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Interpolation of locally Hölder operators

by

LECH MALIGRANDA (Poznań)

Dedicated to my Teacher, Professor Wladyslaw Orlicz on the occasion of his 80th birthday

Abstract. The paper gives some generalizations of non-linear interpolation theorems of Lions [4], Peetre [7], Tartar [8], Bona–Scott [2] and the author [5].

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Let A be any Banach space. We denote by $L^p_*(A)$ the quasi-Banach space of all (classes of) functions $a\colon (0, \infty) \to A$ strongly measurable with respect to the measure dt/t for which the norm $\|a(\cdot)\|_{L^p_*(A)}$ is finite, where

$$\|a(\cdot)\|_{L^p_*(A)} = \begin{cases} \left(\int_0^\infty \|a(t)\|_A^p \, dt/t\right)^{1/p}, & 0 < p < \infty, \\ \operatorname*{ess\,sup}_{0<t<\infty} \|a(t)\|_A, & p = \infty. \end{cases} \tag{1}$$

If A is a real or complex number system, for $L^p_*(A)$ we write briefly L^p_*.

Recall the K-method of interpolation [1]. Let A_0 and A_1 be Banach spaces such that $A_1 \hookrightarrow A_0$ (the symbol $\hookrightarrow$ denotes continuous inclusion). For $a \in A_0$ and $t > 0$ define

$$K(t, a) \equiv K(t, a; A_0, A_1) \equiv \inf\{\|a-b\|_{A_0} + t\|b\|_{A_1}\colon b \in A_1\}. \tag{2}$$

For each fixed $a \in A_0$, $K(t, a)$ is a continuous non-decreasing concave function of t.

For $0 < \theta < 1$ and $0 < p \leqslant \infty$, we denote by $(A_0, A_1)_{\theta,p} \equiv A_{\theta,p}$ a quasi-Banach space with quasi-norm

$$\|a\|_{\theta,p} = \|t^{-\theta} K(t, a)\|_{L^p_*}. \tag{3}$$

Then

$$A_1 \hookrightarrow A_{\theta_0,p_0} \hookrightarrow A_{\theta_1,p_1} \hookrightarrow A_0 \tag{4}$$

provided that $\theta_0 > \theta_1$ or that $\theta_0 = \theta_1$ and $p_0 \leqslant p_1$. Finally we introduce the convention that $A_{0,p} \equiv A_0$ for all p.

Proposition 1 (Peetre, cf. [1], Theorem 3.12.1). *Let $a \in A_0$, $0 < \theta < 1$ and $0 < p_0, p_1 \leqslant \infty$. Suppose that for all $t > 0$ there are $u_i(t) \in A_i$ such that $a = u_0(t) + u_1(t)$ with $t^{i-\theta} u_i(t) \in L^{p_i}_*(A_i)$ for $i = 0, 1$.*

Then $a \in A_{\theta,p}$ and

$$\|a\|_{\theta,p} \leqslant C \|t^{-\theta} u_0(t)\|^{1-\theta}_{L^{p_0}_*(A_0)} \|t^{1-\theta} u_1(t)\|^{\theta}_{L^{p_1}_*(A_1)}, \tag{5}$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$ and $0 < p_0, p_1 < \infty$ or $p_0 = p_1 = p = \infty$.

Proposition 2 (Density). *Suppose that $0 < \theta < 1$ and $0 < p < \infty$. Then A_1 is dense in $A_{\theta,p}$.*

We now prove a result to be used to derive our main theorem.

Lemma 3. *Let $\varepsilon > 0$, $a \in A_0$ and suppose $a(t) \in A_1$ is such that*

$$\|a - a(t)\|_{A_0} + t\|a(t)\|_{A_1} \leqslant (1+\varepsilon) K(t, a) \qquad \forall t > 0.$$

Then, for any $\lambda > 0$ and $0 \leqslant \theta < 1$, $0 < p \leqslant \infty$, such that $a \in A_{\theta,p}$, we have

$$\|a(t^\lambda)\|_{\theta,p} \leqslant (2+\varepsilon) \|a\|_{\theta,p} \tag{6}$$

and

$$\|a - a(t^\lambda)\|_{\theta,p} \leqslant (2+\varepsilon) \|a\|_{\theta,p}. \tag{7}$$

Proof. For $0 < \theta < 1$ it is sufficient to prove that, for all $s > 0$,

$$K(s, a(t^\lambda)) \leqslant (2+\varepsilon) K(s, a) \tag{6'}$$

and

$$K(s, a - a(t^\lambda)) \leqslant (2+\varepsilon) K(\min(s, t^\lambda), a). \tag{7'}$$

If 1° $0 < s \leqslant t^\lambda$, then

$$\begin{aligned} K(s, a(t^\lambda)) &\leqslant s\|a(t^\lambda)\|_{A_1} = s t^{-\lambda} t^\lambda \|a(t^\lambda)\|_{A_1} \\ &\leqslant (1+\varepsilon) s t^{-\lambda} K(t^\lambda, a) \leqslant (1+\varepsilon) K(s, a) \end{aligned}$$

and for all $b \in A_1$

$$\begin{aligned} K(s, a - a(t^\lambda)) &\leqslant \big\|a - a(t^\lambda) - (b - a(t^\lambda))\big\|_{A_0} + s\|b - a(t^\lambda)\|_{A_1} \\ &\leqslant \|a - b\|_{A_0} + s\|b\|_{A_1} + s\|a(t^\lambda)\|_{A_1}, \end{aligned}$$

i.e.,

$$\begin{aligned} K(s, a - a(t^\lambda)) &\leqslant K(s, a) + s\|a(t^\lambda)\|_{A_1} \\ &\leqslant K(s, a) + (1+\varepsilon) s t^{-\lambda} K(t^\lambda, a) \leqslant K(s, a) + (1+\varepsilon) K(s, a) \\ &= (2+\varepsilon) K(s, a). \end{aligned}$$

If 2° $s > t^\lambda$, then for all $b \in A_1$

$$K(s, a(t^\lambda)) \leqslant \|a(t^\lambda) - b\|_{A_0} + s\|b\|_{A_1} \leqslant \|a(t^\lambda) - a\|_{A_0} + \|a - b\|_{A_0} + s\|b\|_{A_1},$$

i.e.,

$$\begin{aligned} K(s, a(t^\lambda)) &\leqslant \|a(t^\lambda) - a\|_{A_0} + K(s, a) \leqslant (1+\varepsilon) K(t^\lambda, a) + K(s, a) \\ &\leqslant (2+\varepsilon) K(s, a) \end{aligned}$$

and

$$K(s, a - a(t^\lambda)) \leqslant \|a - a(t^\lambda)\|_{A_0} \leqslant (1+\varepsilon) K(t^\lambda, a).$$

For $\theta = 0$ we have

$$\begin{aligned} \|a(t^\lambda)\|_{A_0} &\leqslant \|a - a(t^\lambda)\|_{A_0} + \|a\|_{A_0} \leqslant (1+\varepsilon) K(t^\lambda, a) + \|a\|_{A_0} \\ &\leqslant (2+\varepsilon) \|a\|_{A_0} \end{aligned}$$

and

$$\|a - a(t^\lambda)\|_{A_0} \leqslant (1+\varepsilon) K(t^\lambda, a) \leqslant (1+\varepsilon) \|a\|_{A_0}.$$

Theorem 1. *Let $A_1 \hookrightarrow A_0$, $B_1 \hookrightarrow B_0$ and $0 \leqslant \mu < 1$, $0 < r \leqslant \infty$. Suppose T is a mapping such that*

(i) T: $A_{\mu,r} \to B_0$ and for $a, b \in A_{\mu,r}$

$$\|Ta - Tb\|_{B_0} \leqslant f(\|a\|_{\mu,r}, \|b\|_{\mu,r}, \|a-b\|_{\mu,r}) \|a - b\|^{\alpha_0}_{A_0}$$

and

(ii) T: $A_1 \to B_1$ and for $a \in A_1$

$$\|Ta\|_{B_1} \leqslant g(\|a\|_{\mu,r}) \|a\|^{\alpha_1}_{A_1},$$

where f: $\boldsymbol{R}^3_+ \to \boldsymbol{R}_+$ is continuous, non-decreasing in each variable and g: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ is continuous, non-decreasing. Then if $\theta > \mu$ or $\theta = \mu$ and $p \leqslant r$, T maps $A_{\theta,p}$ into $B_{\eta,q}$ and for $a \in A_{\theta,p}$

$$\|Ta\|_{\eta,q} \leqslant C h(\|a\|_{\mu,r}) \|a\|^{\alpha}_{\theta,p},$$

where

$$\eta = \theta\alpha/\alpha_1, \quad \alpha = (1-\eta)\alpha_0 + \eta\alpha_1, \quad p = \alpha q \quad \text{and} \quad h(t) = f(t, 2t, 2t)^{1-\eta} g(2t)^\eta. \tag{8}$$

Proof. Let $a \in A_{\theta,p}$ and $\varepsilon > 0$. For each $t > 0$ choose $a(t) \in A_1$ such that

$$\|a - a(t)\|_{A_0} + t\|a(t)\|_{A_1} \leqslant (1+\varepsilon) K(t, a).$$

Putting

$$Ta = b_0(t) + b_1(t), \qquad \text{where} \qquad b_1(t) = T(a(t^\lambda)),$$

$\lambda = \eta/\theta\alpha_0 = (1-\eta)/(1-\theta)\alpha_1$ we have by Lemma 3

$$\begin{aligned}\|b_0(t)\|_{B_0} &= \|Ta - T(a(t^\lambda))\|_{B_0}\\ &\leqslant f(\|a\|_{\mu,r}, \|a(t^\lambda)\|_{\mu,r}, \|a-a(t^\lambda)\|_{\mu,r})\|a-a(t^\lambda)\|_{A_0}^{\alpha_0}\\ &\leqslant f(\|a\|_{\mu,r}, (2+\varepsilon)\|a\|_{\mu,r}, (2+\varepsilon)\|a\|_{\mu,r})(1+\varepsilon)^{\alpha_0} K(t^\lambda, a)^{\alpha_0},\\ \|b_1(t)\|_{B_1} &= \|T(a(t^\lambda))\|_{B_1} \leqslant g(\|a(t^\lambda)\|_{\mu,r})\|a(t^\lambda)\|_{A_1}^{\alpha_1}\\ &\leqslant g((2+\varepsilon)\|a\|_{\mu,r})(1+\varepsilon)^{\alpha_1} t^{-\alpha_1\lambda} K(t^\lambda, a)^{\alpha_1}.\end{aligned}$$

Hence

$$\begin{aligned}&\int_0^\infty (t^{-\eta}\|b_0(t)\|_{B_0})^{p/\alpha_0}\frac{dt}{t}\\ &\leqslant f(\|a\|_{\mu,r}, (2+\varepsilon)\|a\|_{\mu,r}, (2+\varepsilon)\|a\|_{\mu,r})^{p/\alpha_0}(1+\varepsilon)^p \int_0^\infty (t^{-\eta/\alpha_0} K(t^\lambda, a))^p\frac{dt}{t}\\ &= \frac{(1+\varepsilon)^p}{\lambda} f(\|a\|_{\mu,r}, (2+\varepsilon)\|a\|_{\mu,r}, (2+\varepsilon)\|a\|_{\mu,r})^{p/\alpha_0}\|a\|_{\theta,p}^p,\end{aligned}$$

$$\begin{aligned}\int_0^\infty (t^{1-\eta}\|b_1(t)\|_{B_1})^{p/\alpha_1}\frac{dt}{t} &\leqslant (1+\varepsilon)^p g((2+\varepsilon)\|a\|_{\mu,r})^{p/\alpha_1}\int_0^\infty (t^{-\lambda+(1-\eta)/\alpha_1} K(t^\lambda, a))^p\frac{dt}{t}\\ &= \frac{(1+\varepsilon)^p}{\lambda} g((2+\varepsilon)\|a\|_{\mu,r})^{p/\alpha_1}\|a\|_{\theta,p}^p.\end{aligned}$$

Applying Proposition 1, we have $Ta \in B_{\eta,q}$ and

$$\begin{aligned}\|Ta\|_{\eta,q} &\leqslant C\|t^{-\eta}b_0(t)\|^{1-\eta}_{L_*^{p/\alpha_0}(B_0)}\|t^{1-\eta}b_1(t)\|^{\eta}_{L_*^{p/\alpha_1}(B_1)}\\ &\leqslant C(1+\varepsilon)^{p/q}\cdot\lambda^{-1/q} f(\|a\|_{\mu,r}, (2+\varepsilon)\|a\|_{\mu,r}, (2+\varepsilon)\|a\|_{\mu,r})^{1-\eta}\times\\ &\qquad\times g((2+\varepsilon)\|a\|_{\mu,r})^{\eta}\|a\|_{\theta,p}^{\alpha_0(1-\eta)+\alpha_1\eta}.\end{aligned}$$

From the continuity of f and g we have the theorem.

This theorem, in particular cases, was obtained by many authors:

$\mu = 0$ and $f = g = \text{const}$, Lions [4], Theorem 3.1, with $\alpha_0 = \alpha_1 = \alpha = 1$, and Peetre [7], Theorem 3.1;

$\mu = 0$ and $f(u, v, w) = C(u, v)$, Tartar [8], Theorem 2;

$f(u, v, w) = C(u+v)$ and $\alpha_0 = \alpha_1 = 1$, Bona and Scott [2], Theorem 1;

$f(u, v, w) = C(u, v)$, Maligranda [5], Theorem 7.4.

DEFINITION. Let $0 < \theta < 1$ and $0 < p \leqslant \infty$. We say that the pair A_0, A_1 has a (θ, p) *approximate identity* if there is a family of continuous mappings $S_t\colon A_{\theta,p} \to A_1$ for $0 < t \leqslant 1$, such that

1° $\|S_t a\|_{\theta,p} + t^{1-\theta}\|S_t a\|_{A_1} \leqslant C\|a\|_{\theta,p}$ for all $a \in A_{\theta,p}$ and $t \in (0, 1]$,

and

2° $\|S_t a - a\|_{\theta,p} + t^{-\theta}\|S_t a - a\|_{A_0} \to 0$ as $t \downarrow 0$ for $a \in A_{\theta,p}$, and uniformly on compact subsets of $A_{\theta,p}$.

EXAMPLE (Bona and Scott [2]). The pair $L^2(\boldsymbol{R})$, $H^m(\boldsymbol{R})$, where m is a positive integer, has a $(\theta, 2)$ approximate identity.

THEOREM 2. *Let $A_0, A_1, B_0, B_1, \mu, r, \eta, q$ and T be as in Theorem 1. Assume additionally that the pair A_0, A_1 has a (θ, p) approximate identity $\{S_t\}$ for some $\theta > \mu$ or $\theta = \mu$ and $p \leqslant r$, and that*

(iii) *T is continuous as a map of A_1 to B_1.*

Then T is a continuous map from $A_{\theta,p}$ to $B_{\eta,q}$.

Proof. (The proof of this theorem is analogous to that of [2], Theorem 2 or [5], Theorem 7.5.)

1° It is first demonstrated that $TS_t a \to Ta$ in $B_{\eta,q}$ as $t \downarrow 0$ uniformly for a in compact subsets of $A_{\theta,p}$.

For each $s > 0$, let $a(s) \in A_1$ be such that

$$\|a - a(s)\|_{A_0} + s\|a(s)\|_{A_1} \leqslant (1+\varepsilon)K(s, a).$$

For $\lambda = \eta/\theta\alpha_0$ and $t \in (0, 1]$ we define $b_1(s)$ by

$$b_1(s) = \begin{cases} TS_t a - T(a(s^\lambda)), & \text{if} \quad s^\lambda \leqslant t,\\ 0, & \text{if} \quad s^\lambda > t.\end{cases}$$

Then $b_1(s) \in B_1$ for each $s > 0$. Set $b_0(s) = TS_t a - Ta - b_1(s)$. For $s^\lambda \leqslant t$,

$$\|b_0(s)\|_{B_0} = \|T(a(s^\lambda)) - Ta\|_{B_0} \leqslant f(\|a\|_{\mu,r}, 2\|a\|_{\mu,r}, 2\|a\|_{\mu,r})K(s^\lambda, a)^{\alpha_0}.$$

For $s^\lambda > t$,

$$\begin{aligned}\|b_0(s)\|_{B_0} &= \|TS_t a - Ta\|_{B_0} \leqslant f(\|S_t a\|_{\mu,r}, \|a\|_{\mu,r}, \|S_t a - a\|_{\mu,r})\|S_t a - a\|_{A_0}^{\alpha_0}\\ &\leqslant f(CC'\|a\|_{\theta,p}, \|a\|_{\mu,r}, 2CC'\|a\|_{\theta,p})\|S_t a - a\|_{A_0}^{\alpha_0},\end{aligned}$$

where C is constant in condition 1° of the definition of an approximate identity and C' is the norm of the inclusion $A_{\theta,p} \hookrightarrow A_{\mu,r}$.

Thus for $p < \infty$

$$\begin{aligned}M_0(t)^{p/\alpha_0} &:= \int_0^\infty (s^{-\eta}\|b_0(s)\|_{B_0})^{p/\alpha_0}\frac{ds}{s}\\ &\leqslant \lambda^{-1}\cdot f(\|a\|_{\mu,r}, 2\|a\|_{\mu,r}, 2\|a\|_{\mu,r})^{p/\alpha_0}\int_0^t (s^{-\theta}K(s, a))^p\frac{ds}{s} +\\ &\quad+\frac{\alpha_0}{\eta p} f(CC'\|a\|_{\theta,p}, \|a\|_{\mu,r}, 2CC'\|a\|_{\theta,p})^{p/\alpha_0} t^{-\theta p}\|S_t a - a\|_{A_0}^p,\end{aligned}$$

i.e.,

$$M_0(t) \to 0 \quad \text{as} \quad t \downarrow 0, \quad \text{uniformly on compact subsets of } A_{\theta,p}.$$

Now consider $b_1(s)$. For $s^\lambda \leqslant t$,

$$\|b_1(s)\|_{B_1} = \|TS_t a - T(a(s^\lambda))\|_{B_1} \leqslant \|TS_t a\|_{B_1} + \|T(a(s^\lambda))\|_{B_1}$$
$$\leqslant g(\|S_t a\|_{\mu,r})\|S_t a\|_{A_1}^{\alpha_1} + g(\|a(s^\lambda)\|_{\mu,r})\|a(s^\lambda)\|_{A_1}^{\alpha_1}$$
$$\leqslant C^{\alpha_1} g(CC'\|a\|_{\theta,p}) t^{(\theta-1)\alpha_1}\|a\|_{\theta,p}^{\alpha_1} + g(2\|a\|_{\mu,r}) s^{-\lambda\alpha_1} K(s^\lambda, a)^{\alpha_1}$$

and

$$M_1(t) := \left(\int_0^\infty (s^{1-\eta}\|b_1(s)\|_{B_1})^{p/\alpha_1}\frac{ds}{s}\right)^{\alpha_1/p}$$
$$\leqslant C_1\cdot C^{\alpha_1} g(CC'\|a\|_{\theta,p})\|a\|_{\theta,p}^{\alpha_1} +$$
$$+ C_2 g(2\|a\|_{\mu,r})\left(\int_0^t (s^{-\theta}K(s, a))^p\frac{ds}{s}\right)^{\alpha_1/p}.$$

Thus $M_1(t)$ is seen to be bounded on bounded sets in $A_{\theta,p}$, and hence certainly on compact subsets of $A_{\theta,p}$.

Proposition 1 permits the conclusion

$$\|TS_t a - Ta\|_{\eta,q} \leqslant CM_0(t)^{1-\eta}M_1(t)^\eta.$$

Thus $TS_t a \to Ta$ in $B_{\eta,q}$ uniformly on compact subsets of $A_{\theta,p}$.

2° We prove the continuity of T.

Let $\{a_n\} \subset A_{\theta,p}$ and $a_n \to a$ in $\|\cdot\|_{\theta,p}$. Then if $t > 0$,

$$\|Ta - Ta_n\|_{\eta,q} \leqslant \|Ta - TS_t a\|_{\eta,q} + \|TS_t a - TS_t a_n\|_{\eta,q} + \|TS_t a_n - Ta_n\|_{\eta,q}.$$

Let $\varepsilon > 0$ be given. Since the set $\{a\} \cup \{a_n: n = 1, 2, \ldots\}$ is compact in $A_{\theta,p}$, there is a t_0 such that

$$\|Ta - TS_{t_0}a\|_{\eta,q} \leqslant \tfrac{1}{3}\varepsilon \quad \text{and} \quad \|Ta_n - TS_{t_0}a_n\|_{\eta,q} \leqslant \tfrac{1}{3}\varepsilon$$

for $n = 1, 2, \ldots$ Condition (iii) ensures that TS_{t_0} is continuous from $A_{\theta,p}$ to B_1, i.e., there is an N such that $n \geqslant N$ implies

$$\|TS_{t_0}a - TS_{t_0}a_n\|_{\eta,q} \leqslant C''\|TS_{t_0}a - TS_{t_0}a_n\|_{B_1} \leqslant \tfrac{1}{3}\varepsilon,$$

where C'' is the norm of $B_1 \hookrightarrow B_{\eta,q}$. Thus if $n \geqslant N$,

$$\|Ta - Ta_n\|_{\eta,q} \leqslant \varepsilon,$$

and so T is continuous as a mapping of $A_{\theta,p}$ to $B_{\eta,q}$.

THEOREM 3. *Let $A_1 \hookrightarrow A_0$, $B_1 \hookrightarrow B_0$ and $0 \leqslant \mu < 1$, $0 < p$, $r < \infty$. Suppose T is a mapping such that*

(i) *T: $A_{\mu,r} \to B_0$ and for $a, b \in A_{\mu,r}$*

$$\|Ta - Tb\|_{B_0} \leqslant f(\|a-b\|_{\mu,r})\|a-b\|_{A_0}^{\alpha_0}$$

and

(ii) *T: $A_1 \to B_1$ and for $a, b \in A_1$*

$$\|Ta - Tb\|_{B_1} \leqslant g(\|a-b\|_{\mu,r})\|a-b\|_{A_1}^{\alpha_1},$$

where f, g: $\mathbf{R}_+ \to \mathbf{R}_+$ are continuous non-decreasing functions.

Then if $\theta > \mu$ or $\theta = \mu$ and $p \leqslant r$, T maps $A_{\theta,p}$ into $B_{\eta,q}$ and for $a, b \in A_{\theta,p}$

$$\|Ta - Tb\|_{\eta,q} \leqslant Ch(\|a-b\|_{\mu,r})\|a-b\|_{\theta,p}^{\alpha},$$

where

(9) $\eta = \theta\alpha/\alpha_1, \quad \alpha = (1-\eta)\alpha_0 + \eta\alpha_1, \quad p = \alpha q \quad$ *and* $\quad h(t) = f(2t)^{1-\eta}g(2t)^\eta.$

Proof. For any fixed $a_1 \in A_1$ we set for $a \in A_{\mu,r}$

$$Sa = T(a + a_1) - Ta_1.$$

Then

$$\|Sa - Sb\|_{B_0} = \|T(a+a_1) - T(b+a_1)\|_{B_0}$$
$$\leqslant f(\|a-b\|_{\mu,r})\|a-b\|_{A_0}^{\alpha_0} \quad \forall a, b \in A_{\mu,r}$$
$$\|Sa\|_{B_1} = \|T(a+a_1) - Ta_1\|_{B_1} \leqslant g(\|a\|_{\mu,r})\|a\|_{A_1}^{\alpha_1} \quad \forall a \in A_1.$$

From Theorem 1 we get

$$\|Sa\|_{\eta,q} \leqslant Ch(\|a\|_{\mu,r})\|a\|_{\theta,p}^{(1-\eta)\alpha_0+\eta\alpha_1}.$$

Hence

$$\|Ta - Ta_1\|_{\eta,q} = \|S(a-a_1)\|_{\eta,q} \leqslant Ch(\|a-a_1\|_{\mu,r})\|a-a_1\|_{\theta,p}^{\alpha}.$$

Since $p, r < \infty$, A_1 is dense in $A_{\theta,p}$ (Proposition 2). Taking any $a, b \in A_{\theta,p}$ there are sequences (a_n), (b_n) in A_1 convergent to a and b, respectively, in the $\|\cdot\|_{\theta,p}$ quasi-norm. Hence

$$C_1\|Ta - Tb\|_{\eta,q} \leqslant \|Ta - Ta_n\|_{\eta,q} + \|Ta_n - Tb_n\|_{\eta,q} + \|Tb_n - Tb\|_{\eta,q}$$
$$\leqslant C_2[h(\|a-a_n\|_{\mu,r})\|a-a_n\|_{\theta,p}^{\alpha} + h(\|a_n-b_n\|_{\mu,r})\|a_n-b_n\|_{\theta,p}^{\alpha} +$$
$$+ h(\|b_n-b\|_{\mu,r})\|b_n-b\|_{\theta,p}^{\alpha}].$$

Taking $n \to \infty$, we get

$$\|Ta - Tb\|_{\eta,q} \leqslant C_3 h(\|a-b\|_{\mu,r})\|a-b\|_{\theta,p}^{\alpha}.$$

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MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES
POZNAŃ, POLAND

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Removability of ideals in commutative Banach algebras

by

VLADIMÍR MÜLLER (Praha)

Abstract. A countable family of removable ideals in a commutative Banach algebra is removable.

Introduction. Let A be a commutative Banach algebra with unit. An ideal I in A is called *removable* if there exists a superalgebra $B \supset A$ (i.e., B is a commutative Banach algebra with unit and there is an isometric unit preserving isomorphism $f\colon A \to B$) such that I is not contained in a proper ideal in B. A family $\{I_j\}_{j\in J}$ is called *removable* if there is a superalgebra $B \supset A$ such that I_j is not contained in a proper ideal in B for every $j \in J$.

These notion were introduced by R. Arens [1] where also the following question was raised: Is every (every finite) family of removable ideals removable?

In general the answer is negative as was shown by B. Bollobás [2]. He presented an example of an uncountable family of removable ideals which is not removable. There was also shown that we can adjoin inverses to any countable family of elements of A which are not permanently singular (i.e., which are not topological divisors of zero).

Removable ideals were further studied, e.g. in [4] and [5].

For finite families the answer to the question of R. Arens is affirmative. This was shown in [3] as a consequence of the characterization of non-removable ideals: an ideal I is non-removable if and only if it consists of joint topological divisors of zero (i.e., for every $x_1, \ldots, x_n \in I$ there exists a sequence $\{z_k\}_{k=1}^{\infty} \subset A$, $\|z_k\| = 1$, $\lim\limits_{k\to\infty} \sum_{i=1}^{n} \|z_k x_i\| = 0$).

The aim of this paper is to fill the gap, namely to consider the countable case (see also Problem 3 of [4]). We show that any countable family of removable ideals is removable.

THEOREM 1. *Let A be a commutative Banach algebra with unit, let $p_1, p_2, \ldots$ be positive integers and $K_1, K_2, \ldots$ positive real numbers such that $2 \leqslant p_l \leqslant l+1$, $K_l^{p_l} \leqslant l$, $p_l^{p_1} \leqslant 4\cdot l$ $(l = 1, 2, \ldots)$ (these conditions are only technical). Let $u_{rs} \in A$, $\|u_{rs}\| = 1$ $(r = 1, 2, \ldots, 1 \leqslant s \leqslant p_r)$ and*

$$(1) \qquad \|x\| \leqslant K_r \sum_{s=1}^{p_r} \|u_{rs} x\| \quad (r = 1, 2, \ldots, x \in A).$$