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Weighted norm inequalities and Schur's Lemma

by

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Abstract. New proofs of known results on boundedness of the Hardy–Littlewood maximal operator on weighted L^p spaces are given for both one and two weights. The proofs involve the explicit construction of auxiliary functions ((2.9) and (3.3)) required for the application of Schur's Lemma. In contrast to recent work on this problem, the "reverse Hölder" property of A_p weights arises quite naturally in this approach. The proof in the case of two weights rests upon what may be viewed as a generalization of this "reverse Hölder" property.

1. Introduction. Coifman, Jones, and Rubio de Francia [3] have recently given an easy proof of the factorization of A_p weights, using a rather general construction and relying only on the boundedness of the Hardy–Littlewood maximal operator on weighted L^p spaces, with weight in A_p. They have also used the same ideas to give a new real-variable proof of a weak version of the Helson–Szegö Theorem. Here we show that the same circle of ideas leads naturally to a new proof of the boundedness of the Hardy–Littlewood maximal operator on weighted L^p spaces.

A non-negative function $w \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ is an A_p weight for some $1 < p < \infty$ if

$$(1.1) \qquad \sup_Q \Big[|Q|^{-1} \int w(x)\,dx\Big] \cdot \Big[|Q|^{-1} \int_Q w(x)^{-1/(p-1)}\,dx\Big]^{p-1} \leqslant C < \infty .$$

The supremum is taken over all cubes $Q \subset \boldsymbol{R}^n$; $|Q|$ denotes the measure of Q. w is an A_∞ weight if there exist $\delta > 0$ and $\varepsilon > 0$ such that for any measurable $E \subset Q$,

$$(1.2) \qquad |E| < \delta \cdot |Q| \Rightarrow w(E) < (1-\varepsilon) \cdot w(Q).$$

Here $w(E) = \int_E w$. It is well known and easily checked that $w \in A_p \Rightarrow w \in A_\infty$ if $1 < p < \infty$ ([2]).

We present new proofs of the following two theorems:

THEOREM 1.1 [2]. *$w \in A_p$ if and only if there exists $C < \infty$ such that*

$$\int Mf(x)^p w(x)\,dx \leqslant C \int |f(x)|^p w(x)\,dx \quad \textit{for all } f \in L^1_{\mathrm{loc}}.$$

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THEOREM 1.2 [6]. *Given $w(x)$, $v(x) \geqslant 0$ and $1 < p < \infty$, there exists $C < \infty$ such that $\int Mf(x)^p w(x)\,dx \leqslant C \int |f(x)|^p v(x)\,dx$ for all f if and only if there is $A < \infty$ such that*

$$\int_Q M(v^{1-p'} \cdot \chi_Q)(x)^p w(x)\,dx \leqslant A \int_Q v^{1-p'}(x)\,dx < \infty \quad \text{for all cubes } Q. \tag{1.3}$$

Here and henceforth $1/p + 1/p' = 1$, χ_Q is the characteristic function of Q, and M denotes the Hardy–Littlewood maximal operator. Theorem 1.1 does not follow at once from Theorem 1.2, since it must first be verified that the A_p condition implies (1.3) when $v = w$.

The basis for the proofs and the link between this paper and the ideas of Coifman, Rubio de Francia and Jones is

SCHUR LEMMA. *Suppose $Kf(x) = \int k(x, y) f(y)\,dy$, where $k(x, y)$ is measurable and non-negative. Suppose $1 < p < \infty$. Then K is a bounded operator on L^p if and only if there exists a non-negative function $u(x)$ finite almost everywhere and satisfying*

$$K(u^{p'})(x) \leqslant Cu^{p'}(x) \tag{1.4}$$

and

$$K^*(u^p)(x) \leqslant Cu^p(x) \quad \text{for almost all } x. \tag{1.5}$$

(K^* denotes the formal adjoint of K.)

We shall prove Theorems 1.1 and 1.2 by explicitly constructing the auxiliary functions $u(x)$.

An interesting feature of this approach in the case of two weights is a generalization to this situation of the reverse Hölder inequality. This property of A_∞ weights and its corollary, the implication "$A_p \Rightarrow A_{p-\varepsilon}$", played a crucial role in the approach of Coifman and Fefferman [2]. However, Sawyer's solution [6] to the two-weight problem relied on no such inequality, and indeed (1.3) may hold for any particular value of p without holding for any smaller p. In this paper the reverse Hölder inequality arises rather naturally in the case of one weight, and we give a simple proof of its validity. This proof is closely related to [1]. In the two-weight situation, the main point of our proof is an inequality (embodied in Lemmas 3.1 and 3.2) which when specialized to the case $v = w \in A_\infty$, is equivalent to the reverse Hölder inequality. Thus in a sense there is a weak analogue in the two-weight situation of this reverse Hölder inequality.

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2. One weight. As observed by Fefferman and Stein [4], in the proof of Theorem 1.1 it suffices to treat the dyadic maximal operator instead of the Hardy–Littlewood maximal operator. Then it suffices to consider a fixed linearization $\mathcal{M}$ of the dyadic maximal operator. That is, for each x we choose a fixed dyadic cube Q_x containing x, and define

$$\mathcal{M}f(x) = |Q_x|^{-1} \int_{Q_x} f(y)\,dy \quad \text{for all } f.$$

With a fixed $w \in A_p$ and a fixed linearization $\mathcal{M}$ we construct a Schur auxiliary function u; this will be done in such a way that the constants appearing in (1.4) and (1.5) depend only on the A_p bound of w (and the dimension). This demonstrates that all linearized maximal operators are uniformly bounded.

By considering the Calderón–Zygmund decomposition of an arbitrary function in L^1 with compact support, we may suppose that there is a collection $\{Q_j^k\}_{k \geqslant 0}$ of dyadic cubes such that

$$\mathcal{M}f(x) = \sum_{j,k} \chi_{E_j^k}(x) \cdot |Q_j^k|^{-1} \int_{Q_j^k} f(y)\,dy, \tag{2.1}$$

where

$$\mathring{Q}_i^k \cap \mathring{Q}_j^k = \emptyset \quad \text{unless} \quad i = j, \tag{2.2}$$

$$\text{If } \mathring{Q}_i^k \cap \mathring{Q}_j^l \neq \emptyset \text{ and } k < l, \text{ then } Q_j^l \subsetneqq Q_i^k, \tag{2.3}$$

$$E_j^k = Q_j^k \setminus \Big(\bigcup_{l > k} \bigcup_i Q_i^l\Big), \tag{2.4}$$

$$\text{For fixed } i \text{ and } k, \quad \sum_{j: Q_j^{k+l} \subset Q_i^k} |Q_j^{k+l}| \leqslant C2^{-l} \cdot |Q_i^k| \tag{2.5}$$

(C depends only on the dimension n),

$$\text{If } k > 0, \text{ then each } Q_i^k \text{ is contained in some } Q_j^{k-1}. \tag{2.6}$$

Without loss of generality we may assume that there are only finitely many Q_j^k. $\mathring{Q}$ denotes the interior of a cube Q.

Fix $1 < p < \infty$ and suppose $w \in A_p$. The operator $Kf(x) = w(x)^{1/p} \cdot \mathcal{M}(w^{-1/p} f)(x)$ has the same norm on $L^p(dx)$ as does $\mathcal{M}$ on $L^p(w(x)\,dx)$, and is of the type to which Schur's Lemma applies. Hence it suffices to construct $\mu(x) \geqslant 0$ such that

$$\mathcal{M}(w^{-1/p} \cdot \mu^{p'}) \leqslant Cw^{-1/p} \cdot \mu^{p'} \tag{2.7}$$

and

$$\mathcal{M}^*(w^{1/p} \cdot \mu^p) \leqslant Cw^{1/p} \cdot \mu^p, \tag{2.8}$$

where C depends only on p, n and the A_p bound of w. Define

$$\begin{aligned} \mu(x) &= \sum_{j,k} 2^{\eta k} \chi_{E_j^k}(x) \cdot [w(x)^{-1/p^2} + w(x)^{1/p \cdot p'} \mathcal{M}(w^{1-p'})^{1/p'}(x)] \\ &= \mu_1(x) + \mu_2(x), \end{aligned} \tag{2.9}$$

where $\eta > 0$ is a constant to be determined below. This construction succeeds because of the linearized reverse Hölder inequality:

LEMMA 2.1. *Suppose $v \in A_\infty$. If $\eta > 0$ is sufficiently small, there exists $C < \infty$ such that for any j, k,*

$$\int_{Q_j^k} \sum_{l \geq k} \sum_i 2^{\eta(l-k)} \chi_{Q_i^l}(x) v(x)\, dx \leq C \int_{Q_j^k} v(x)\, dx.$$

Proof. If m is sufficiently large and δ is the constant in (1.2), then $|(\bigcup_i Q_i^{k+m}) \cap Q_j^k| < \delta |Q_j^k|$ by (2.5). Thus $v((\bigcup_i Q_i^{k+m}) \cap Q_j^k) < (1-\varepsilon) \cdot v(Q_j^k)$, and iterating yields $v((\bigcup_i Q_i^l) \cap Q_j^k) \leq C \cdot (1-\varepsilon)^{l-k} \cdot v(Q_j^k)$ for a smaller value of ε, for all $l > k$. Then

$$\int_{Q_j^k} \sum_{l \geq k} 2^{\eta(l-k)} \sum_i \chi_{Q_i^l}(x) v(x)\, dx = \sum_{l \geq k} 2^{\eta \cdot (l-k)} \cdot v((\bigcup_i Q_i^l) \cap Q_j^k)$$

$$\leq (\sum_{l \geq k} 2^{\eta(l-k)} \cdot C \cdot (1-\varepsilon)^{l-k}) \cdot v(Q_j^k) = C \cdot v(Q_j^k), \quad \text{if } 2^\eta \cdot (1-\varepsilon) < 1. \quad \square$$

COROLLARY (reverse Hölder inequality) [2]. *If $v \in A_\infty$, then there exist $\delta > 0$ and $C < \infty$ such that for any Q,*

$$(|Q|^{-1} \int_Q v(x)^{1+\delta}\, dx)^{1/(1+\delta)} \leq C \cdot |Q|^{-1} \int_Q v(x)\, dx.$$

Proof. We may assume that $\int_Q v = 1$, $|Q| = 1$ and Q is dyadic (by scaling and translating the dyadic grid). Let M be the dyadic maximal operator, and let

$$\{x\colon M(v \cdot \chi_Q) > 2^k\} = \bigcup_i Q_i^k \quad \text{for} \quad k \geq 0.$$

Then majorizing v by Mv pointwise gives

$$\int_Q v(x)^{1+\delta}\, dx \leq C \cdot \int_Q \sum_k \sum_i 2^{\delta k} \cdot \chi_{E_i^k}(x) v(x)\, dx \leq C \int_Q v(x)\, dx$$

by Lemma 2.1, if δ is sufficiently small. $\square$

Using the linearized reverse Hölder inequality it is easy to verify that (2.7) holds with μ defined as in (2.9). Recall that $w \in A_p \Rightarrow v = w^{1-p'} \in A_{p'} \subset A_\infty$. Fix (j, k). If $x \in E_j^k$, then

$$\mathcal{M}(w^{-1/p} \cdot \mu_1^{p'})(x) = |Q_j^k|^{-1} \int_{Q_j^k} (\sum_{i,l} 2^{\eta \cdot p' \cdot l} \chi_{E_i^l}(y) \cdot w^{1-p'}(y))\, dy$$

$$\leq C \cdot 2^{\eta p' k} \cdot |Q_j^k|^{-1} \int_{Q_j^k} w(y)^{1-p'}\, dy = C \cdot w(x)^{-1/p} \cdot \mu_2(x)^{p'};$$

the inequality follows from Lemma 2.1 applied to $w^{1-p'}$; if η is sufficiently small.

Similarly if $x \in E_j^k$

$$\mathcal{M}(w^{-1/p} \mu_2^{p'})(x) = |Q_j^k|^{-1} \int_{Q_j^k} \sum_{i,l} 2^{\eta p' l} \cdot \chi_{E_i^l}(y) \cdot (|Q_i^l|^{-1} \int_{Q_i^l} w(z)^{1-p'}\, dz)\, dy$$

$$\leq |Q_j^k|^{-1} \sum_{i,l: Q_i^l \subset Q_j^k} 2^{\eta p' l} \cdot \int_{Q_i^l} w(z)^{1-p'}\, dz$$

$$\leq C \cdot |Q_j^k|^{-1} 2^{\eta p' k} \cdot \int_{Q_j^k} w(y)^{1-p'}\, dy = C \cdot w(x)^{-1/p} \cdot \mu_2(x)^{p'},$$

again by Lemma 2.1. Thus (2.7) holds.

It remains only to verify (2.8); but the weighting factors $2^{\eta k}$ have been introduced precisely to ensure that (2.8) will follow easily.

$$\mathcal{M}^*(w^{1/p} \cdot \mu_1^p)(x) = \sum_{\substack{i,l: \\ Q_j^k \subset Q_i^l}} |Q_i^l| \int_{E_i^l} 2^{\eta p l} w(y)^0\, dy$$

so that if $x \in E_j^k$,

$$\mathcal{M}^*(w^{1/p} \cdot \mu_1^p)(x) = \sum_{\substack{i,l: \\ Q_j^k \subset Q_i^l}} |Q_i^l| \int_{E_i^l} 2^{\eta p l} w(y)^0\, dy.$$

For each $l \leq k$ there is at most one i so that $Q_j^k \subset Q_i^l$. Hence

$$\mathcal{M}^*(w^{1/p} \cdot \mu_1^p)(x) \leq \sum_{l \leq k} 2^{\eta l p} \cdot |E_i^l| \cdot |Q_i^l|^{-1} \leq \sum_{l \leq k} 2^{\eta p l} \leq C \cdot 2^{\eta p k} = C \cdot w(x)^{1/p} \cdot \mu_1(x)^p.$$

In the same way

$$\mathcal{M}^*(w^{1/p} \cdot \mu_2^p)(x) = \sum_{i,l:} |Q_i^l|^{-1} \int_{E_i^l} w(y)^{1/p} \cdot 2^{\eta p l} \cdot w(y)^{1/p'} \cdot (|Q_i^l|^{-1} \int_{Q_i^l} w(z)^{1-p'}\, dz)^{p/p'}\, dy$$

$$= \sum_{\substack{i,l \\ Q_j^k \subset Q_i^l}} 2^{\eta p l} \cdot (|Q_i^l|^{-1} \int_{E_i^l} w(y)\, dy) \cdot (|Q_i^l|^{-1} \int_{Q_i^l} w(y)^{1-p'}\, dy)^{p-1}$$

$$\leq C \cdot \sum_{l \leq k} 2^{\eta p l} = C \cdot 2^{\eta k p} = C \cdot w(x)^{1/p} \cdot \mu_1(x)^p,$$

since $w \in A_p$. This completes the proof of Theorem 1.1.

3. Two weights. Notation is as in the preceding section. The construction of the Schur function μ is changed in only one essential way: in the case of one weight the reverse Hölder inequality allowed us to introduce auxiliary weights $2^{\eta k}$ which grow by a constant factor with each move from Q_j^k to $Q_i^{k+1} \subset Q_j^k$. In the two-weight situation these factors may no longer be taken to be constant. Instead, at each transition from Q_j^k to a subcube Q_i^{k+1} we multiply by a factor between 1 and 2^η, depending on the behavior of the pair of weights (w, v) on Q_j^k.

In order to prove Theorem 1.2 we must construct $\mu(x)$ such that

$$w^{1/p}(x)\,\mathcal{M}(v^{-1/p}\mu^{p'})(x)\leqslant C\mu^{p'}(x) \tag{3.1}$$

and

$$v^{-1/p}(x)\,\mathcal{M}^{*}(w^{1/p}\mu^{p})(x)\leqslant C\mu^{p}(x). \tag{3.2}$$

Let

$$a_{k,j}=\Big[|Q_j^k|^{-1}\int_{E_j^k} w(x)\,dx\Big]\cdot\Big[|Q_j^k|^{-1}\int_{Q_j^k} v^{1-p'}(x)\Big]^{p-1},$$

$$b_{k,j}=|Q_j^k|^{-1}\int_{E_j^k} w^{1/p}(x)\,v^{-1/p}(x)\,dx,$$

and

$$\gamma_{k,j}^{p}=\prod_{\substack{l\leqslant k\\ Q_j^k\subset Q_i^l}}\big(1+\eta\cdot(a_{l,i}+b_{l,i})\big),$$

where $\eta>0$ is a small constant to be specified below. Then define

$$\mu(x)=\sum_{k,j}\gamma_{k,j}\cdot\chi_{E_j^k}(x)\cdot\Big[v(x)^{-1/p^2}+w(x)^{1/p\cdot p'}\cdot\Big\{|Q_j^k|^{-1}\cdot\int_{Q_j^k} v^{1-p'}(y)\,dy\Big\}^{1/p'}\Big] \tag{3.3}$$
$$=\mu_1(x)+\mu_2(x).$$

Lemma 3.1. *Suppose that (w, v) satisfies the two-weight condition (1.3). Then there exist $A, B<\infty$ such that if η is sufficiently small, then for any Q_j^k,*

$$\int_{Q_j^k}\sum_{l\geqslant k}\sum_{i}\gamma_{l,i}^{p'}\cdot v^{1-p'}(y)\cdot\chi_{E_i^l}(y)\,dy\leqslant A\gamma_{k,j}^{p'}\int_{Q_j^k} v^{1-p'}+B\gamma_{k,j}^{p'}\cdot\int_{Q_j^k}\mathcal{M}(v^{1-p'})^{p}\cdot w.$$

By the two-weight hypothesis, the second term on the right-hand side of this inequality is dominated by a constant times the first. However, the following argument does not go through if the second term is omitted.

Proof. The proof is by descending induction on k. It is no loss of generality to assume that $k=0$ and the lemma is true for $k+1=1$. Fix j and let $Q^0=Q_j^0$, $E^0=E_j^0$ and so on. Note that

$$(1+\eta\cdot(a_{k,j}+b_{k,j}))^{p'/p}\leqslant 1+C\eta(a_{k,j}+b_{k,j}),$$

since $a_{k,j}$ and $b_{k,j}$ are bounded as a consequence of the two-weight hypothesis.

$$\int_{Q^0}\sum_{i,l}\gamma_{l,i}^{p'}\cdot v^{1-p'}(y)\cdot\chi_{E_i^l}(y)\,dy$$
$$=\Big[\sum_j\int_{Q_j^1}\sum_{i,l}\gamma_{l,i}^{p'}\cdot v^{1-p'}(y)\cdot\chi_{E_i^l}(y)\,dy\Big]+(1+\eta(a_0+b_0))^{p'/p}\cdot\int_{E^0} v^{1-p'}=\mathrm{I}+\mathrm{II}.$$

By induction,

$$\mathrm{I}\leqslant(1+\eta(a_0+b_0))^{p'/p}\cdot\Big(A\int_{Q^0\setminus E^0} v^{1-p'}+B\int_{Q^0\setminus E^0}\mathcal{M}(v^{1-p'})^{p}\cdot w\Big)$$
$$\leqslant A\int_{Q^0\setminus E^0} v^{1-p'}+B\int_{Q^0\setminus E^0}\mathcal{M}(v^{1-p'})^{p}w+C\cdot\eta(a_0+b_0)\Big(A\int_{Q^0\setminus E^0} v^{1-p'}+$$
$$+B\int_{Q^0\setminus E^0}\mathcal{M}(v^{1-p'})^{p}\cdot w\Big).$$

By (1.3), the term involving η is $\leqslant C\cdot\eta(a_0+b_0)(A+CB)\int_{Q^0} v^{1-p'}$. Observe that since $\int_{Q^0} v^{1-p'}=|Q^0|\cdot\mathcal{M}(v^{1-p'})(y)$ for all $y\in E^0$,

$$a_0\cdot\int_{Q^0} v^{1-p'}=|Q^0|^{-p}\cdot\Big(\int_{Q^0} v^{1-p'}\Big)^{p}\cdot\Big(\int_{E^0} w\Big)=\int_{E^0}\mathcal{M}(v^{1-p'})^{p}\cdot w.$$

Also $b_0\leqslant\big(|Q^0|^{-1}\int_{E^0} w\big)^{1/p}\cdot\big(|Q^0|^{-1}\int_{E^0} v^{1-p'}\big)^{1/p'}$ by Hölder's inequality so that as above,

$$b_0\cdot\int_{Q^0} v^{1-p'}\leqslant\Big(\int_{E^0}\mathcal{M}(v^{1-p'})^{p}\cdot w\Big)^{1/p}\cdot\Big(\int_{E^0} v^{1-p'}\Big)^{1/p'}\leqslant\int_{E^0}\mathcal{M}(v^{1-p'})^{p}w+\int_{E^0} v^{1-p'}.$$

Hence altogether

$$(a_0+b_0)\int_{Q^0} v^{1-p'}\leqslant 2\int_{E^0}\mathcal{M}(v^{1-p'})^{p}w+\int_{E^0} v^{1-p'}. \tag{3.4}$$

Therefore

$$\mathrm{I}\leqslant A\int_{Q^0\setminus E^0} v^{1-p'}+B\int_{Q^0\setminus E^0}\mathcal{M}(v^{1-p'})^{p}\cdot w+$$
$$+2C\cdot(A+CB)\cdot\eta\cdot\Big[\int_{E^0}\mathcal{M}(v^{1-p'})^{p}\cdot w+\int_{E^0} v^{1-p'}\Big].$$

Using some of the above estimates we also have

$$\mathrm{II}\leqslant\int_{E^0} v^{1-p'}+C\eta\cdot\Big[2\int_{E^0}\mathcal{M}(v^{1-p'})^{p}\cdot w+\int_{E^0} v^{1-p'}\Big]$$
$$\leqslant(1+2C\eta)\int_{E^0} v^{1-p'}+2C\eta\int_{E^0}\mathcal{M}(v^{1-p'})^{p}\cdot w.$$

In order to complete the inductive step we therefore need to choose the constants so that

$$1+2C\eta+2C\eta(A+CB)\leqslant A\qquad\text{and}\qquad 2C\eta\cdot(1+A+CB)\leqslant B.$$

If $B=1$ and $A=2$, this holds for sufficiently small η. $\square$

LEMMA 3.2. *If* (w, v) *satisfies* (1.3), *then there exist* A, $B < \infty$ *such that for sufficiently small* η, *for any* Q_j^k,

$$\int_{Q_j^k} \sum_{i,l} \gamma_{l,i}^{p'} \cdot \chi_{E_i^l}(y) \cdot \mathcal{M}(v^{1-p'})^p(y) \cdot w(y)\,dy \leqslant A\gamma_{k,j}^{p'} \cdot \int_{Q_j^k} \mathcal{M}(v^{1-p'})^p \cdot w + B \cdot \gamma_{k,j}^{p'} \int_{Q_j^k} v^{1-p'}.$$

Proof. We proceed by descending induction on k as in the last lemma. Again assume $k = 0$. Then

$$\int_{Q^0} \sum_{i,l} \gamma_{l,i}^{p'} \cdot \chi_{E_i^l}(y) \cdot \mathcal{M}(v^{1-p'})^p \cdot w(y)\,dy$$
$$\leqslant \sum_j \int_{Q_j^1} \sum_{i,l} \gamma_{l,i}^{p'} \chi_{E_i^l(y)} \cdot \mathcal{M}(v^{1-p'})^p(y) \cdot w(y)\,dy + (1 + C\eta(a_0+b_0)) \cdot \int_{E^0} \mathcal{M}(v^{1-p'})^p \cdot w = \mathrm{I} + \mathrm{II}.$$

By induction,

$$\mathrm{I} \leqslant (1 + C\eta(a_0+b_0)) \cdot \Big[A \int_{Q^0 \setminus E^0} \mathcal{M}(v^{1-p'})^p \cdot w + B \int_{Q^0 \setminus E^0} v^{1-p'}\Big].$$

Using the estimate (1.3) yields

$$\mathrm{I} \leqslant A \int_{Q^0 \setminus E^0} \mathcal{M}(v^{1-p'})^p \cdot w + B \int_{Q^0 \setminus E^0} v^{1-p'} + \int_{E^0} \mathcal{M}(v^{1-p'})^p \cdot w + C\eta(a_0+b_0) \cdot (B + CA + C) \int_{Q^0} v^{1-p'}.$$

By (3.4),

$$\mathrm{I} \leqslant A \int_{Q^0 \setminus E^0} \mathcal{M}(v^{1-p'})^p \cdot w + B \int_{Q^0 \setminus E^0} v^{1-p'} + C\eta \cdot (B + CA + C) \cdot \int_{E^0} v^{1-p'} + (1 + 2C\eta(B + CA + C)) \int_{E^0} \mathcal{M}(v^{1-p'})^p \cdot w.$$

In the same way

$$\mathrm{II} \leqslant \int_{E^0} \mathcal{M}(v^{1-p'})^p \cdot w + C\eta(a_0+b_0) \int_{Q^0} v^{1-p'}$$
$$\leqslant \int_{E^0} \mathcal{M}(v^{1-p'})^p \cdot w + C\eta \cdot \Big(2 \int_{E^0} \mathcal{M}(v^{1-p'})^p \cdot w + \int_{E^0} v^{1-p'}\Big)$$
$$= C\eta \int_{E^0} v^{1-p'} + (1 + 2C\eta) \int_{E^0} \mathcal{M}(v^{1-p'})^p \cdot w.$$

Thus to conclude the proof we must choose the constants so that $C\eta \cdot (B + CA + C) \leqslant B$ and $2 + C\eta \cdot (2C + 2(B + CA + C)) \leqslant A$, which is again possible. □

These two lemmas play the role of the linearized reverse Hölder inequality in the proof of (3.1). Suppose that $x \in E_j^k$. Since $-1/p - p'/p^2 = 1 - p'$,

$$\mathcal{M}(v^{-1/p} \cdot \mu_1^{p'})(x) = |Q_j^k|^{-1} \int_{Q_j^k} v^{1-p'}(y) \cdot \sum_{l,i} \gamma_{l,i}^{p'} \cdot \chi_{E_i^l}(y)\,dy$$
$$\leqslant A\gamma_{k,j}^{p'} \int_{Q_j^k} v^{1-p'} + B\gamma_{k,j}^{p'} \int_{Q_j^k} \mathcal{M}(v^{1-p'})^p \cdot w$$
$$\leqslant (A + CB) \cdot \gamma_{k,j}^{p'} \cdot \int_{Q_j^k} v^{1-p'} = (A + CB) \cdot w^{1/p}(x) \cdot \mu_2^{p'}(x).$$

Similarly

$$\mathcal{M}(v^{-1/p} \cdot \mu_2^{p'})(x) = |Q_j^k|^{-1} \int_{Q_j^k} \sum_{i,l} \gamma_{l,i}^{p'} \cdot \chi_{E_i^l}(y) \cdot \Big[v^{-1/p}(y) \cdot w^{1/p}(y) \cdot |Q_i^l|^{-1} \cdot \int_{Q_i^l} v^{1-p'}\Big]\,dy.$$

By Hölder's inequality this is dominated by the product of

$$\Big[|Q_j^k|^{-1} \int_{Q_j^k} \sum_{l,i} \gamma_{l,i}^{p'} \cdot \chi_{E_i^l}(y) \cdot v^{1-p'}(y)\,dy\Big]^{1/p'}$$

and

$$\Big[|Q_j^k|^{-1} \int_{Q_j^k} \sum_{l,i} \gamma_{l,i}^{p'} \cdot \chi_{E_i^l}(y) \cdot \mathcal{M}(v^{1-p'})^p(y) \cdot w(y)\,dy\Big]^{1/p}.$$

By Lemmas 3.1 and 3.2 and by (1.3), this is dominated by

$$C \cdot \gamma_{k,j}^{p'} \cdot \int_{Q_j^k} v^{1-p'} = C \cdot w^{-1/p}(x) \cdot \mu_2^{p'}(x).$$

This completes the proof of (3.1). As in the case of one weight, (3.2) follows easily from the definition of $\gamma_{k,j}$. Indeed, suppose that $x \in E_j^k$. For $0 \leqslant l \leqslant k$ there is a unique Q_i^l containing Q_j^k. We write $Q^l = Q_i^l$, $E^l = E_i^l$, $\gamma_{l,i} = \gamma_l$ and so on. Then

$$\mathcal{M}^*(w^{1/p} \mu_1^p)(x) = \sum_{l \leqslant k} |Q^l|^{-1} \int_{E^l} \gamma_l^p w^{1/p}(y) v^{-1/p}(y)\,dy$$
$$= \sum_{l \leqslant k} \gamma_l^p \cdot b_l = \sum_{l \leqslant k} \Big(b_l \cdot \prod_{m \leqslant l} (1 + \eta \cdot (a_m + b_m))\Big)$$
$$\leqslant C \sum_{0 \leqslant l \leqslant k} \Big((a_l + b_l) \cdot \prod_{0 \leqslant m < l} (1 + \eta(a_m + b_m))\Big),$$

since $1+\eta(a_l+b_l)\leqslant C$ by the two-weight hypothesis (1.3). We claim that

$$\sum_{l\leqslant k}\Big((a_l+b_l)\cdot\prod_{m<l}\big(1+\eta\cdot(a_m+b_m)\big)\Big)\leqslant\eta^{-1}\cdot\prod_{0\leqslant l\leqslant k}\big(1+\eta(a_l+b_l)\big). \tag{3.5}$$

This follows at once by induction. We thus have for $x\in E_j^k$,

$$\mathscr{M}^*(w^{1/p}\mu_1^p)(x)\leqslant C\eta^{-1}\cdot\prod_{l\leqslant k}\big(1+\eta\cdot(a_l+b_l)\big)=C\cdot\eta^{-1}\cdot v^{1/p}(x)\cdot\mu_1^p(x).$$

In the same fashion

$$\mathscr{M}^*(w^{1/p}\mu_2^p)(x)=\sum_{l\leqslant k}|Q^l|^{-1}\int_{E^l}\gamma_l^p\cdot\Big[|Q^l|^{-1}\int_{Q^l}v^{1-p'}\Big]^{p/p'}\cdot w^{1/p}(y)\cdot w^{1/p'}(y)\,dy$$
$$=\sum_{l\leqslant k}\gamma_l^p\cdot a_l\leqslant\sum_{l\leqslant k}\gamma_l^p\cdot(a_l+b_l).$$

Using (3.5) and (1.3) we see that this last expression is no larger than $C\cdot\eta^{-1}\cdot\gamma_k^p=C\eta^{-1}\cdot v^{1/p}(x)\cdot\mu_1^p(x)$. This completes the proof of (3.2) and hence of Theorem 1.3. $\square$

Definition (3.3) of the Schur auxiliary function $\mu(x)$ is easily motivated. One begins with the function $\mu^{(0)}(x)=v^{-1/p^2}(x)$ (since this leads to a relatively simple μ in the case of one weight). The requirement $\mathscr{M}(v^{-1/p}\cdot\mu^{p'})\leqslant C\cdot w^{-1/p}\cdot\mu^{p'}$ forces the modification

$$\mu^{(1)}(x)=v^{-1/p^2}(x)+w^{1/pp'}(x)\cdot\Big(|Q_j^k|^{-1}\int_{Q_j^k}v^{1-p'}\Big)^{1/p'}$$
$$=\mu^{(0)}+w^{1/p}\cdot\mathscr{M}\big(v^{-1/p}\cdot(\mu^{(0)})^{p'}\big)\quad\text{on } E_j^k.$$

Next calculating $\mathscr{M}^*(w^{1/p}\cdot(\mu^{(1)})^p)$ leads to the further modification

$$\mu^{(2)}(x)=\mu^{(1)}(x)+\sum_{k,j}\chi_{E_j^k}(x)\cdot\Big(\sum_{\substack{i,l:\\ Q_j^k\subset Q_i^l}}(a_i^l+b_i^l)\Big)\cdot\mu^{(1)}(x).$$

By considering infinitely many iterations of the operator $f\to[v^{-1/p}\cdot\mathscr{M}^*(w^{1/p}f^p)]^{1/p}$ one is finally led to our definition of $\mu(x)$.

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