a common dense domain. In this paper an other method is presented, which can be used for commutative algebras, namely a method of connecting them with spectral measures. This is the subject of the paper.

Following Dixon [4], by a *-algebra of closed operators we understand a set $\mathscr{A}$ of closed, densely defined linear operators in a given Hilbert space H (with perhaps different domains) which satisfies the following conditions:

(i) $\mathscr{A}$ contains identity and the zero operator;

(ii) for any operators A, B from the set $\mathscr{A}$ their sum $A+B$ and product AB are closable;

(iii) for any operators A, B from the set $\mathscr{A}$ and any complex number λ the operators

$$A+B \overset{\text{df}}{=} \overline{A+B}, \quad A\circ B \overset{\text{df}}{=} \overline{AB}, \quad \lambda\cdot A \overset{\text{df}}{=} \overline{\lambda A},$$

and A^* belong to the set $\mathscr{A}$;

(iv) the set $\mathscr{A}$ together with the operations $(+, \circ, \cdot, *)$ form an algebra with involution.

For any *-algebra of closed operators $\mathscr{A}$ we will write $\mathscr{A}^b$ to denote the set of all bounded operators from $\mathscr{A}$, i.e., $\mathscr{A}^b = \mathscr{A} \cap B(H)$ ($B(H)$ denotes the

This paper can be classified under the AMS Subject Classification Scheme number 47 D 40.

algebra of all bounded linear operators mapping the space H into itself).

A *-algebra of closed operators $\mathscr{A}$ is called *symmetric* if and only if for any operator A that belongs to this algebra the operator $(I+A^*A)^{-1}$ lies also in $\mathscr{A}$.

A symmetric *-algebra of closed operators $\mathscr{A}$ is called an *EC^*-algebra* (an *EW^*-algebra*) if and only if its bounded part $\mathscr{A}^b$ is a C^*-algebra (respectively a W^*-algebra).

2. Algebras of spectral integrals.

2. Let E be a spectral measure on a certain measure space (Ω, M) into a Hilbert space H and define

$$\mathscr{A}_E \stackrel{\mathrm{df}}{=} \{\int_\Omega f\,dE\colon f\colon \Omega \to C\text{-measurable}\}.$$

PROPOSITION 1. *For any spectral measure E the set $\mathscr{A}_E$ is a commutative EC^*-algebra.*

Proof. To prove that $\mathscr{A}_E$ is a *-algebra of closed operators it is sufficient to show that for the spectral integrals the following equalities are true:

$$\text{(i)}\quad \int_\Omega (f+g)\,dE = \overline{\int_\Omega f\,dE + \int_\Omega g\,dE}, \qquad \text{(ii)}\quad \int_\Omega fg\,dE = \overline{(\int_\Omega f\,dE)(\int_\Omega g\,dE)},$$

$$\text{(iii)}\quad \int_\Omega \lambda f\,dE = \overline{\lambda\int_\Omega f\,dE}, \qquad \text{(iv)}\quad \int_\Omega \bar{f}\,dE = (\int_\Omega f\,dE)^*,$$

where f and g are measurable functions and λ is a complex number.

Let $\sigma_n = \{z\in\Omega\colon n-1 \leqslant (1+|f(z)|)(1+|g(z)|) < n\}$, $n = 1, 2, 3, \ldots$ and $H_n = E(\sigma_n)H$. As the sets σ_n are pairwise disjoint and cover the whole set Ω, we get a decomposition of our Hilbert space H onto an orthogonal sum of subspaces H_n, $H = \bigoplus_{n=1}^{\infty} H_n$. Any subspace H_n is contained in the domains of all the eight operators on both sides of (i), (ii), (iii) and (iv) and, what is more, these operators are bounded on H_n and map the spaces H_n into themselves. But these operators are closed, and so they are uniquely determined by their parts in H_n (as direct sums of parts in H_n). Hence to obtain equalities (i), (ii), (iii), (iv) it suffices to check them only on the spaces H_n, which we immediately get from the relations

$$\int_\Omega (f+g)\chi_n\,dE = \int_\Omega f\chi_n\,dE + \int_\Omega g\chi_n\,dE, \qquad \int_\Omega \lambda f\chi_n\,dE = \lambda\int_\Omega f\chi_n\,dE,$$

$$\int_\Omega fg\chi_n\,dE = (\int_\Omega f\chi_n\,dE)(\int_\Omega g\chi_n\,dE), \qquad \int_\Omega \bar{f}\chi_n\,dE = (\int_\Omega f\chi_n\,dE)^*,$$

valid as the functions involved in them are bounded (χ_n denotes the characteristic function of the set σ_n).

The algebra $\mathscr{A}_E$ is symmetric because the operator $[I+(\int_\Omega f\,dE)^*(\int_\Omega f\,dE)]^{-1}$ is equal to $\int_\Omega (1+|f|^2)^{-1}\,dE$.

To see that the bounded part $\mathscr{A}_E^b$ is closed in the norm topology we shall only consider that $\|\int_\Omega f\,dE\| = E\text{-ess sup}\,|f|$ and that the algebra of all measurable functions on Ω is complete in the topology of the norm $\|f\| = E\text{-ess sup}\,|f|$.

3. Nevertheless $\mathscr{A}_E$ does not always have to be an EW^*-algebra – an example will be given later (Example 5). Now let us formulate a sufficient condition for the algebra $\mathscr{A}_E$ to be EW^*. Before we do that we shall introduce some measure terminology. For details reader can consult [2], from which this terminology is taken.

If μ and ν are positive finite measures on a measure space (Ω, M) we write $\mu \prec \nu$ to denote that the measure μ is absolutely continuous with respect to the measure ν and $\mu \sim \nu$ if both $\mu \prec \nu$ and $\nu \prec \mu$. Later on the measures are regarded up to the equivalence $\sim$. The partial order $\prec$ gives us in the set of all the classes of equivalence for all finite positive measures on (Ω, M) a lattice structure. The supremum $\mu \vee \nu$ is the class of equivalence for the measure $\mu+\nu$ and the infimum $\mu \wedge \nu$ is the class of equivalence for the measure μ' from the Lebesgue decomposition $\mu = \mu'+\mu''$ of μ into parts absolutely continuous and singular with respect to ν. A set J of measures is called a *σ-ideal of measures* if, together with the measure μ, there belongs to J any measure absolutely continuous with respect to μ and if for a given countable family $\{\mu_n\}_{n=1,2,\ldots}$ of measures from J also their supremum $\bigvee_{n=1}^{\infty} \mu_n$ is in J.

For any spectral measure E the set

$$J_E \stackrel{\mathrm{df}}{=} \{\mu_x\colon x\in H,\ \mu_x(\sigma) \stackrel{\mathrm{df}}{=} (E(\sigma)x, x),\ \sigma\in M\}$$

is a σ-ideal of measures. This ideal J_E is σ-generated by a family $\{\mu_s\}_{s\in S}$ of measures from J_E if for any measure μ from J_E there exist measures μ_{s_n}, $s_n\in S$, $n = 1, 2, \ldots$, such that $\mu \prec \bigvee_{n=1}^{\infty} \mu_{s_n}$.

4. PROPOSITION 2. *If for a given spectral measure E its σ-ideal of measures J_E has the following property:*

(+) *there exists a family of measures $\{\mu_s\}_{s\in S} \subset J_E$ and a family $\{\omega_s\}_{s\in S}$ of measurable subsets of Ω, each ω_s being a support set for the measure μ_s, such that the family $\{\mu_s\}_{s\in S}$ σ-generates the ideal J_E and the sets $\{\omega_s\}_{s\in S}$ are pairwise disjoint,*

then the algebra $\mathscr{A}_E$ is a commutative EW^*-algebra.

Proof. Recalling Proposition 1, we only have to show that $\mathscr{A}_E^b$ is a W^*-algebra. We will show that it is closed in the strong topology. For this purpose take a net $\{f_\alpha\}_{\alpha\in A}$ of measurable bounded functions on Ω such that the spectral integrals $A_\alpha = \int_\Omega f_\alpha \, dE$ strongly converge to a bounded operator A. We should find that A is of the form $\int_\Omega f\, dE$ for some function f. The proof of our theorem will consist in the construction of this function.

For any vectors $y_1, \ldots, y_n$ in H, set $\mu_{y_1,\ldots,y_n} = \sum_{k=1}^{n} \mu_{y_k}$. Then:

$$\int_\Omega |f_\alpha - f_\beta|^2 \, d\mu_{y_1,\ldots,y_n} = \sum_{k=1}^{n} \int_\Omega |f_\alpha - f_\beta|^2 \, d\mu_{y_k}$$
$$= \sum_{k=1}^{n} \|A_\alpha y_k - A_\beta y_k\|^2 \xrightarrow[\alpha,\beta\in A]{} 0$$

because the net $\{A_\alpha\}_{\alpha\in A}$ strongly converges. This means that the net $\{f_\alpha\}_{\alpha\in A}$ regarded as a net of functions from $L^2(\mu_{y_1,\ldots,y_n})$ is a Cauchy net there; hence there is a function $f_{y_1,\ldots,y_n}$ in $L^2(\mu_{y_1,\ldots,y_n})$ (depending of course on the choice of the vectors $y_1, \ldots, y_n$) such that the net $\{f_\alpha\}_{\alpha\in A}$ converges to this function in the space $L^2(\mu_{y_1,\ldots,y_n})$.

5. Before going further, it will be useful to prove the following simple

LEMMA 3. *If $\mu_y \prec \mu_x$ for some $x, y\in H$, then the net $\{f_\alpha\}_{\alpha\in A}$ converges to the function f_x in the space $L^2(\mu_y)$ (in the space $L^2(\mu_x)$ it converges to f_x by the definition of f_x).*

To see this let us remark that $f_x \xrightarrow[\alpha\in A]{} f_{x,y}$ in the space $L^2(\mu_{x,y})$; hence $f_\alpha \xrightarrow[\alpha\in A]{} f_{x,y}$ in $L^2(\mu_x)$ and also $f_\alpha \xrightarrow[\alpha\in A]{} f_{x,y}$ in $L^2(\mu_y)$. From the definition of the function f_x we know that $f_\alpha \xrightarrow[\alpha\in A]{} f_x$ in the space $L^2(\mu_x)$, and so we must have $f_x = f_{x,y}$ a.e. μ_x. Similarly $f_y = f_{x,y}$ a.e. μ_y. But we have $\mu_y \prec \mu_x$, and so the equality $f_x = f_{x,y}$ is not only a.e. μ_x but also a.e. μ_y, which gives $f_x = f_y$ a.e. μ_y and thus $f_\alpha \xrightarrow[\alpha\in A]{} f_y$ in $L^2(\mu_y)$, which is what was required.

6. Now we can define the function $f\colon \Omega \to C$ which we seek as follows. Let $\{\mu_s\}_{s\in S}$ be the family of measures and $\{\omega_s\}_{s\in S}$ the family of sets as in condition $(+)$ and let $\mu_s = \mu_{x_s}$, $x_s \in H$, $s\in S$. We set $f(z) = f_{x_s}(z)$ for $z\in\omega_s$, $s\in S$, and $f(z) = 0$ elsewhere. As the sets ω_s are disjoint, the function f is a well-defined measurable function. As ω_s is a support set for the measure μ_s, we have $f = f_{x_s}$ a.e. μ_s for any index $s\in S$. This shows that the net $\{f_\alpha\}_{\alpha\in A}$ converges to the function f in each space $L^2(\mu_s)$. The measures μ_s, $s\in S$, are pairwise singular, and so for any countable family of indices $\{s_n\}_{n=1,2,\ldots} \subset S$ we have

$$\bigvee_{n=1}^{\infty} \mu_{s_n} \sim \sum_{n=1}^{\infty} 2^{-n}\|x_n\|^{-2}\mu_{s_n} = \mu_x$$

(see [2], Theorem 6.6), where $x = \sum_{n=1}^{\infty} 2^{-n}\|x_n\|^{-2} x_n$, $x_n = x_{s_n}$.

For such a vector x we get

$$\mu_x(\{z\in\Omega\colon f(z) \neq f_x(z)\}) = \sum_{n=1}^{\infty} 2^{-n}\|x_n\|^{-2}\mu_{s_n}(\{z\in\Omega\colon f(z)\neq f_x(z)\}) = 0;$$

the reason is that the lemma implies $f_x = f_{x_n}$ a.e. μ_{s_n}, and the equality $f = f_{x_n}$ a.e. μ_{s_n} is obtained directly from the definition of our function f. Thus we have obtained $f = f_x$ almost everywhere μ_x, i.e., in other terms, we have found that the net $\{f_\alpha\}_{\alpha\in A}$ converges to the function f in the space $L^2(\mu_x)$.

Using the lemma once more and having in mind the σ-generating property of the family $\{\mu_s\}_{s\in S}$, we conclude that, for any vector z from H, $f_\alpha \xrightarrow[\alpha\in A]{} f$ in the space $L^2(\mu_z)$. Thus f must be in each space $L^2(\mu_z)$, which means that any vector z from H is in the domain of the operator $\int_\Omega f\, dE$ and (if we use the polarization formula) the net $\{A_\alpha\}_{\alpha\in A}$ weakly converges to the operator $\int_\Omega f\, dE$. But this net converges to the operator A, and so A is equal to the operator $\int_\Omega f\, dE$ and thus belongs to $\mathscr{A}_E^b$. This ends the proof of Proposition 2.

7. Remark 4. As for a separable Hilbert space, there exists a single measure $\mu_x \in J_E$ generating just the ideal J_E, condition $(+)$ is automatically fulfilled, and so the algebra $\mathscr{A}_E$ is always EW^* in a separable space, as Brown has already pointed out in [2], Theorem 10.1. For a non-separable space this is not necessarily so, as is shown by the following example:

8. EXAMPLE 5. Let us take the square $[0, 1]^2$ and the algebra of all Borel subsets of this square as a measure space. Let μ_α and ν_β denote the measures defined as follows:

$$\mu_\alpha(\sigma) \stackrel{\mathrm{df}}{=} m_1(\sigma\cap a_\alpha), \quad \text{where} \quad a_\alpha = \{\alpha\}\times[0, 1],$$
$$\nu_\beta(\sigma) \stackrel{\mathrm{df}}{=} m_1(\sigma\cap b_\beta), \quad \text{where} \quad b_\beta = [0, 1]\times\{\beta\}.$$

$0\leqslant \alpha, \beta \leqslant 1$, σ is a Borel subset of the square $[0, 1]^2$ and m_1 is a one-dimensional Lebesgue measure on vertical and horizontal segments a_α, b_β.

For a Hilbert space let

$$H = \Big(\bigoplus_{0\leqslant\alpha\leqslant 1} L^2(\mu_\alpha)\Big) \oplus \Big(\bigoplus_{0\leqslant\beta\leqslant 1} L^2(\nu_\beta)\Big)$$

and let E be the direct sum of standard spectral measures, i.e., let $E(\sigma)$ be the operator of multiplication by the characteristic function of the set σ, for any Borel subset of the square $[0, 1]^2$.

Then for the projection P on the first part of H, i.e., on $\bigoplus_{0\leqslant\alpha\leqslant 1} L^2(\mu_\alpha)$, we find that P belongs to $\overline{\mathscr{A}_E^b}$ (closure in the strong topology) since P is the direct sum of projections onto $L^2(\mu_\alpha)$, each being of the form $E(a_\alpha)$ and thus belonging to the algebra $\mathscr{A}_E^b$.

From the definition of P we also get the relations $P(\chi_{a_\alpha}) = \chi_{a_\alpha}$ in $L^2(\mu_\alpha)$ and $P(\chi_{b_\beta}) = 0$ in $L^2(\nu_\beta)$ for the characteristic functions of the segments a_α, b_β. If P is of the form $E(\sigma)$ for some Borel set σ, we should have

$$P(\chi_{a_\alpha}) = \chi_{a_\alpha}\chi_\sigma = \chi_{\sigma\cap a_\alpha} = \chi_{a_\alpha} \quad \text{in } L^2(\mu_\alpha)$$

and

$$P(\chi_{b_\beta}) = \chi_{b_\beta}\chi_\sigma = \chi_{\sigma\cap b_\beta} = 0 \quad \text{in } L^2(\nu_\beta),$$

i.e., $m_1(\sigma\cap a_\alpha) = 1$ and $m_1(\sigma\cap b_\beta) = 0$ for $0\leqslant\alpha, \beta\leqslant 1$.

Thus we get a contradiction:

$$1 = \int_0^1 1\,d\alpha = \int_0^1 m_1(\sigma\cap a_\alpha)\,d\alpha = m_2(\sigma) = \int_0^1 m_1(\sigma\cap b_\beta)\,d\beta = \int_0^1 0\,d\beta = 0$$

(we have used the Fubini theorem, and m_2 denotes the two-dimensional Lebesgue measure on the square $[0, 1]^2$).

Thus $\mathscr{A}_E^b \neq \overline{\mathscr{A}_E^b}$ and the algebra $\mathscr{A}_E$ is not an EW^*-algebra.

9. An interesting property of EW^*-algebras can be deduced from [7] and [8]. In these papers the following problem is considered: given a spectral measure E, what properties an operator should have in order to be of the form $\int f\,dE$?

In both papers, in order to obtain a sufficient condition for that, some assumptions are needed about the measure E. In [7] one assumes that for the measure E there exists a countable set of vectors such that the values of projections $E(\sigma)$ on these vectors are dense in the Hilbert space H. This leads to our condition (+) with the additional assumption that the set of indices S is countable and, owing to Proposition 2, the result of this assumption will be that $\mathscr{A}_E$ is an EW^*-algebra.

The assumption about the measure which is made in [8], i.e., the completeness of the lattice of subspaces $\{E(\sigma)H\}_{\sigma\in M}$, is directly equivalent to $\mathscr{A}_E$ being an EW^*-algebra (see [6], XVII.3, Corollary 8). The main theorem of [8] says that for a spectral measure E such that the lattice of subspaces $\{E(\sigma)H\}_{\sigma\in M}$ is complete (i.e., such that the algebra $\mathscr{A}_E$ is EW^*) to the algebra $\mathscr{A}_E$ belongs any closed operator T such that:

(a) T leaves invariant any subspace of H that is invariant for all the projections $E(\sigma)$,

(b) T commutes with these projections, i.e., $E(\sigma)T\subset TE(\sigma)$.

But the inverse is also true. If, for a spectral measure E, to the algebra $\mathscr{A}_E$ belongs any closed operator T as above, then $\mathscr{A}_E$ is an EW^*-algebra. This is obvious since properties (a), (b) are preserved when we take a strong limit of bounded operators fulfilling these conditions (and such are the operators from $\mathscr{A}_E^b$), and so the bounded part $\mathscr{A}_E^b$ of the algebra $\mathscr{A}_E$ is closed in the strong topology.

Thus we have obtained another characterization of EW^*-algebras of type $\mathscr{A}_E$:

For a given spectral measure E the algebra $\mathscr{A}_E$ is EW^* if and only if any closed operator which leaves invariant every invariant subspace for E and commutes with E belongs to $\mathscr{A}_E$.

3. General form of commutative EC^*-algebras.

10. In this section we shall consider the inverse problem: can we find such a spectral measure E, for a given commutative EC^* or EW^*-algebra, that this algebra will be of the form $\mathscr{A}_E$?

The following corollary will be useful:

COROLLARY 6. *For two spectral measures E, F defined on measure spaces (Ω_1, M_1) and (Ω_2, M_2) respectively and acting into the same Hilbert space H we set $E\leqslant F$ if and only if for any $\sigma\in M_1$ there exists an $\omega\in M_2$ such that $E(\sigma) = F(\omega)$. Then the following conditions are equivalent:*

(i) $E\leqslant F$,

(ii) $\mathscr{A}_E^b\subset\mathscr{A}_F^b$,

(iii) $\mathscr{A}_E\subset\mathscr{A}_F$,

(iv) *for any $\sigma\in M_1$ there is $E(\sigma)\in\mathscr{A}_F$.*

The proof is easy, and can be left to the reader as an exercise. In the implication (ii)$\Rightarrow$(iii) we can use the decomposition of the integral $\int_{\Omega_1} f\,dE$ into the direct sum of its pieces on spaces $E(\sigma_n)H$, where $\sigma_n = \{z\in\Omega_1: n-1\leqslant|f(z)| < n\}$,

11. Now we can prove the main result of this section.

PROPOSITION 7. *For any commutative EC^*-algebra $\mathscr{A}$ there exists a spectral measure E on Borel subsets of some compact space Ω such that:*

(i) $\mathscr{A}\subset\mathscr{A}_E$,

(ii) $\overline{\mathscr{A}^b} = \mathscr{A}_E^b$ *(closure in the strong topology).*

Remark 8. The proposition implies that each commutative EC^*-algebra is a subalgebra of some EW^*-algebra of the form $\mathscr{A}_E$ and that for any commutative EW^*-algebra there is a biggest commutative EW^*-algebra which has the same set of bounded operators and is of the form $\mathscr{A}_E$ for some measure E.

12. Proof of Proposition 7. As the algebra $\mathscr{A}$ is commutative, then each operator A from $\mathscr{A}$ is normal (i.e., A is closed, densely defined and $A^*A = AA^*$) and thus there exists a spectral measure E_A associated with A such that the operator A is of the form $A = \int_C \lambda\, dE_A(\lambda)$.

For the bounded operators from $\overline{\mathscr{A}^b}$, since $\overline{\mathscr{A}^b}$ is a W^*-algebra, the values of their spectral measures all belong to $\overline{\mathscr{A}^b}$, being approximable in the strong topology by polynomials of A, A^*. (See also Proposition 4.75 of [5].) We will show that it is also valid for unbounded operators from $\mathscr{A}$.

The algebra $\mathscr{A}$ is symmetric, and so, together with a fixed operator A from $\mathscr{A}$, the operator $B = (I+A^*A)^{-1}$ belongs to $\mathscr{A}$. Thus we have in $\overline{\mathscr{A}^b}$, as has been said above, all the values of the spectral measure E_B associated with the operator B. There will also be all the operators of the form $\int u\, dE_B$, where u is any E_B-essentially bounded function, as they are approximable in the norm topology by the linear combinations of the projections E_B.

Let $C = AB$. Using the property $AA^* = A^*A$, we can find that $C^* = A^*B$. Thus we obtain two inclusions: $AB = C = C^{**} = (A^*B)^* \supset BA^{**} = BA$ and $A^*B = C^* = (AB)^* \supset BA^*$ (the operator B is self-adjoint). Next let us take sets $\sigma_n = [1/(n+1), 1/n)$ and define functions $u_n(t)$, setting $u_n(t) = t^{-1}$ for $t \in \sigma_n$ and $u_n(t) = 0$ elsewhere, $n = 1, 2, \ldots$ From the definition of these functions we get

$$E_B(\sigma_n) = B\int u_n\, dE_B = \left(\int u_n\, dE_B\right) B.$$

We also have $H = \bigoplus_{n=1}^{\infty} H_n$, where $H_n = E_B(\sigma_n)H$. Now, using the inclusion obtained before, we obtain

$$\begin{aligned} AE_B(\sigma_n) &= AB\int u_n\, dE_B = C\int u_n\, dE_B = \left(\int u_n\, dE_B\right)C = \left(\int u_n\, dE_B\right)AB \\ &\supset \left(\int u_n\, dE_B\right)BA = E_B(\sigma_n)A. \end{aligned}$$

We have used the commutativity of the algebra $\overline{\mathscr{A}^b}$, from which come the operators B, $\int u_n\, dE_B$, C. Similarly we can find that $A^*E_B(\sigma_n) \supset E_B(\sigma_n)A^*$. The operators $AE_B(\sigma_n)$ are bounded since they are equal to the product $C\cdot\int u_n\, dE_B$ of two operators from $\overline{\mathscr{A}^b}$. Thus the operator A is the direct sum of its parts in the subspaces H_n (reducing A, which is what we have just proved).

Let A_n be the restriction of A to the subspace H_n. A_n is a bounded normal operator in H_n. If E_n is a spectral measure for A_n, then the spectral measure E_A given by the formula $E_A(\sigma) = \bigoplus_{n=1}^{\infty} E_n(\sigma)$ will be the spectral measure for the operator $A = \bigoplus_{n=1}^{\infty} A_n$.

What we have done is in fact a brief proof of the spectral theorem for unbounded normal operators. We can easily reach our goal proceeding as follows:

The operators $AE_B(\sigma_n)$ and $A^*E_B(\sigma_n)$ are from the algebra $\overline{\mathscr{A}^b}$, and so in this algebra will also be found any operator of the form $p(A, A^*)E_B(\sigma_n)$, where p is a complex polynomial of two variables. Hence we get $p(A_n, A_n^*)E_B(\sigma_n) = p(A, A^*)E_B(\sigma_n) \in \overline{\mathscr{A}^b}$.

Each measure E_n has a compact support, and so for any Borel set σ there is a net of polynomials $\{p_s\}_{s\in S}$ such that $p_s(A_n, A_n^*) \xrightarrow[s\in S]{} E_n(\sigma)$ in the strong topology. Therefore we have $p_s(A_n, A_n^*)E_B(\sigma_n)x \xrightarrow[s\in S]{} E_n(\sigma)E_B(\sigma_n)x$ for any vector x from H. We have proved that the operators $p_s(A_n, A_n^*)E_B(\sigma_n)$ are in the algebra $\overline{\mathscr{A}^b}$; hence in $\overline{\mathscr{A}^b}$ will be found also the projections $E_n(\sigma)E_B(\sigma_n)$. But $E_A(\sigma)x = \sum_{n=1}^{\infty} E_n(\sigma)E_B(\sigma_n)x$ for any $x \in H$, and so the algebra $\overline{\mathscr{A}^b}$ contains any projection $E_A(\sigma)$, σ being a Borel set and E_A the spectral measure connected with some, perhaps unbounded, operator A from the algebra $\mathscr{A}$.

13. Before we go further, we will prove the following important lemma, perhaps interesting in itself.

Lemma 9. *Let $\{E_s\}_{s\in S}$ be a family of regular spectral measures on compact spaces Ω_s, acting into the same Hilbert space H and commuting with one another (i.e., their values commute) and let Ω be a Cartesian product $\Omega = \mathbf{P}_{s\in S}\Omega_s$.*

Then there is a unique regular spectral measure E on Borel subsets of Ω such that

$$E\left(\omega \times \mathbf{P}_{s\neq t}\Omega_s\right) = E_t(\omega), \qquad t \in S,\ \omega \text{ is a Borel subset of } \Omega_t$$

(such a measure is called the product or amalgamate of E_s).

For such a measure we have $\int_\Omega g\circ p_s\, dE = \int_{\Omega_s} g\, dE_s$, where $p_s\colon \Omega \to \Omega_s$ is the projection on Ω_s from the Cartesian product Ω and g is any measurable complex function on Ω_s.

The proof of this lemma in the case of two measures the reader can find in [1], Theorems 33 and 34. By means of induction it extends to the case of a finite set of measures. When this set is uncountable, then the σ-algebra of

Borel sets in $\Omega = \mathop{\mathbf{P}}\limits_{s\in S} \Omega_s$ is the union of the σ-algebras of type $B_Q \overset{\mathrm{df}}{=} \{\sigma \times \times \mathop{\mathbf{P}}\limits_{s\in S\setminus Q} \Omega_s\colon \sigma$ – Borel subset of $\mathop{\mathbf{P}}\limits_{s\in Q} \Omega_s\}$, where Q is any countable subset of S. Therefore if we prove the lemma for a countable set of measures, we will be able to define the measure E as follows: $E(\sigma') \overset{\mathrm{df}}{=} E_Q(\sigma)$, where $\sigma' \in B_Q$, $\sigma' = \sigma \times \mathop{\mathbf{P}}\limits_{s\in S\setminus Q} \Omega_s$, E_Q being the (countable) product of measures $\{E_s\}_{s\in Q}$. The definition will be proper because of uniqueness of the product.

Thus it remains to prove the lemma for a family $\{E_n\}_{n=1,2\ldots}$ of regular, commuting spectral measures defined on compact sets Ω_n.

Let us denote by $B^{(n)}$ the σ-algebra of all sets of the form $\sigma \times \times \mathop{\mathbf{P}}\limits_{k=n+1}^{\infty} \Omega_k$, σ being a Borel subset of $\Omega_1 \times \Omega_2 \times \ldots \times \Omega_n$, and let $B = \bigcup\limits_{n=1}^{\infty} B^{(n)}$, B being a ring of subsets of Ω generating the σ-algebra of all Borel subsets of $\Omega = \mathop{\mathbf{P}}\limits_{n=1}^{\infty} \Omega_n$. From the uniqueness of the finite product $E^{(n)}$ of spectral measures $E_1, E_2, \ldots, E_n$ the definition

$$E(\sigma') \overset{\mathrm{df}}{=} E^{(n)}(\sigma), \qquad \sigma' \in B^{(n)}, \qquad \sigma' = \sigma \times \mathop{\mathbf{P}}\limits_{k=n+1}^{\infty} \Omega_k$$

establishes a well-defined function E on the ring B. This function is additive on disjoint sets, its values are projections in H and $E(\sigma \cap \omega) = E(\sigma)E(\omega)$, for the reason that the restriction of E to any σ-algebra $B^{(n)}$ is a spectral measure.

Let $\{\sigma_m\}_{m=1,2\ldots}$ be a family of sets from B such that $\sigma_m \subset \sigma_{m-1}$ and $\bigcap\limits_{m=1}^{\infty} \sigma_m = \emptyset$. We can assume without loss of generality that $\sigma_m \in B^{(m)}$. Fix a positive $\varepsilon > 0$ and a vector x from H. The measures $E^{(m)}$ are regular, and so there exist compact sets K_m such that $K_m \subset \sigma_m$, $K_m \in B^{(m)}$ and $(E(\sigma_m \setminus K_m)x, x) \leqslant \varepsilon \cdot 2^{-m}$ (we use the assumption of compactness of the sets Ω_n). Then we have $\bigcap\limits_{m=1}^{\infty} K_m \subset \bigcap\limits_{m=1}^{\infty} \sigma_m = \emptyset$; so, since the sets K_m are compact, there has to be a natural number N such that $\bigcap\limits_{m=1}^{N} K_m = \emptyset$. But the function E is finitely additive on disjoint sets and hence we get, for any $M \geqslant N$,

$$\begin{aligned}(E(\sigma_M)x, x) &= \Big(E\big(\sigma_M \setminus \bigcap_{m=1}^{M} K_m\big)x, x\Big) \leqslant \Big(E\big(\bigcup_{m=1}^{M} (\sigma_m \setminus K_m)\big)x, x\Big) \\ &\leqslant \sum_{m=1}^{M} (E(\sigma_m \setminus K_m)x, x) \leqslant \sum_{m=1}^{M} \varepsilon \cdot 2^{-m} < \varepsilon.\end{aligned}$$

Thus we have found that, for any $x \in H$, $\lim\limits_{m\to\infty} (E(\sigma_m)x, x) = 0$, i.e., together with what we have said previously, that E is a spectral measure on the ring B. It expands uniquely to a regular spectral measure on the σ-algebra of Borel subsets of Ω which is generated by the ring B (see [1], Theorem 7, p. 15).

In this way we have proved the first part of the lemma. The second states that $\int\limits_{\Omega} g \circ p_s \, dE = \int\limits_{\Omega_s} g \, dE_s$ and is an immediate result of the definition of the the product of measures and the definition of the spectral integral.

In our proof we have used the compactness of the sets Ω_n. This assumption can be replaced by the assumption that Ω_n are Polish spaces, but this makes the proof slightly more difficult.

14. Now we can continue the proof of Proposition 7. As the measure E which we need let us take the product of all the measures $\{E_A\}_{A\in\overline{\mathscr{A}^b}}$ on $\Omega = \mathop{\mathbf{P}}\limits_{A\in\mathscr{A}^b} \sigma(A)$. Its existence is ensured by the lemma, as all the values of these measures belong to a commutative algebra $\overline{\mathscr{A}^b}$. Ω is compact because the spectra $\sigma(A)$ are compact for the operators from the algebra $\overline{\mathscr{A}^b}$, which are bounded.

The second part of the lemma shows that for any operator A belonging to $\overline{\mathscr{A}^b}$ we have $\mathscr{A}_{E_A} \subset \mathscr{A}_E$, and thus we obtain $\overline{\mathscr{A}^b} \subset \mathscr{A}_E$.

If an operator A is from the algebra $\mathscr{A}$ but is unbounded, we have proved that all the projections $E_A(\sigma)$ are in the algebra $\overline{\mathscr{A}^b}$ (contained in $\mathscr{A}_E$), and so, using Corollary 6, we will get $\mathscr{A}_{E_A} \subset \mathscr{A}_E$, in particular $A \in \mathscr{A}_E$. Hence we finally infer that the whole algebra $\mathscr{A}$ is contained in $\mathscr{A}_E$, i.e., we infer condition (i) of Proposition 7. To get (ii) we have already found that $\overline{\mathscr{A}^b} \subset \mathscr{A}_E^b$. It remains to show the inclusion $\mathscr{A}_E^b \subset \overline{\mathscr{A}^b}$, but since $\overline{\mathscr{A}^b}$ is a W^*-algebra it will be sufficient to prove that any projection $E(\sigma)$ is in $\overline{\mathscr{A}^b}$ (other operators from $\mathscr{A}_E^b$ are approximable by the linear combinations of such projections in the norm topology).

Let N denote the family of all Borel subsets σ of Ω that satisfy $E(\sigma) \in \overline{\mathscr{A}^b}$. If ω is a Borel subset of $\sigma(A)$ for some $A \in \overline{\mathscr{A}^b}$, then we have $E_A(\omega) = E(\sigma_\omega)$ for the set σ_ω of the form $\sigma_\omega = \omega \times \mathop{\mathbf{P}}\limits_{B\in\mathscr{A}^b\setminus\{A\}} \sigma(B)$. We have proved that $E_A(\omega) \in \overline{\mathscr{A}^b}$, which allows us to include in the family N all the sets of the above form, and thus the cylindric sets.

The properties of a spectral measure: $E(\Omega \setminus \sigma) = I - E(\sigma)$, $E(\sigma_1 \cap \ldots \cap \sigma_n) = E(\sigma_1) \cdot \ldots \cdot E(\sigma_n)$, $E\big(\bigcup\limits_{n=1}^{\infty} \sigma_n\big) = \sum\limits_{n=1}^{\infty} E(\sigma_n)$, $\sigma_n \cap \sigma_m = \emptyset$, together with the fact that such operations do not lead outside the W^*-algebra $\overline{\mathscr{A}^b}$; let us conclude that N is a σ-algebra of sets. But in N are all the cylindric sets, and so N must coincide with the family of all Borel subsets of Ω.

This ends the proof of Proposition 7.

4. Some relations to GB^*-algebras.

15. Proposition 7 characterizes commutative EC^*-algebras from the point of view of spectral measures. Such algebras can also be characterized from the algebraical point of view as presented in a paper of Dixon [3].

For a complete topological algebra $\mathscr{A}$ with a unit and involution take its biggest subset B which has the following properties: B is closed, bounded, contains a unit, $B \cdot B \subset B$ and $B^* = B$. Following [3] the algebra $\mathscr{A}$ is called a *GB^*-algebra* if this set B is absolutely convex and the subalgebra of $\mathscr{A}$ generated by B is a B^*-algebra under the norm induced by B (i.e., the norm for which B is a unit ball).

A commutative EC^*-algebra $\mathscr{A}$ can be equipped with a topology in which it is a GB^*-algebra. We will briefly show how it can be done. Proposition 7 allows us to regard $\mathscr{A}$ as a subalgebra of the algebra $\mathscr{A}_E$ for a certain spectral measure E on a measure space (Ω, M). The mapping $\varphi\colon M_E(\Omega) \ni f \to \int_\Omega f\,dE \in A_E$ is a $*$-isomorphism from the algebra $M_E(\Omega)$ of all measurable complex functions on Ω (regarded up to E-almost everywhere equality) onto the EW^*-algebra A_E.

In the algebra $M_E(\Omega)$ we define topology by the family of semimetrics

$$d_x(f, g) \overset{\mathrm{df}}{=} \int \frac{|f-g|}{1+|f-g|}\, d\mu_x, \qquad x \in H.$$

It is easy to show that with this topology the algebra $M_E(\Omega)$ becomes an algebra with continuous multiplication, usually not locally convex. The involution $f^*(z) = \overline{f(z)}$ is continuous in this topology. The biggest subset B_1 of this algebra which fulfils the following conditions: B_1 is bounded, B_1 is closed, $1 \in B_1$, $B_1 \cdot B_1 \subset B_1$, $B_1^* = B_1$, is the set

$$B_1 = \{f \in M_E(\Omega)\colon\ E\text{-ess sup}\,|f| \leqslant 1\}$$

and is absolutely convex. It generates the subspace

$$L_E^\infty(\Omega) = \{f \in M_E(\Omega)\colon\ E\text{-ess sup}\,|f| < \infty\},$$

which with the norm induced by B_1 (i.e., the norm $\|f\| = E\text{-ess sup}\,|f|$) is isometrically isomorphic (by the isomorphism φ) with the W^*-algebra $\mathscr{A}_E^b$. If B is a subset of our EC^*-algebra $\mathscr{A}$ (we equip this algebra with the topology induced from $M_E(\Omega)$), which fulfils the same conditions as B_1, then $B = A \cap \varphi(B_1)$. Hence B generates the algebra $\mathscr{A} \cap \varphi(L_E^\infty(\Omega)) = \mathscr{A} \cap \mathscr{A}_E^b = \mathscr{A}^b$, which is a C^*-algebra. Thus we conclude that $\mathscr{A}$ with the topology induced from $M_E(\Omega)$ is a commutative GB^*-algebra.

16. For GB^*-algebras Dixon has proved, with the additional assumption that they are locally convex, that they are isomorphic with EC^*-algebras of operators having a common dense reducing subspace (see [3], Theorem 7.11).

In our case the GB^*-topology is not locally convex in general, and so we can suspect operators from the algebra $\mathscr{A}_E$ not to have such a subspace. Indeed, we have for a regular spectral measure E on a Polish space Ω and a vector x in H: $x \in \bigcap_{f \in M_E(\Omega)} D(\int f\,dE)$ if and only if for any $f \in M_E(\Omega)$, we have $\int |f|^2\, d\mu_x < \infty$ ($D(\int f\,dE)$ denotes the domain of the operator $\int f\,dE$). The above is possible if and only if there is no countable family of disjoint sets $\{\sigma_n\}_{n=1,2,\ldots}$ such that $\mu_x(\sigma_n) > 0$ for any n, because the function $f = \sum_{n=1}^{\infty} \chi_{\sigma_n} \cdot \mu_x(\sigma_n)^{-2}$ would have the integral $\int |f|^2\, d\mu_x$ equal to ∞. For the measure μ_x on the Polish space Ω this is equivalent to $\mu_x = \sum_{k=1}^{m} \lambda_k \delta_{z_k}$, where z_k are certain points of Ω, λ_k are positive coefficients and δ_{z_k} are Dirac measures, i.e., $\delta_{z_k}(\sigma) = 1$ if $z_k \in \sigma$ and $\delta_{z_k}(\sigma) = 0$ if $z_k \notin \sigma$.

From the equation

$$(E(\{z_1, \ldots, z_m\})x, x) = \mu_x(\{z_1, \ldots, z_m\}) = \mu_x(\Omega) = \|x\|^2$$

we obtain

$$x = E(\{z_1, \ldots, z_m\})x = \sum_{k=1}^{m} E(\{z_k\})x,$$

and this leads to the conclusion that

$$\bigcap_{A \in \mathscr{A}_E} D(A) = \operatorname{span} \bigcup_{z \in \Omega} E(\{z\})H.$$

The right-hand side does not have to be dense in H. In fact, there are spectral measures for which it contains only zero, for example the spectral measure of a normal operator having no point spectrum.

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Received January 27, 1982 (1732)

STUDIA MATHEMATICA, T. LXXVIII. (1984)

Weak type inequalities for the maximal ergodic function and the maximal ergodic Hilbert transform in weighted spaces

by

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Abstract. In this paper we show that the maximal ergodic function associated to an invertible, measure preserving ergodic transformation on a probability space is of weak type (1,1) with respect to $wd\mu$, where w is a positive integrable function, if and only if w satisfies Muckenhoupt condition A_1. We also prove the same result for the maximal ergodic Hilbert transform.

1. Introduction. Let $(X, \mathfrak{F}, \mu)$ be a non-atomic probability space and T an ergodic, invertible measure preserving point transformation from X onto itself. We will denote by f^* the non-centered maximal ergodic function

$$f^*(x) = \sup_{n,m\geqslant 0} (n+m+1)^{-1} \sum_{i=-n}^{m} |f(T^i x)|, \quad n, m \in \boldsymbol{Z}, \tag{1.1}$$

and by

$$Hf(x) = \sup_{s,t\geqslant 0} \Big| \sum_{s<|h|<t} \frac{f(T^h x)}{h} \Big|, \quad s, t \in \boldsymbol{Z},$$

the maximal ergodic Hilbert transform.

In [1] and [2] it was shown that the operators $f \to f^*$ and $f \to Hf$ are bounded on $L^p(wd\mu)$, $p > 1$, if and only if the positive integrable function w satisfies the condition:

(A_p') There exists a constant M such that for a.e. x

$$k^{-1} \sum_{i=0}^{k-1} w(T^i x) \cdot \big[k^{-1} \sum_{i=0}^{k-1} (w(T^i x))^{-1/(p-1)}\big]^{p-1} \leqslant M \tag{1.2}$$

for all positive integers k.

Condition A_p' is the natural analogue of Muckenhoupt condition for the Hardy–Littlewood maximal operator [4].

In this paper our main result is given by the following theorem.

(1.1) THEOREM. *Let w be a positive integrable function. Then*

(i) *The operator $f \to f^*$ is of weak type* (1.1) *with respect to $wd\mu$ if and only if $w \in A_1$.*