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A functional calculus for Rockland operators on nilpotent Lie groups

by

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Abstract. Let G be a homogenous Lie group and let L be a positive Rockland operator. Let

$$Lf = \int_0^\infty \lambda \, dE(\lambda) f$$

be the spectral resolution of L on $L^2(G)$. It is shown that if $m \in \mathcal{S}(\boldsymbol{R}^+)$, then if

$$T_m f = \int_0^\infty m(\lambda) \, dE(\lambda) f,$$

then T_m is of the form $T_m f = f * M$, where $M \in \mathcal{S}(G)$.

Let G be a nilpotent Lie group and let L be a hypoelliptic, positive, left-invariant differential operator on G which satisfies a subelliptic estimate:

For every left-invariant differential operator ∂ on G there exist an integer $\sigma(\partial)$ and a constant C such that

(0.1) $\|\partial u\|_{L^2(G)} \leqslant C \|(1+L)^{\sigma(\partial)} u\|_{L^2(G)}$ *for* $u \in \mathrm{Dom}(\bar{L}^{\sigma(\partial)})$ *and, consequently, for an integer S and a constant C* $\sup_{x \in G} |u(x)| \leqslant C \|(1+L)^S u\|_{L^2(G)}$ *for* $u \in \mathrm{Dom}(\bar{L}^S)$.

Let $E(\lambda)$ be the spectral resolution of a positive self-adjoint extension of L, which in fact is unique and equal to the closure of L, and let

$$T^t f = \int_0^\infty e^{-t\lambda} \, dE(\lambda) f, \qquad f \in L^2(G), \tag{0.2}$$

be the semi-group of operators on $L^2(G)$ generated by $-\bar{L}$.

Following the program of E. M. Stein formulated in [10] we investigate operators

$$T_m f = \int_0^\infty m(\lambda) \, dE(\lambda) f, \tag{0.3}$$

where m is a bounded function $\boldsymbol{R}^+$, on other spaces of functions on G under suitable conditions on m. For instance, a Marcinkiewicz–Myhlin type theorem has been proved by E. M. Stein and the author in the case when L is the sublaplacian on a stratified nilpotent Lie group, cf. [2].

Also in [2] G. B. Folland and E. M. Stein proved that for positive Rockland operators (see definition below) on graded nilpotent groups which by [3] are hypoelliptic and satisfy (0.1) the semi-group (0.2) is of the form

$$T^t f = f * \varphi_t, \quad \text{where } \varphi_t \in \mathcal{S}(G). \tag{0.4}$$

The aim of this paper is to show that for a positive hypoelliptic L which satisfies (0.1) and the semi-group (0.2) is of the form (0.4), $m \in \mathcal{S}(\boldsymbol{R}^+)$ implies that the operator (0.3) is of the form

$$T_m f = f * M, \quad \text{where } M \in \mathcal{S}(G). \tag{0.5}$$

As a matter of fact, we get an evaluation of the number of derivatives and moments for m which guarantee that M has a given number of derivatives and moments.

The main method used in the paper is a C^k functional calculus for L^2 functions on a group of polynomial growth which decay at infinity as $(1+|x|)^{-\alpha}$ for a fixed but rather large α, as in the previous papers (e.g. [4], [6], [2]) a C^k functional calculus for exponentially decaying functions has been used.

Some corollaries follow.

If the semi-group satisfies (0.4), then φ_t depend holomorphically on t, i.e. $\boldsymbol{R}^+ \ni t \to \varphi_t \in \mathcal{S}(G)$ extends to a holomorphic map $\{z\colon \operatorname{Re} z > 0\} \ni z \to \varphi_z \in \mathcal{S}(G)$, which suggests that for Rockland operators the functions φ_t should be real analytic, as they are if the group is $\boldsymbol{R}^d$.

Using present functional calculus one can obtain the following Marcinkiewicz–Myhlin multiplier theorem in the same way as it has been done for the sublaplacian in [2].

If L is a positive Rockland operator, there exists a number k such that if

$$\sup |\lambda^j m^{(j)}(\lambda)| < \infty \quad \text{for} \quad j = 0, 1, \ldots, k,$$

then T_m (as defined by (0.3)) is of weak-type (1, 1) and thus bounded on every $L^p(G)$, $1 < p < \infty$.

We should perhaps also mention that in analogy with [1] and [6] Riesz–Bochner and other summability methods for the expansions in eigenfunctions of Rockland operators are available.

1. Preliminaries. A simply connected nilpotent Lie group G is called *graded* if its (left-invariant) Lie algebra $\boldsymbol{g}$ is endowed with a vector space decomposition

$$\boldsymbol{g} = \oplus \sum V_j \tag{1.1}$$

such that $[V_i, V_j] \subset V_{i+j}$. Then $\boldsymbol{g}$ admits a family $\{\delta_r\}_{r>0}$ of dilations which are automorphisms of $\boldsymbol{g}$:

$$\text{if} \quad a = \min\{j\colon V_j \neq 0\} \quad \text{we define} \quad \delta_r X = r^{j/a} X \quad \text{if} \quad X \in V_j.$$

Putting

$$\delta_r \exp X = \exp \delta_r X$$

we obtain a family of automorphisms — dilations — of G.

Let Q be the positive number (the homogeneous dimension of G) defined by

$$f(\delta_r x)\,dx = r^{-Q} \int f(x)\,dx, \quad f \in L^1(G).$$

There exists a continuous function

$$G \ni x \to |x| \in \boldsymbol{R}^+$$

such that $|x| = 0$ iff $x = e$, $|x| = |x^{-1}|$, for a constant γ, $|xy| \leqslant \gamma(|x|+|y|)$ and $|\delta_r x| = r|x|$.

Let G be an arbitrary locally compact group and let U be a fixed symmetric compact neighbourhood of e. If G is a graded nilpotent Lie group we let

$$U = \{x \in G\colon |x| \leqslant 1\}.$$

We define a subadditive function

$$\tau(x) = \min\{n\colon x \in U^n\}.$$

The following lemma is used in [2].

LEMMA 1.1. *If G is a graded nilpotent group there exist positive constants a, b, c, C such that*

$$c\tau(x)^a \leqslant |x| \leqslant C\tau(x)^b \quad \textit{for} \quad |x| \geqslant 1. \tag{1.2}$$

Remark. It has been proved by Joe Jenkins [7] that a, b can be taken as equal to 1 if and only if G is stratified, i.e. if the smallest Lie subalgebra of $\boldsymbol{g}$ containnig V_1 is equal to $\boldsymbol{g}$.

We write

$$w(x) = 1 + \tau(x).$$

Then for every $\alpha \geqslant 0$ we have

$$w^\alpha(x) \geqslant 1, \quad w^\alpha(x) = w(x^{-1}), \quad w^\alpha(xy) \leqslant w^\alpha(x)\,w^\alpha(y) \tag{1.3}$$

and also

$$w^\alpha(xy) \leqslant C_\alpha\big(w^\alpha(x) + w^\alpha(y)\big). \tag{1.4}$$

Let $M(G)$ be the Banach *-algebra of Borel measures on G and let

$$M_\alpha = \{\mu \in M(G)\colon \int w^\alpha(x)\, d|\mu|(x) = \|\mu\|_{M_\alpha} < \infty\}.$$

In virtue of (1.3) M_α is a Banach *-algebra. We write

$$L_\alpha = \{f \in L^1(G)\colon \int |f(x)|\, w^\alpha(x)\, dx = \|f\|_{L_\alpha} < \infty\}.$$

L_α is, of course, a Banach *-subalgebra of M_α.

A locally compact group is called *of polynomial growth*, if for every compact subset U the Haar measure $|U^n|$ of U^n satisfies

(1.5) $$|U^n| = O(n^R) \quad \text{as } n \to \infty.$$

By Lemma 1.1, if G is a graded nilpotent group, then G is of polynomial growth and (1.5) holds for an R such that

$$[aQ] \leqslant R \leqslant [bQ]+1.$$

Also,

(1.6) if G is of polynomial growth, then w^{-R-2} is integrable.

It has been proved by T. Pytlik [9] that if G is of polynomial growth, then L_α, $\alpha > 0$, is a symmetric Banach *-algebra. This implies that for every commutative Banach *-subalgebra A of L_α every multiplicative linear functional on A is bounded on the C^*-algebra generated by the operators

$$L^2(G) \ni \xi \to \xi * f \in L^2(G), \quad f \in A.$$

Let G be a nilpotent Lie group and let $X_1, \ldots, X_n$ be a basis of its Lie algebra $\mathfrak{g}$. If G is graded, we assume that this basis is selected according to (1.1). For a multi-index $(i_1, \ldots, i_n) = I$ we write

$$X^I = X_1^{i_1} \ldots X_n^{i_n}.$$

Let $|I| = i_1 + \ldots + i_n$.

For a nilpotent, simply connected group G, if

$$\frac{\partial}{\partial x_j} f(x) = \frac{\partial}{\partial x_j} f(\exp(x_1 X_1 + \ldots + x_n X_n))$$

with $x = \exp(x_1 X_1 + \ldots + x_n X_n)$ and

$$\left(\frac{\partial}{\partial x}\right)^I = \left(\frac{\partial}{\partial x_1}\right)^{i_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{i_n},$$

we have

(1.7) $$X^I f(x) = \sum_{|J| \leqslant |I|} a_J(x) \left(\frac{\partial}{\partial x}\right)^J f(x),$$

$a_J(\exp(x_1 X_1 + \ldots + x_n X_n))$ being polynomials in $x_1, \ldots, x_n$. Consequently,

(1.8) $$|a_J(x)| \leqslant C_J\, w(x)^N, \quad N = N(J)$$

(cf. e.g. [2]).

For a function u we define

$$\|u\|_{\alpha,k,p} = \Big(\sum_{|I| \leqslant k} \int |X^I u(x)|^p\, w(x)^{p\alpha}\, dx\Big)^{1/p}.$$

The following lemma is an immediate consequence of (1.6), (1.7), (1.8) and the ordinary Sobolev lemma.

LEMMA 1.2. *For every $1 \leqslant p, q \leqslant \infty$ and α, k there exist α', k' and a constant c such that*

$$\|u\|_{\alpha,k,p} \leqslant c\, \|u\|_{\alpha',k',q}.$$

We define $\mathscr{S}_{\alpha,k,p}(G)$ as the space of functions f for which $\|f\|_{\alpha,k,p}$ is finite and

$$\mathscr{S}(G) = \bigcap_{\alpha,k} \mathscr{S}_{\alpha,k,p}(G).$$

In virtue of Lemma 1.2, $\mathscr{S}(G)$ does not depend on p, $1 \leqslant p \leqslant \infty$.

We also note

(1.9) If $\mu \in M_\alpha$ and $f \in \mathscr{S}_{\alpha,k,1}(G)$, then $f * \mu \in \mathscr{S}_{\alpha,k,1}(G)$.

By a Rockland operator on a graded Lie group G (cf. [2]) we mean a differential left-invariant operator L on G which is homogeneous with respect to the dilations and for every irreducible unitary representation π of G the operator π_L is injective on the space of C^∞ vectors. It has been proved by B. Helffer and J. Nourrigat [3] that such operators are hypoelliptic. Moreover, if L is positive, it satisfies (0.1).

Let L be a positive Rockland operator. As we have mentioned before, G. B. Folland and E. M. Stein proved in [2] that the semi-group generated by $-\bar{L}$ is of the form (0.4). Moreover, if L is homogeneous of degree D, then

$$\varphi_t(x) = t^{-Q/D}\, \varphi_1(\delta_{t^{-1/D}} x)$$

and consequently,

$$\{\varphi_t\}_{t \to 0} \text{ is an approximate identity in } \mathscr{S}_{\alpha,k,p}(G).$$

Finally we note that if L satisfies (0.1), then for every d and k there is a positive number $s(d, k)$ such that

(1.10) $$\|(1+L)^k X^I u\|_{L^2(G)} \leqslant C\, \|(1+L)^{s(d,k)} u\|_{L^2(G)}$$

for all $|I| \leqslant d$.

2. Functional calculus. Let B be a Banach *-algebra. We say that a function F operates on an element f in B if the Gelfand transform of f with respect to the smallest commutative Banach subalgebra A containing f is real and there exists a g in A such that $F\circ\hat{f} = \hat{g}$. We then write

$$g = F\cdot f.$$

For f in B let

$$e(f) = \sum_{j=1}^{\infty} \frac{(if)^j}{j!}.$$

Suppose that

$$\|e(nf)\|_B = O(|n|^l) \quad \text{as } |n|\to\infty$$

and let $|\hat{f}(\lambda)| < a$. Then for every $F\in C_c^k[-a, a]$ with $k > l+1$ and $F(0) = 0$, F operates on f and

$$F\cdot f = \sum_{n\in\mathbf{Z}} F(n)\, e(nf), \quad \text{whence} \quad \|F\cdot f\|_B \leqslant C_f \|F\|_{C^k[-a,a]}, \tag{2.1}$$

cf. e.g. [8] for the details.

In the present paper functional calculus is used in the following way.

Suppose for some $\alpha > 0$ and all $t > 0$ $\varphi_t = \varphi_t^* \in L_\alpha$ and, moreover, $\varphi_{s+t} = \varphi_s * \varphi_t$ and $\lim_{t\to 0} \|f*\varphi_t - \varphi_t\|_{L^2(G)} = 0$ for f in $L^2(G)$.

Let B be the smallest closed Banach *-subalgebra of L_α containing all φ_t, $t > 0$. Then B is a commutative, semisimple, symmetric Banach *-algebra and the Gelfand space of B can be identified with $\mathbf{R}^+$ in such a way that $\hat{\varphi}_t(\lambda) = e^{-t\lambda}$. Moreover, there is a spectral measure on $L^2(G)$ such that for M in B

$$f*M = \int_0^\infty \hat{M}(\lambda)\, dE(\lambda) f,$$

where $\hat{M}$ is the Gelfand transform of M. Then, clearly, if a function F operates on $M\in B$, then

$$f*F\cdot M = \int_0^\infty F(\hat{M}(\lambda))\, dE(\lambda) f.$$

THEOREM 2.1. *Suppose G is of polynomial growth, let U and R be defined as in* (0.5). *Suppose $\varphi^* = \varphi\in L_\alpha\cap L^2(G)$ with $\alpha > \beta + R/2 + 1$, then*

$$\|e(n\varphi)\|_{L_\beta} = O(|n|^{3(\beta+R/2)+4}) \quad \text{as } n\to\infty.$$

Proof. First we note the following fact:

If $\alpha\geqslant\beta$ and $g\in L_\alpha$ with $\operatorname{supp} g\subset G\setminus U^l$, then

$$\|g\|_{L_\beta} \leqslant l^{\beta-\alpha} \|g\|_{L_\alpha}. \tag{2.2}$$

In fact,

$$\|g\|_{L_\beta} = \int_{G\setminus U^l} |g|\, w^\beta = \int_{G\setminus U^l} |g|\, w^\alpha w^{\beta-\alpha} \leqslant \|g\|_{L_\alpha} l^{\beta-\alpha}.$$

Since L_α is a Banach algebra, $e(\varphi) = \psi\in L_\alpha$ and also $\psi\in L^2(G)$. Moreover, for $n\geqslant 0$ we have

$$e(n\varphi) = (e+\psi)^{*n} - e,$$

where e is the unit element in $M(G)$ — the delta measure at the unit element of G.

Now we fix n and replacing φ by $-\varphi$, if necessary, we assume that $n > 0$.

Let

$$f = \psi\, \mathbf{1}_{U^{n^2}}, \quad g = \psi\, \mathbf{1}_{G\setminus U^{n^2}}. \tag{2.3}$$

Hence, since $\alpha\geqslant 1$, by (2.2), we have

$$\|e+\psi-e+f\|_{L^1(G)} = \|g\|_{L^1(G)} \leqslant \|g\|_{L_\alpha} n^{-2} \leqslant \|\psi\|_{L_\alpha} n^{-2}. \tag{2.4}$$

For a measure μ in $M(G)$ let

$$\lambda(\mu) = \sup\{\|\mu*\xi\|_{L^2(G)}\colon \|\xi\|_{L^2(G)} = 1\}.$$

Consequently, since

$$e+\psi = \sum_{j=0}^{\infty} \frac{(i\varphi)^{*j}}{j!}$$

and $\varphi = \varphi^*$, by the spectral theorem, $\lambda(e+\psi) = 1$, whence, by (2.4) we have

$$\lambda(e+f) \leqslant 1 + \|\psi\|_{L_\alpha} n^{-2}. \tag{2.5}$$

We write

$$\begin{aligned} \|e+\psi)^{*n}\|_{M_\beta} &= \|((e+f)+g)^{*n}\|_{M_\beta} \\ &\leqslant \sum_{m=0}^{n} \sum_{a,b} \|(e+f)^{a_1} * g^{b_1} * \ldots * (e+f)^{a_n} * g^{b_n}\|_{M_\beta}, \end{aligned} \tag{2.6}$$

where the inner summation extends over all sequences $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$ of zeros and ones such that $a_1 + \ldots + a_n = m$, $b_1 + \ldots + b_n = n-m$.

Now we fix two such sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ and we estimate

$$\begin{aligned} &\|(e+f)^{a_1} * g^{b_1} * \ldots * (e+f)^{a_n} * g^{b_n}\|_{M_\beta} \\ &\leqslant \int \ldots \int \|(e+f)^{a_1} s_1 * \ldots * (e+f)^{a_n} s_n\|_{M_\beta} |g^{b_1}(s_1)| \ldots |g^{b_n}(s_n)|\, ds_1 \ldots ds_n \\ &\leqslant \sum_{q=0}^{\infty} \int_{\Omega(q)} \|(e+f)^{a_1} s_1 * \ldots * (e+f)^{a_n} s_n\|_{M_\beta} |g^{b_1}(s_1)| \ldots |g^{b_n}(s_n)|\, ds_1 \ldots ds_n, \end{aligned} \tag{2.7}$$

where for a measure μ and $M\subset G$ we write $\mu s(M) = \mu(Ms)$ and

$$\Omega(q) = \{(s_1, \ldots, s_n)\in G\times\ldots\times G\colon \max_j \tau(s_j^{b_j}) = q\}.$$

For a $(s_1, \ldots, s_n)$ in $\Omega(q)$ we have

$$(2.8)\quad \|(e+f)^{a_1} s_1^{b_1} * \ldots * (e+f)^{a_n} s_n^{b_n}\|_{M_\beta}$$
$$\leqslant \begin{cases} \sum\limits_{a_j=1} \|(e+f)^{a_1} s_1^{b_1} * \ldots * (e+f)^{a_{j-1}} s_{j-1}^{b_{j-1}} * f s_j^{b_j} \ldots s_n^{b_n}\|_{L_\beta} & \text{if} \quad \boldsymbol{a} \neq 0, \\ \|s_1 \ldots s_n\|_{M_\beta} \leqslant (1+nq)^\beta & \text{if} \quad \boldsymbol{a} = 0. \end{cases}$$

But, by (2.3),

$$\operatorname{supp}(e+f)^{a_1} s_1^{b_1} * \ldots * (e+f)^{a_{j-1}} s_{j-1}^{b_{j-1}} * f s_j^{b_j} \ldots s_n^{b_n}$$
$$\subset U^{a_1 n^2} U^q \ldots U^{a_j n^2} U^q \ldots U^q \subset U^{n^3+nq}.$$

On the other hand, if $\xi \in L^2(G)$ and $\operatorname{supp} \xi \subset U^m$, then

$$\|\xi\|_{L_\beta} = \int_{U^m} |\xi|\, w^\beta \leqslant (1+m)^\beta |U^m|^{1/2} \|\xi\|_{L^2(G)}.$$

Consequently, by (2.5),

$$\|(e+f)^{a_1} s_1^{b_1} * \ldots * (e+f)^{a_{j-1}} s_j^{b_{j-1}} * f s_j^{b_j} \ldots s_n^{n}\|_{L_\beta}$$
$$\leqslant (1+n^3+nq)^\beta \lambda(e+f)^{a_1+\ldots+a_{j-1}} (n^3+nq)^{r/2} \|f\|_{L^2(G)}$$
$$\leqslant (1+\|\psi\|_{L_\alpha} n^{-2})^n \|\psi\|_{L^2(G)} (1+n^3+nq)^{\beta+R/2}$$

and so, by (2.8), for a constant C depending on G and ψ only

$$\|(e+f)^{a_1} s_1^{b_1} * \ldots * (e+f)^{a_n} s_n^{b_n}\|_{M_\beta} \leqslant C(1+q)^{\beta+R/2} (2+n)^{3(\beta+R/2)+1}.$$

On the other hand, since

$$\Omega(q) \subset \bigcup_{b_j=1} \{(s_1, \ldots, s_n):\ s_j^{b_j} \in U^q \setminus U^{q-1}\},$$

by (2.2), we have

$$\int_{\Omega(q)} |g^{b_1}(s_1)| \ldots |g^{b_n}(s_n)|\, ds_1 \ldots ds_n$$
$$\leqslant \begin{cases} 0, & \text{if} \quad \boldsymbol{b} = 0 \text{ and } q > 1, \\ 1, & \text{if} \quad \boldsymbol{b} = 0 \text{ and } q = 1, \\ (n-m)\|g\|_{L^1(G)}^{n-m-1} \|g\|_{L_\alpha} (q-1)^{-\alpha}, & \text{if} \quad \boldsymbol{b} \neq 0. \end{cases}$$

Thus, since $\alpha > \beta + R/2 + 1$, for $\boldsymbol{b} \neq 0$ we have

$$\int \ldots \int \|(e+f)^{a_1} s_1 * \ldots * (e+f)^{a_n} s_n\|_{M_\beta} |g^{b_1}(s_1)| \ldots |g^{b_n}(s_n)|\, ds_1 \ldots ds_n$$
$$\leqslant C \|\psi\|_{L_\alpha} \|g\|_{L^1(G)}^{n-m-1} (2+n)^{3(\beta+R/2)+1} (n-m) \sum_{q=1}^{\infty} (1+q)^{\beta+R/2} (q-1)^{-\alpha}$$
$$\leqslant C' \|g\|_{L^1(G)}^{n-m-1} n^{3(\beta+R/2)+2},$$

where C' is independent of n, $\boldsymbol{a}$ and $\boldsymbol{b}$.

Hence, by (2.6),

$$\|(e+\psi)^{*n}\|_{M_\beta} \leqslant C' n^{3(\beta+R/2)+2} \sum_{m=1}^{n} \binom{n}{m} \|g\|_{L^1(G)}^{n-m-1}.$$

But, by (2.2),

$$\|g\|_{L^1(G)} \leqslant \|\psi\|_{L_\alpha} n^{-2},$$

whence

$$\|(e+\psi)^{*n}\|_{M_\beta} \leqslant C' n^{3(\beta+R/2)+2} \|\psi\|_{L_\alpha}^{-1} n^2 \sum_{m=0}^{n} \binom{n}{m} (\|\psi\|_{L_\alpha} n^{-2})^{n-m}$$
$$\leqslant C'' n^{3(\beta+R/2)+4} (1+\|\psi\|_{L_\alpha} n^{-2})^n,$$

which completes the proof of Theorem 2.1.

COROLLARY 2.2. *Suppose $\varphi = \varphi^* \in L_\alpha \cap L^2(G) \subset L_\beta$, $\lambda(\varphi) < a$, $\alpha > \beta + R/2 + 1$, $F \in C_c^k(-a, a)$ with $k > 3(\beta + R/2 + 2)$ and $F(0) = 0$, then F operates on φ in L and there exists a measure μ in M_β such that*

$$F \cdot \varphi = \varphi * \mu.$$

Proof. We write $\psi = e(\varphi)$ and

$$(e+\psi)^{*n} - e = \psi * [e + (e+\psi) + \ldots + (e+\psi)^{*n-1}].$$

Of course,

$$\psi = \varphi * v, \quad \text{where} \quad v = i \sum_{j=0}^{\infty} \frac{(i\varphi)^{*j}}{(j+1)!} \in M_\alpha \subset M_\beta.$$

Hence, if

$$\mu_n = v * [e + (e+\psi) + \ldots + (e+\psi)^{*n-1}],$$

by Theorem 2.1,

$$\|\mu_n\|_{M_\beta} \leqslant C n^{3(\beta+R/2)+5},$$

whence, by assumption on F and (2.1), we obtain the result with

$$\mu = \sum_{n \in \mathbf{Z}} \hat{F}(n) \mu_n.$$

The following corollary is an immediate consequence of Corollary 2.2 and (1.8).

COROLLARY 2.3. *If $\varphi = \varphi^* \in \mathscr{S}_{\alpha,d,1}(G) \cap L^2(G)$ with $\lambda(\varphi) < a$, $\alpha > \beta + R/2 + 1$ and $F \in C_c^k(-a, a)$ with $k > 3(\beta + R/2 + 2)$ and $F(0) = 0$, then F operates on φ in L_β and $F \cdot \varphi \in \mathscr{S}_{\beta,d,1}$ and the map*

$$C^k[-a, a] \ni F \to F \cdot \mathscr{S}_{\beta,d,1} \quad \textit{is continuous.}$$

On a nilpotent Lie group G (or more generally a Lie group of polynomial growth) let L be a hypoelliptic, left invariant differential operator which satisfies (0.1). Moreover, let the semi-group T^t defined by (0.2) satisfy (0.4).

For a function $m \in C^k(\boldsymbol{R}^+)$ we write

$$\|m\|_{k,k'} = \sup\{(1+\lambda)^{k'}|m^{(j)}(\lambda)|:\ j = 0, \ldots, k,\ \lambda \in \boldsymbol{R}^+\}$$

and by $\mathscr{S}_{k,k'}(\boldsymbol{R}^+)$ we denote the space of functions m for which $\|m\|_{k,k'}$ is finite.

Let, finally, T_m be the operator defined by (0.3) for a bounded function m on $\boldsymbol{R}^+$ and let S and $s(d, k)$ be the numbers of (0.1) and (1.10).

Then the following theorem holds.

THEOREM 2.4. *Given β and d. If $k \geqslant 3(\beta+R/2+2)$ and $k' \geqslant 2+3(\beta+R/2+2)(s(2d, S)+S)$, then $m \in \mathscr{S}_{k,k'}(\boldsymbol{R}^+)$ implies that $T_m f = f * M$, where $M \in \mathscr{S}_{\beta,d,1}(G)$ and $\|M\|_{\beta,d,1} \leqslant C\|m\|_{k,k'}$, where C is independent of m in $\mathscr{S}_{k,k'}(\boldsymbol{R}^+)$.*

First we prove

LEMMA 2.5. *Let*

$$K(x) = \int_0^\infty e^{-t}\varphi_t(x)\,dt. \tag{2.9}$$

Then for every $\alpha \geqslant 0$

$$\int |K(x)|\,w^\alpha(x)\,dx < \infty \tag{2.10}$$

and, if $l \geqslant s(2d, S)+S$,

$$K^{*l} \in S_{\alpha,d,1} \quad \text{for all } \alpha \geqslant 0. \tag{2.11}$$

Proof. It is easy to verify (cf. e.g. [5], Lemma 5.1) that if a function $\varphi\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ satisfies

$$\begin{gathered}\varphi(s+t) \leqslant C\big(\varphi(s)+\varphi(t)\big), \qquad \varphi(s+t) \leqslant \varphi(s)\,\varphi(t),\\ \sup\{\varphi(t):\ t \in (0, 1]\} < \infty,\end{gathered} \tag{2.12}$$

then for constants C' and k

$$\varphi(t) \leqslant C'(1+t)^k.$$

To verify (2.10) we note that by (1.3) and (1.4) the function

$$\varphi(t) = \langle|\varphi_t|, w^\alpha\rangle$$

satisfies (2.12). Hence

$$\int |K(x)|\,w^\alpha(x)\,dx \leqslant \int_0^\infty e^{-t}\langle|\varphi_t|, w^\alpha\rangle\,dt \leqslant C'\int_0^\infty e^{-t}(1+t)^k\,dt < \infty$$

and (2.10) follows.

To prove (2.11) we note first that by (2.9), in virtue of (0.2) and (0.4)

$$(1+L)\,u * K = u \quad \text{for} \quad u \in C_c^\infty(G).$$

From this it follows that

$$X^I K^{*l} \in L^2(G) \quad \text{for} \quad |I| \leqslant d \quad \text{and} \quad l \geqslant s(d, S). \tag{2.13}$$

In fact, by (0.1) and (1.10), since $u * X^I K^{*l} = X^I(u * K^{*l})$,

$$\begin{aligned}|\langle u^*, X^I K^{*l}\rangle| &= |u * X^I K^{*l}(e)| \leqslant C\,\|(1+L)^S(u * X^I K^{*l})\|_{L^2(G)}\\ &\leqslant C'\,\|(1+L)^{s(d,S)}(u * K^{*l})\|_{L^2(G)} \leqslant C''\,\|u^*\|_{L^2(G)}.\end{aligned}$$

In particular, $K^{*S} \in L^2(G)$ and so

$$X^I K^{*l} \in L^\infty(G) \quad \text{for} \quad l \geqslant s(d, S)+S. \tag{2.14}$$

We have to show that

$$\int |X^I K^{*l}(x)|\,w^\alpha(x)\,dx < \infty \quad \text{for} \quad l \geqslant s(2d, S)+S \quad \text{and} \quad |I| \leqslant d. \tag{2.15}$$

Let f be a fixed function in $C_c^\infty(G)$ such that $f(x) = f(x^{-1}) \geqslant 0$ and $\int f(x)\,w^{-4\alpha}(x)\,dx = 1$. Let

$$w'(x) = w^{4\alpha} * f(x).$$

Then, by (1.3),

$$\begin{aligned}w'(x) &\geqslant w^{4\alpha}(x)\int w^{-4\alpha}(y) f(y)\,dy = w^{4\alpha}(x),\\ w'(x) &\leqslant w^{4\alpha}(x)\int w^{4\alpha}(y) f(y)\,dy = Cw^{4\alpha}(x).\end{aligned} \tag{2.16}$$

Also for $X \in \boldsymbol{g}$,

$$|Xw'(x)| = |w^{4\alpha} * Xf| \leqslant w^{4\alpha}(x)\int w^{4\alpha}(y)\,|Xf(y)|\,dy = C_X\,w^{4\alpha}(x). \tag{2.17}$$

Now we proceed by induction on $|I|$. For $|I| = 0$ (2.15) is simply (2.10). Suppose $X^I = X_j X^J$ with $|J| < d$. We may also assume that α is so big that

$$\int w^{-2\alpha}(x)\,dx < \infty.$$

We have

$$\begin{aligned}\langle|X^I K^{*l}|, w^\alpha\rangle &= \langle|X^I K^{*l}|\,w^{-3\alpha}, w^{4\alpha}\rangle\\ &\leqslant \langle(X^I K^{*l})^2, w^{4\alpha}\rangle^{1/2}\big(\textstyle\int w^{-2\alpha}(x)\,dx\big)^{1/2}.\end{aligned}$$

But, by (2.16) and (2.17),

$$\begin{aligned}\langle(X^I K^{*l})^2, w^{4\alpha}\rangle &\leqslant \langle(X^I K^{*l})^2, w'\rangle = \langle X^I K^{*l}, (X_j X^J K^{*l})\,w'\rangle\\ &\leqslant |\langle X^I K^{*l}, X_j(X^J K^{*l} w')\rangle| + |\langle X^I K^{*l}, X^J K^{*l} X_j w'\rangle|\\ &\leqslant C\,\langle|X_j X^I K^{*l}|, |X^J K^{*l}|\,w'\rangle + C_{X_j}\langle|X^I K^{*l}|, |X^J K^{*l}|\,w'\rangle\\ &\leqslant C\,\langle|X_j X^I K^{*l}|, |X^J K^{*l}|\,w^{4\alpha}\rangle + CC_{X_j}\langle|X^I K^{*l}|, |X^J K^{*l}|\,w^{4\alpha}\rangle.\end{aligned}$$

Since $1+|I| \leqslant 2d$, by (2.14), both $X_j X^I K^{*l}$ and $X^I K^{*l}$ belong to $L^1(G)$ and, by inductive hypothesis, $|X^J K^{*l}| w^{4\alpha}$ belongs to $L^1(G)$. Thus (2.15) follows and the proof of Lemma 2.5 is complete.

Proof of Theorem 2.4. Let

$$F(\xi) = \begin{cases} m(\xi^{-1/l}-1) & \text{for} \quad 0 < \xi \leqslant 1, \\ 0 & \text{for} \quad -\pi \leqslant \xi \leqslant 0. \end{cases}$$

It is clear that F can be extended to a function in $C_c^k[-\pi, \pi]$ if and only if

$$m \in C^k(\boldsymbol{R}^+) \quad \text{and} \quad \lim_{\lambda \to \infty} (1+\lambda)^{1+jl} m^{(j)} = 0 \quad \text{for} \quad j = 0, \ldots, k.$$

By Lemma 2.5, if $l \geqslant s(2d, S)+S$, then $K^{*l} \in \mathscr{S}_{\alpha,d,1}(G)$ for all $\alpha \geqslant 0$, and, by Corollary 2.3, if $k > 3(\beta+R/2+2)$, then F operates on K^{*l} and

$$M = F \cdot K^{*l} \in \mathscr{S}_{\beta,d,1}(G).$$

In other words, if $k' \geqslant 2+3(\beta+R/2+2)(s(2d, S)+S)$, then $m \in \mathscr{S}_{k,k'}(\boldsymbol{R}^+)$ implies that $M \in \mathscr{S}_{\beta,d,1}(G)$. But an easy calculation shows that

$$\hat{M}(\lambda) = F\left((1+\lambda)^{-l}\right) = m(\lambda),$$

which completes the proof.

Another application of the functional calculus is the following

THEOREM 2.6. *Suppose on a Lie group of polynomial growth*

$$T^t f = f * \varphi_t, \qquad \varphi_t = \varphi_t^* \in \mathscr{S}(G)$$

is a semi-group of operators strongly continuous on $L^2(G)$. *Then the mapping*

$$\boldsymbol{R}^+ \ni t \to \varphi_t \in \mathscr{S}(G)$$

extends to a holomorphic map

$$\{z\colon \operatorname{Re} z > 0\} \ni z \to \varphi_z \in \mathscr{S}(G).$$

Proof. For a fixed β we let A to be the commutative $*$-subalgebra of L_β generated by φ_t, $t > 0$. As we know, the Gelfand space of A is $\boldsymbol{R}^+$ and $\hat{\varphi}_t(\lambda) = e^{-t\lambda}$.

For a fixed number $a > 0$ we let

$$\boldsymbol{C}_a = \{z \in \boldsymbol{C}\colon \operatorname{Re} z > a\}.$$

Let $k > 3(\beta+R/2+2)$ and let

$$m = [k/a]+1.$$

For z in $\boldsymbol{C}_a$ we select a function $F_z \in C_c^k[-\pi, \pi]$ in such a way that

$$F_z(x) = x^{mz} \quad \text{for} \quad 0 < x \leqslant 1$$

and the map

$$\boldsymbol{C}_a \ni z \to F_z \in C^k[-\pi, \pi] \quad \text{is continuous.} \tag{2.18}$$

Since $\varphi_{1/m} \in \mathscr{S}(G)$, by Corollary 2.3, $F_z \cdot \varphi_{1/m} \in \mathscr{S}_{\beta,l,1}(G)$ for all l and, moreover, by (2.18), the map

$$\boldsymbol{C}_a \ni z \to F_z \cdot \varphi_{1/m} \in \mathscr{S}_{\beta,l,1} \subset L_\beta \quad \text{is continuous.} \tag{2.19}$$

We put

$$\varphi_z = F_z \cdot \varphi_{1/m}$$

and we see that

$$\hat{\varphi}_z(\lambda) = e^{-z\lambda}$$

which shows that φ_z does not depend on the selection of m and the map $z \to \varphi_z$ is an extension of $t \to \varphi_t$, $t > a$.

By (2.19), for every C^1 curve γ in $\boldsymbol{C}_a$ we have

$$\int_\gamma \varphi_z \, dz \in \mathscr{S}_{\beta,l,1}(G) \subset L_\beta$$

and for every $\lambda \geqslant 0$

$$\int_\gamma \hat{\varphi}_z(\lambda)\, dz = 0, \quad \text{whence} \quad \int_\gamma \varphi_z \, dz = 0$$

which shows that $\boldsymbol{C}_a \ni z \to \varphi_z \in \mathscr{S}_{\beta,l,1}(G)$ is holomorphic. Since a, l, β are arbitrary, the proof is complete.

References

[1] J. Długosz, *Almost everywhere convergence of Riesz means of Laguerre expansions*, to appear.

[2] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, 1982.

[3] B. Helffer et J. Nourrigat, *Caractérisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie gradué*, Comm. in Partial Differential Equations, 4 (8) (1978), 899–958.

[4] A. Hulanicki, *Commutative subalgebra of $L^1(G)$ associated with a subelliptic operator on a Lie group*, Bull. Amer. Math. Soc. 81 (1975), 121–124.

[5] —, *A class of convolution semi-groups of measures on a Lie group*, Lecture Notes in Math. 828 (1980), 82–101.

[6] — and Joe W. Jenkins, *Almost everywhere summability on nilmanifolds*, Trans. Amer. Math. Soc. 278 (1983), 703–715.

[7] Joe W. Jenkins, *Dilations and gauges on nilpotent Lie groups*, Colloq. Math. 41 (1979), 91–101.

[8] Y. Katznelson, *An introduction to harmonic analysis*, John Wiley and Sons, Inc., 1968.

[9] T. Pytlik, *On the spectral radius of elements in group algebras*, Bull. Acad. Polon. Sci. 21 (1973), 899–902.
[10] E. M. Stein, *Topics in harmonic analysis related to the Littlewood–Paley theory*, Ann. of Math. Studies, Princeton Univ. Press, Princeton 1970.

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An example of a continuum of pairwise non-isomorphic spaces of C^∞-functions

by

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Abstract. There is given a family (K_τ) of compact sets K_τ in the Euclidean plane with τ ranging in a real interval such that the Whitney spaces $\mathscr{E}(K_\tau)$ are pairwise non-isomorphic. A successful distinction of the topological structures which is sufficient for this result is managed by a certain topological property involving an increasing monotone function on $\boldsymbol{R}_+$. After Zaharyuta had presented a continuum of pairwise non-isomorphic spaces of analytic functions the open question for an analogous example in the frame of C^∞-functions is clarified by a positive answer.

In papers [2], [3], [8] and [9], e.g., linear topological invariants or special properties of locally convex spaces were used to distinguish the topological structures of (F)-spaces. Zaharyuta presented in his paper [9] a continuum of pairwise non-isomorphic spaces of analytic functions, i.e. a family $\{G_\tau\}$ of domains G_τ such that τ ranges in $[0, 1]$, e.g., and $\mathscr{O}(G_\tau)$ is not isomorphic to $\mathscr{O}(G_\sigma)$ for different σ, τ. The existence of such a continuum of spaces of C^∞-functions, however, is an open problem till now.

This paper gives an affirmative answer to this question by presenting a family $\{K_\tau |\ \tau \in [a, b]\}$ of compact sets K_τ in $\boldsymbol{R}^2$ which describes a continuum of this kind consisting of Whitney spaces $\mathscr{E}(K_\tau)$. The method applied in the paper is different from Zaharyuta's one and – due to an idea of D. Vogt – makes use of certain properties of (F)-spaces called $(\mathbf{DN}_\varphi)$ here (cf. [1], [6]).

The sets K_τ are given by the graphs of monotonically increasing (real) analytic functions Φ_τ on $\boldsymbol{R}_+$ such that the family $\{\Phi_\tau\}$ is monotone in τ. The parameter τ plays an essential role only in the boundary behaviour of Φ_τ near the point 0. Larger values of τ cause extremely faster convergence of $\Phi_\tau(x)$ to 0 with $x \to 0+$. All the K_τ have not the extension property, i.e. there exists no extension operator from $\mathscr{E}(K_\tau)$ to $\mathscr{E}(\boldsymbol{R}^2)$ (see [4], Beispiel 2). This is a necessary consequence if the K_τ shall have interior points. Since if $K \subseteq \boldsymbol{R}^n$ has at least one interior point and has the extension property, then $\mathscr{E}(K)$ is isomorphic to the space s of rapidly decreasing functions (see [7], Satz 4.1).

Definition 1 (cf. [6]). Let φ: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a monotonically increasing function. An (F)-space E is said to have the *property* $(\mathbf{DN}_\varphi)$ if the following