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Received June 21, 1983 (1772)

STUDIA MATHEMATICA, T. LXXVIII. (1984)

An example of a continuum of pairwise non-isomorphic spaces of C^∞-functions

by

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Abstract. There is given a family (K_τ) of compact sets K_τ in the Euclidean plane with τ ranging in a real interval such that the Whitney spaces $\mathscr{E}(K_\tau)$ are pairwise non-isomorphic. A successful distinction of the topological structures which is sufficient for this result is managed by a certain topological property involving an increasing monotone function on $\boldsymbol{R}_+$. After Zaharyuta had presented a continuum of pairwise non-isomorphic spaces of analytic functions the open question for an analogous example in the frame of C^∞-functions is clarified by a positive answer.

In papers [2], [3], [8] and [9], e.g., linear topological invariants or special properties of locally convex spaces were used to distinguish the topological structures of (F)-spaces. Zaharyuta presented in his paper [9] a continuum of pairwise non-isomorphic spaces of analytic functions, i.e. a family $\{G_\tau\}$ of domains G_τ such that τ ranges in $[0, 1]$, e.g., and $\mathscr{O}(G_\tau)$ is not isomorphic to $\mathscr{O}(G_\sigma)$ for different σ, τ. The existence of such a continuum of spaces of C^∞-functions, however, is an open problem till now.

This paper gives an affirmative answer to this question by presenting a family $\{K_\tau |\ \tau \in [a, b]\}$ of compact sets K_τ in $\boldsymbol{R}^2$ which describes a continuum of this kind consisting of Whitney spaces $\mathscr{E}(K_\tau)$. The method applied in the paper is different from Zaharyuta's one and — due to an idea of D. Vogt — makes use of certain properties of (F)-spaces called $(\mathbf{DN}_\varphi)$ here (cf. [1], [6]).

The sets K_τ are given by the graphs of monotonically increasing (real) analytic functions Φ_τ on $\boldsymbol{R}_+$ such that the family $\{\Phi_\tau\}$ is monotone in τ. The parameter τ plays an essential role only in the boundary behaviour of Φ_τ near the point 0. Larger values of τ cause extremely faster convergence of $\Phi_\tau(x)$ to 0 with $x \to 0+$. All the K_τ have not the extension property, i.e. there exists no extension operator from $\mathscr{E}(K_\tau)$ to $\mathscr{E}(\boldsymbol{R}^2)$ (see [4], Beispiel 2). This is a necessary consequence if the K_τ shall have interior points. Since if $K \subseteq \boldsymbol{R}^n$ has at least one interior point and has the extension property, then $\mathscr{E}(K)$ is isomorphic to the space s of rapidly decreasing functions (see [7], Satz 4.1).

Definition 1 (cf. [6]). Let φ: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a monotonically increasing function. An (F)-space E is said to have the *property* $(\mathbf{DN}_\varphi)$ if the following assertion is true for an increasing system $(|\ |_n)$ of semi-norms describing the topology of E:

$$\exists q\in N\ \forall k\in N\ \exists j,\ n\in N,\ C>0\ \forall r>0,\qquad |\ |_k \leqslant r^j |\ |_q + C\varphi(C/r)|\ |_n.$$

Remark. Obviously, this definition is independent of the choice of the system $(|\ |_n)$ and thus property $(\mathbf{DN}_\varphi)$ is a topological invariant.

Definition 2. Let $\Phi\colon [0,1]\to \boldsymbol{R}_+$ be a monotonically increasing C^1-function with the property $\Phi(x)<x$ for every $x\in]0,1]$. Then the appropriate generalized cone D_Φ is defined to be the following set:

$$D_\Phi := \{(x,y)\in \boldsymbol{R}^2|\ 0\leqslant x\leqslant 1;\ 0\leqslant y\leqslant \Phi(x)\}.$$

Proposition 1. *Let $\Phi\colon \boldsymbol{R}_+\to\boldsymbol{R}_+$ be a convex monotone bijection which has on the interval $[0,1]$ the properties of Definition 2. Then the (F)-space $\mathscr{E}(D_\Phi)$ has property $(\mathbf{DN}_\varphi)$ for the function $\varphi := \Phi^{-1}$.*

Proof. By the continuous differentiability of Φ, Whitney's condition is satisfied by the compact set D_Φ and so the following norms describe the topology of $\mathscr{E}(D_\Phi)$:

$$|f|_m := \sup_{\substack{z\in D\\ |\alpha|\leqslant m}} |f^{(\alpha)}(z)|;\qquad m=0,1,2,\ldots$$

In any case, let $C(n)$ denote a suitable increasing function of the natural numbers n. During a series of modifications of $C(n)$ we do not change the notation $C(n)$.

(i) Let d_Φ denote the following subset of D_Φ:

$$d_\Phi := \{(x,y)\in D_\Phi|\ x\leqslant 1/2\}.$$

If $g\in\mathscr{E}(d_\Phi)$ is such that $|g|_0\leqslant 1$ and $|g|_m\leqslant N\geqslant 1$, then by Lemma 1 in [5] (cf. [4], Lemma 4.1) we have for $\mu=1,\ldots,m-1$:

$$|g^{(0,\mu)}(x,0)|\leqslant C(m)\big(N^{\mu/m}+\Phi(x)^{-\mu}\big);\ x\in]0,\tfrac{1}{2}]. \tag{1}$$

If $f\in\mathscr{E}(d_\Phi)$ admits the bounds $|f|_0\leqslant 1$ and $|f|_n\leqslant M\geqslant 1$, then by the same lemma we have the estimates

$$|f^{(\lambda,0)}|_0\leqslant C(n)M^{\lambda/n},\qquad \lambda=1,\ldots,n-1.$$

In the following considerations we fix integers n, k and a pair $\alpha=(\lambda,\mu)\in N_0^2$ and assume $0<|\alpha|\leqslant k<n$ with $|\alpha|$ defined as $|\alpha| := \lambda+\mu$. For later use, we write $C_1(n) := C(n)^n$ with the $C(n)$ obtained in this moment, assuming further that $C(n)\geqslant 1$. Let $M>C_1(n)$ and $f\in\mathscr{E}(d_\Phi)$ admit the bounds previously required. By the substitutions

$$g := \frac{1}{C(n)M^{\lambda/n}}f^{(\lambda,0)},\qquad N := \frac{M}{C(n)M^{\lambda/n}},\qquad m := n-\lambda,$$

the estimate (1) reads

$$\frac{1}{C(n)M^{\lambda/n}}|f^{(\lambda,\mu)}(x,0)|\leqslant C(m)\big[C(n)^{-\mu/m}M^{(1-\lambda/n)\mu/m}+\Phi(x)^{-\mu}\big]. \tag{2}$$

Estimating the exponent of M

$$\frac{\lambda}{n}+\left(1-\frac{\lambda}{n}\right)\frac{\mu}{m}=\frac{1}{n}(\lambda+\mu)\leqslant\frac{k}{n},$$

we draw from (2)

$$|f^{(\lambda,\mu)}(x,0)|\leqslant C(n)\big(M^{k/n}+\Phi(x)^{-\mu}M^{\lambda/n}\big)$$

with a new meaning for $C(n)$, of course.

If $M^{1/n}\leqslant\Phi(x)^{-1}$, the term $\Phi(x)^{-\mu}M^{\lambda/n}$ is estimated by $\Phi(x)^{-k}$, in the other cases $M^{1/n}>\Phi(x)^{-1}$, and the term is estimated by $M^{(\lambda+\mu)/n}\leqslant M^{k/n}$. Thus, in both cases we obtain

$$|f^{(\lambda,\mu)}(x,0)|\leqslant C(n)\big(M^{k/n}+\Phi(x)^{-k}\big). \tag{3}$$

(ii) In the notations of (i) we have $k+1\leqslant n$ and therefore $|f^{(\lambda+1,\mu)}|\leqslant M$, which gives the estimate

$$|f^{(\alpha)}(0,0)-f^{(\alpha)}(x,0)|\leqslant xM.$$

In connection with (3) the following estimate results for every $x\in]0,\frac{1}{2}]$:

$$|f^{(\alpha)}(0,0)|\leqslant xM+C(n)\big(M^{k/n}+\Phi(x)^{-k}\big). \tag{4}$$

For every positive x the inequality

$$M^{k/n}\leqslant xM+x^{-\gamma},\qquad \gamma := k/(n-k)$$

is valid. For if $x<M^{-1/(1+\gamma)}$, the term $x^{-\gamma}$ majorizes $M^{\gamma/(1+\gamma)}=M^{k/n}$ and in the other case, $M^{k/n}=M^{1-1/(1+\gamma)}$ is majorized by xM.

By assumption $n-k\geqslant 1$ and $x^{-\gamma}\leqslant x^{-k}\leqslant\Phi(x)^{-k}$ for $x\in]0,1]$. Thus (4) gives

$$|f^{(\alpha)}(0,0)|\leqslant C(n)\big(xM+\Phi(x)^{-k}\big). \tag{5}$$

If $r\geqslant C(n)^{1/k}\Phi(\frac{1}{2})^{-1}$ with $C(n)$ as in (5), then there exists $x\in]0,\frac{1}{2}]$ such that

$$C(n)\Phi(x)^{-k}=r^k$$

and so for this values of r the following is true for $C=C(n)$:

$$|f^{(\alpha)}(0,0)|\leqslant r^k+CM\varphi(C^{1/k}/r). \tag{6}$$

(iii) Let f be an arbitrary element of $\mathscr{E}(d_\Phi)$. If $|f|_n>C_1(n)|f|_0$ the function $|f|_0^{-1}f$ admits the bounds required in (i) with $M := |f|_0^{-1}|f|_n$, so (6) is valid for this function:

(7) $$|f^{(\alpha)}(0, 0)| \leqslant r^k |f|_0 + C\varphi(C^{1/k}/r)|f|_n, \quad r \geqslant C^{1/k}\Phi(\tfrac{1}{2})^{-1}.$$

If, on the other hand, $|f|_n \leqslant C_1(n)|f|_0$, for every $r \geqslant C_1^{1/k}$ we have

$$|f^{(\alpha)}(0, 0)| \leqslant |f|_n \leqslant C_1 |f|_0 \leqslant r^k |f|_0$$

and so (7) is valid in any case for large r depending on n.

(iv) Let $f \in \mathscr{E}(D_\Phi)$ and assume first that $(x, y) \in d_\Phi$. By convexity of Φ there exists a translation $\tilde{d}_\Phi$ of d_Φ such that the point $(0, 0)$ proceeds to (x, y) and $\tilde{d}_\Phi \subseteq D_\Phi$. We can apply the result of (iii) to the restriction of f to $\tilde{d}_\Phi$, yielding

(8) $$|f^{(\alpha)}(x, y)| \leqslant r^k |f|_0 + C\varphi(C/r)|f|_n \quad \forall r \geqslant C_2$$

with suitable $C_2 = C_2(n)$ and $C = C(n)$, by monotonicity of φ.

There exist a linear transformation A of $\boldsymbol{R}^2$ and a family $(U_\nu)_\nu$ of orthogonal transformations of $\boldsymbol{R}^2$ composed with translations such that all the images $P_\nu = U_\nu \circ A(Q)$ of the square $Q := [0, 1]^2$ are enclosed in D_Φ and constitute a covering of $D_\Phi - d_\Phi$. By simple estimations which reduce the problem to the one-dimensional case we first see that for a suitable constant C depending on n only

$$|g|_k \leqslant C |g|_0^{(n-k)/k} |g|_n^{k/n}$$

for all $g \in \mathscr{E}(Q)$ and $k = 0, \ldots, n$. By iterated application of the chain rule the norm $|f|_k$ can be estimated for every $f \in \mathscr{E}(P_\nu)$ by $C_k |g|_k$ with $g = f \circ U_\nu \circ A$ and a constant $C_k = C(k, A)$. So we obtain for $f \in \mathscr{E}(D_\Phi)$, $|\alpha| \leqslant k$, $z \in D_\Phi - d_\Phi$ by a choice of a ν with $z \in P_\nu$:

$$|f^{(\alpha)}(z)| \leqslant C_k |g|_k \leqslant C_k C |g|_0^{(n-k)/n} |g|_n^{k/n} \leqslant C C_k C_n^{k/n} |f|_0^{(n-k)/n} |f|_n^{k/n},$$

using the same argument in the estimation of $|g|_n$. With a new meaning for C this gives

(9) $$|f^{(\alpha)}(z)| \leqslant C |f|_0^{(n-k)/n} |f|_n^{k/n}$$

for $f \in \mathscr{E}(D_\Phi)$, $z \in D_\Phi - d_\Phi$, $|\alpha| \leqslant k$.

By an elementary calculation the right-hand side of (9) can be estimated by

$$C \cdot (r^k |f|_0 + r^{k-n} |f|_n)$$

for every $r > 0$. For $r > 1$ we have by assumption $\Phi(1/r) < 1/r$ and

$$r^{k-n} \leqslant r^{-1} \leqslant \varphi(1/r).$$

So (9) turns to

(10) $$|f^{(\alpha)}(x, y)| \leqslant C \cdot (r^k |f|_0 + \varphi(C/r)|f|_n)$$

for $r \geqslant 1$. By the substitution $r \mapsto C^{-1/k}\varrho$ and modification of C the estimation (8) is generalized to all $(x, y) \in D_\Phi$. Further replacement of C generalizes the

estimation to all $r > 0$ yielding:

(11) $$|f|_k \leqslant r^k |f|_0 + C\varphi(C/r)|f|_n.$$

Choosing $q = 0$, $j = k$, $n = k+1$ and C as in (11), property $(\mathbf{DN}_\varphi)$ is proved.

PROPOSITION 2. *Let Φ and ψ be two functions belonging to $C^1[0, \infty[$ and having the properties in the assumptions of Proposition 1. Assume further that*

$$\forall \lambda, p \in \boldsymbol{N}\ \exists x_1 > 0\ \forall x \in]0, x_1]\quad \psi(x) \leqslant \Phi(x^p)^\lambda.$$

Then $\mathscr{E}(D_\psi)$ does not have property $(\mathbf{DN}_{\Phi^{-1}})$.

Proof. Let $\varphi := \Phi^{-1}$ and assume property $(\mathbf{DN}_\varphi)$ for the space $\mathscr{E}(D_\psi)$. Thus, there exist natural numbers q, j and $n > q+1$ and a positive C such that

(12) $$|f|_{q+1} \leqslant r^j |f|_q + C\Phi^{-1}(C/r)|f|_n$$

for every $r > 0$ and $f \in \mathscr{E}(D_\psi)$.

Because of the density of $\mathscr{E}(D_\psi)$ in $\mathscr{E}^n(D_\psi)$ the inequality (12) is valid for every $f \in \mathscr{E}^n(D_\psi)$ and $r > 0$.

We denote by $P_\nu \in \mathscr{E}^n(\boldsymbol{R})$ and $f_\nu \in \mathscr{E}^n(D_\psi)$ the following functions, for $\nu = 1, 2, 3, \ldots$,

$$P_\nu(x) := \begin{cases} \dfrac{1}{(n+1)!}(1 - x\nu)^{n+1} & \text{if } 0 \leqslant x \leqslant 1/\nu, \\ 0 & \text{if } x > 1/\nu, \end{cases}$$

$$f_\nu(x, y) := \frac{1}{(q+1)!} y^{q+1} P_\nu(x).$$

Let $\alpha = (\lambda, \mu) \in \boldsymbol{N}_0^2$ and $f_\nu^{(\alpha)}(x, y) \neq 0$; this causes the restrictions $\mu \leqslant q+{}$ $+1$, $\lambda \leqslant n+1$ and $x < 1/\nu$. Then we have

$$|f_\nu^{(\alpha)}(x, y)| = \frac{y^{q+1-\mu}}{(q+1-\mu)!} \cdot \frac{|-\nu|^\lambda (1 - \nu x)^{n+1-\lambda}}{(n+1-\lambda)!} \leqslant \psi(x)^{q+1-\mu} \nu^\lambda$$
$$= \psi(x)^{q+1} \nu^{|\alpha|} (\nu\psi(x))^{-\mu}.$$

In the case where $|\alpha| \leqslant q$, the greatest possible value for μ is $|\alpha|$; because of $\nu\psi(x) \leqslant \psi(x)/x < 1$ we have

$$|f_\nu^{(\alpha)}(x, y)| \leqslant \psi(x)^{q+1} (\nu\psi(x))^{-|\alpha|} \nu^{|\alpha|} = \psi(x)^{q+1-|\alpha|} \leqslant \psi(x) \leqslant \psi(1/\nu)$$

for arbitrary $(x, y) \in D_\psi$ and $\alpha \in \boldsymbol{N}_0^2$ with $|\alpha| \leqslant q$, whence the inequality

(13) $$|f_\nu|_q \leqslant \psi(1/\nu)$$

results for $\nu = 1, 2, 3, \ldots$

If $q+1 \leqslant |\alpha| \leqslant n$, the greatest possible value for μ is $q+1$ and in this case we have

$$|f_v^{(\alpha)}(x, y)| \leqslant v^{|\alpha|-q-1} \leqslant v^{n-q-1}$$

and the following estimation is valid for $v = 1, 2, 3, \ldots$

$$|f_v|_n \leqslant v^{n-q-1}. \tag{14}$$

Calculating $f_v^{(0,q+1)}(0, 0)$, we get from (12), (13), and (14):

$$\frac{1}{(n+1)!} \leqslant r^j \psi(1/v) + C\Phi^{-1}(C/r)\, v^{n-q-1} \tag{15}$$

for every $r > 0$, $v \in \boldsymbol{N}$.

Choosing $\lambda = j+1$ and $p = n-q$ by hypothesis there exists a positive x_1 such that for all $v > x_1^{-1}$ we obtain by (15)

$$\frac{1}{(n+1)!} \leqslant r^j \Phi\left(\frac{1}{v^{n-q}}\right)^{j+1} + C\Phi^{-1}(C/r)\, v^{n-q-1}.$$

Setting $r = C\Phi(1/v^{n-q})^{-1}$ this estimation yields

$$\frac{1}{(n+1)!} \leqslant C^j \Phi\left(\frac{1}{v^{n-q}}\right) + C \cdot \frac{1}{v}$$

and gives a contradiction letting $v \to \infty$.

In the sequel we will produce a great number of pairs (Φ, ψ) as in Proposition 2. Let $\log^2$ denote the function $\log \circ \log$ from $]e, \infty[$ onto $\boldsymbol{R}_+$ and, similarly, $\exp^4$ the fourth iterate of exp from $\boldsymbol{R}$ into $\boldsymbol{R}_+$.

Lemma. *For the functions*

$$\tilde{\Phi}_\tau(y) := \exp^4(\tau \log^2 y); \quad y > e, \ \tau \geqslant 1, \tag{16}$$

the following assertion is true:

To every pair (p, λ) of natural numbers p, λ and $\sigma > \tau \geqslant 1$ there exists a number $y_0 > e$ such that the inequality

$$\tilde{\Phi}_\tau(y^p)^\lambda \leqslant \tilde{\Phi}_\sigma(y)$$

is valid for every $y \geqslant y_0$.

Proof. (i) First we prove the

Claim. $\tilde{\Phi}_\tau(y)^\lambda \leqslant \tilde{\Phi}_\tau(y^\lambda)$ for every $\lambda \geqslant 2$, $\tau \geqslant 1$, $y \geqslant \lambda$.

From the inequality $\lambda \leqslant \lambda^\tau$ we obtain by multiplication with $\exp(\tau z)$ and monotonicity of exp

$$[\exp^2(\tau z)]^\lambda \leqslant \exp^2(\tau z + \tau \log \lambda) \tag{17}$$

for arbitrary λ, $\tau \geqslant 1$ and $z \in \boldsymbol{R}$.

By assumptions we have $u := \exp^2(\tau \log^2 y) \geqslant y \geqslant \lambda$ and

$$\log \lambda < \lambda \leqslant \lambda(\lambda-1) \leqslant u(u-1) \leqslant u^\lambda - u. \tag{18}$$

Setting $z = \log^2 y$, the estimation (17) yields

$$u^\lambda \leqslant \exp^2(\tau \log^2 y + \tau \log \lambda) = \exp^2(\tau \log^2 y^\lambda),$$

whence by (18) and (17) with $\tau = 1$, $z = u$ we obtain

$$\tilde{\Phi}_\tau(y)^\lambda = (\exp^2 u)^\lambda \leqslant \exp^2(u + \log \lambda) \leqslant \exp^2 u^\lambda \leqslant \tilde{\Phi}_\tau(y^\lambda).$$

(ii) Assume $\lambda = 1$. Setting $y_0 := \exp(p^{\tau/(\sigma-\tau)})$ we get for $y \geqslant y_0$:

$$\log^2 y \geqslant \frac{\tau}{\sigma - \tau} \log p,$$

$$\sigma \log^2 y \geqslant \tau(\log^2 y + \log p) = \tau \log^2 y^p,$$

whence the claim of the lemma follows in this case.

(iii) Since λ is a natural number, $\lambda \geqslant 2$ in the remaining cases. Setting $y_0 := [\exp((\lambda p)^{\tau/(\sigma-\tau)})] \vee \lambda^{1/p}$, we get for $y \geqslant y_0$ by (i) and (ii):

$$\tilde{\Phi}_\tau(y^p)^\lambda \leqslant \tilde{\Phi}_\tau(y^{p\lambda}) \leqslant \tilde{\Phi}_\sigma(y).$$

Remark. If we set

$$\Phi_\tau(x) := \exp(-\exp^3(\tau \log^2 1/x)); \quad x \in]0, e^{-1}[, \ \tau > 1,$$

the relation $\Phi_\tau(x) = \tilde{\Phi}_\tau(1/x)^{-1}$ is valid for all $x < e^{-1}$ and Φ_τ is a convex C^∞-function on the interval $]0, e^{-1}[$.

Proof. Clearly, the Φ_τ are C^∞. The convexity is proved by an elementary calculation: The functions

$$g_\tau(x) := \exp(-(\log 1/x)^\tau), \quad F(\eta) := \exp(-e^{1/\eta})$$

have positive first and second derivatives on the interval $]0, e^{-1}[$ and g_τ takes its values in this interval too. Thus the composition $\Phi_\tau = F \circ g_\tau$ has a positive second derivative there.

Corollary. *There exists a continuum $\{K_\tau |\ \tau \in [a, b]\}$ of compact sets in $\boldsymbol{R}^2$ such that for each $\sigma, \tau \in [a, b]$ with $\sigma \neq \tau$ the (F)-spaces $\mathscr{E}(K_\sigma)$ and $\mathscr{E}(K_\tau)$ are not isomorphic.*

Proof. Let $1 < a < b$ and let Φ_τ denote the functions as in the remark. Clearly, beyond the conclusions of the remark, $\Phi_\tau(x) < x$ for all positive $x \leqslant e^{-1}$. Choose convex C^∞-extensions of the Φ_τ to all of $\boldsymbol{R}$, also denoted by Φ_τ, and which further satisfy the assumptions of Definition 2 and Proposition 1.

If $\tau < \sigma$, to every pair (p, λ) of natural numbers there exists by the lemma a positive y_1 such that

$$\tilde{\Phi}_\tau(y^p)^\lambda \leqslant \Phi_\sigma(y)$$

for $y > y_1$. By the relation in the remark for $x < y_1^{-1}$ we get

$$\Phi_\tau(x^p)^\lambda \geqslant \Phi_\sigma(x).$$

Applying Proposition 2 to $\Phi := \Phi_\tau$ and $\psi := \Phi_\sigma$, we see that $\mathcal{E}(D_{\Phi_\sigma})$ does not have property $(\mathbf{DN}_{\Phi_\tau^{-1}})$. But, due to Proposition 1, the space $\mathcal{E}(D_{\Phi_\tau})$ has this property and so the spaces are not isomorphic. Setting $K_\tau := D_{\Phi_\tau}$ for $\tau \in [a, b]$ the corollary is proved.

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Received July 2, 1982 (1778)

STUDIA MATHEMATICA, T. LXXVIII. (1984)

The canonical seminorm on Weak L^1

by

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Abstract. For each $f \in \text{Weak } L^1$ let $q_1(f) = \sup_{\alpha>0} \alpha\mu(\{x \mid |f(x)| > \alpha\})$ and let $q(f)$ be the seminorm,

$$q(f) = \inf_{f=f_1+f_2+\ldots+f_n} \sum_{j=1}^{n} q_1(f_j).$$

It is known that q is equivalent to the seminorm I defined by

$$I(f) = \lim_{n\to\infty} \{\sup_{b/a>n} (\log b/a)^{-1} \int_{\{x|a\leqslant|f(x)|\leqslant b\}} |f(x)|\, d\mu\}.$$

It is shown here that in fact $q(f) = I(f)$ and also that the normed quotient space of WeakL^1 generated by q is not complete.

0. Introduction. This note is a sequel to [1]. We shall assume familiarity with the terminology and notation of that paper in which it was shown that, for a non-atomic underlying measure space, the canonical seminorm q on the space of measurable functions WeakL^1 is equivalent to a more "concretely" defined seminorm I. We shall show here that the seminorms q and I are in fact equal, using a refinement of the argument in [1]. We also exhibit another two seminorms which are equivalent to q and show that W, the quotient space of WeakL^1 modulo the functions f satisfying $q(f) = 0$, is not complete, as incorrectly claimed in [1].

We gratefully acknowledge correspondence with Nigel Kalton who expressed doubts about the claim in [1].

1. Equality of q and I. In order to establish that $q(f) = I(f)$ for all $f \in \text{Weak}L^1$ it suffices to show that $q(f) \leqslant I(f)$ for each function of the form $f = \sum_{k=-\infty}^{\infty} \lambda^k \chi_{I_k}$, where $\lambda > 1$, and I_k are disjoint measurable sets of finite measure (cf. [1], pp. 151–152). In [1] the sets I_k were taken as intervals on the real line. However, for our purposes here it is a little simpler to consider them as (disjoint) circles. More specifically we assume that the underlying measure space (X, Σ, μ) contains each I_k and each Lebesgue measurable

* Research supported by the Technion V. P. R. Fund.