STUDIA MATHEMATICA, T. LXXIX. (1984)

Interpolation of analytic families of operators

by

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Abstract. Generalizations of E. M. Stein's theorem on interpolation of analytic families of operators $\{T_z\}_{0 \leqslant \mathrm{Re}\, z \leqslant 1}$ are considered in the context of A. P. Calderón's complex interpolation spaces. For such a family $\{T_z\}$ such that T_z maps $A_0 \cap A_1$ into $B_0 + B_1$ for each z and $T_{j+it} a \in B_j$ with $\|T_{j+it} a\|_{B_j} \leqslant \|a\|_{A_j}$ for each $a \in A_0 \cap A_1$, $t \in \mathbf{R}$, $j = 0, 1$, it follows in some cases that T_θ maps $[A_0, A_1]_\theta$ into $[B_0, B_1]_\theta$. In other cases, depending inter alia on what sort of continuity conditions are imposed on $T_{j+it} a$, we may only be able to assert that $T_\theta([A_0, A_1]_\theta)$ is contained in a space larger than $[B_0, B_1]_\theta$, such as $[B_0, B_1]_\theta^w$ or $[B_0, B_1]^\theta$. For many couples (B_0, B_1), these last three spaces in fact coincide, however an example is given of a couple for which they are all distinct from each other. This also shows that, for certain analytic families $\{T_z\}$ as above, one may have $T_\theta(A_0 \cap A_1) \not\subset [B_0, B_1]_\theta$ or even $T_\theta(A_0 \cap A_1) \not\subset [B_0, B_1]_\theta^w$.

0. Introduction. Let $\bar{S}$ denote the closed strip $\{z\colon 0 \leqslant \mathrm{Re}\, z \leqslant 1\}$ in the complex plane and $A(\bar{S})$ the algebra of bounded continuous functions on $\bar{S}$ that are analytic on the open strip S. This paper treats the following question:

Let (A_0, A_1) and (B_0, B_1) be two compatible couples of Banach spaces and let $\{T_z\}_{z \in \bar{S}}$ be a family of operators on $A_0 \cap A_1$ into $B_0 + B_1$ such that

(A) *for every $a \in A_0 \cap A_1$ and $b^* \in (B_0 + B_1)^*$, $\langle b^*, T_z a \rangle \in A(\bar{S})$*

and

(B) *there exist constants M_0 and M_1 such that*

$$\|T_{j+it} a\|_{B_j} \leqslant M_j \|a\|_{A_j}, \quad j = 0, 1, \ -\infty < t < \infty, \ a \in A_0 \cap A_1.$$

Does this imply that T_θ maps $[A_0, A_1]_\theta$ into $[B_0, B_1]_\theta$?

We will give an example showing that the answer is no. The problem is that although the analyticity (A) implies that T_z is continuous and differentiable on S, the behaviour at the boundary may be quite bad. The example also shows that it is not sufficient to require T_z to be continuous in the $L(A_0 \cap A_1, B_0 + B_1)$ operator norm on $\bar{S}$; some sort of continuity or measura-

* Research supported by the Fund for the Promotion of Research at the Technion.
** The research was carried out while this author visited the Technion, February 1982.

bility in B_0 or B_1 seems necessary. However, the extra condition can be very weak; the following consequence of Theorem 1 below covers most cases that are likely to appear in applications.

If further, for every $a \in A_0 \cap A_1$, $\{T_{it} a\}_{-\infty < t < \infty}$ lies in a separable subspace of B_0, then T_θ can be extended to a linear operator from $[A_0, A_1]_\theta$ to $[B_0, B_1]_\theta$ of norm not exceeding $M_0^{1-\theta} M_1^\theta$. In particular, this holds if $\langle b^*, T_{it} a\rangle$ is continuous for every $b^* \in B_0^*$ (cf. [9], Lemma 8) or indeed for every family $\{T_z\}$ as above if B_0 is separable or $B_0 \supset B_1$.

Note that the extra condition is asymmetric; no condition on B_1 is required. (Of course, B_0 and B_1 may be interchanged.)

This type of theorem was first proved for L^p-spaces by Stein [19]. Various generalizations have been given by several authors, see e.g. [4], [7], [14], [17], but in some cases they have unfortunately neglected to mention the extra conditions which are required on the boundary [17], [4].

A more direct generalization of Stein's theorem is given in Section 3. There we use $[B_0, B_1]^\theta$ as the target space and can completely dispense with continuity on the boundary.

1. Analytic families and interpolation methods. For the sake of this discussion, we define four versions of the complex method.

Let $\mathcal{F}^w(B_0, B_1)$ be the Banach space of all functions $f\colon \bar{S} \to B_0 + B_1$ such that

(i) $\langle b^*, f(z)\rangle \in A(\bar{S})$ for any $b^* \in (B_0+B_1)^*$,

(ii) $f(z) \in B_0$ for $\operatorname{Re} z = 0$ and $f(z) \in B_1$ for $\operatorname{Re} z = 1$,

(iii) $\|f\|_{\mathcal{F}^w} = \sup\{\|f(j+it)\|_{B_j}\colon j = 0, 1, -\infty < t < \infty\} < \infty$.

We will use the following closed subspaces:

$$\mathcal{F}(B_0, B_1) = \{f \in \mathcal{F}^w(B_0, B_1)\colon f(it) \text{ is } B_0\text{-continuous and } f(1+it) \text{ is } B_1\text{-continuous}\},$$

$$\mathcal{F}^\alpha(B_0, B_1) = \{f \in \mathcal{F}^w(B_0, B_1)\colon \{f(it)\}_{-\infty < t < \infty} \text{ lies in a separable subspace of } B_0\},$$

$$\mathcal{F}^\sigma(B_0, B_1) = \{f \in \mathcal{F}^w(B_0, B_1)\colon f(z) \text{ is } B_0+B_1\text{-continuous on } \bar{S}\}.$$

By the maximum principle, $\|f(z)\|_{B_0+B_1} \leqslant \|f\|_{\mathcal{F}^w}$, $z \in \bar{S}$.

LEMMA 1. $\mathcal{F}^\sigma(B_0, B_1) = \{f \in \mathcal{F}^w\colon f \text{ is } B_0+B_1\text{- continuous on } \partial S\}$.

Proof. If the restriction of f to ∂S is B_0+B_1-continuous, let g be its Poisson integral. Then g is B_0+B_1-continuous on $\bar{S}$, but application of linear functionals shows that $g(z) = f(z)$.

Hence $\mathcal{F}$ is the smallest and $\mathcal{F}^w$ the largest of these spaces. Obviously $\mathcal{F}^\alpha = \mathcal{F}^w$ when B_0 is separable. Also, $\mathcal{F}^\alpha = \mathcal{F}^w$ when $B_0 \supset B_1$ [9].

We define the corresponding interpolation spaces $[B_0, B_1]_\theta^w$, $[B_0, B_1]_\theta^\sigma$, $[B_0, B_1]_\theta^\alpha$, $[B_0, B_1]_\theta$ $(0 < \theta < 1)$ to be $\{f(\theta)\colon f \in \mathcal{F}^w(B_0, B_1)\}$, etc. (equipped with the quotient norms).

$[B_0, B_1]_\theta$ is the space defined by A. P. Calderón [3], see also [2] (and the alternative equivalent definitions in [9] and [13]). $[B_0, B_1]_\theta^\sigma$ is the space defined by Lions [12]. A related construction was introduced by S. G. Krein [11].

It is clear that $[B_0, B_1]_\theta \subset [B_0, B_1]_\theta^\sigma \subset [B_0, B_1]_\theta^w$. It was proved in [9] that $[B_0, B_1]_\theta^\alpha = [B_0, B_1]_\theta$ for any couple (B_0, B_1). Hence, all four interpolation spaces coincide if B_0 is separable or $B_0 \supset B_1$. While this covers many cases, the problem of whether $[B_0, B_1]_\theta$ and $[B_0, B_1]_\theta^w$ always coincide was left open in [9]. In the next section, we will answer this negatively by producing an example with $[B_0, B_1]_\theta \neq [B_0, B_1]_\theta^\sigma = [B_0, B_1]_\theta^w$.

The conditions (A) and (B) on a family of operators $\{T_z\}_{z \in \bar{S}}$ as stated in the introduction can be equivalently reformulated by replacing (A) by the requirement that

$$T_z a \in \mathcal{F}^w(B_0, B_1) \quad \text{for every } a \in A_0 \cap A_1.$$

By replacing $\mathcal{F}^w$ by a smaller space here, we impose further conditions on $\{T_z\}$. The connection between the various interpolation spaces and analytic families may be stated as follows. (See also [10], where the further complications that occur for quasi-Banach spaces are discussed.)

THEOREM 1. *Let (A_0, A_1) and (B_0, B_1) be two compatible couples of Banach spaces and let $\{T_z\}_{z \in \bar{S}}$ be a family of linear operators from $A_0 \cap A_1$ into $B_0 + B_1$ such that the boundedness condition* (B) *holds and*

$$T_z a \in \mathcal{F}^w(B_0, B_1) \quad \textit{for every } a \in A_0 \cap A_1.$$

Then T_θ has a unique extension to a bounded operator mapping $[A_0, A_1]_\theta$ into $[B_0, B_1]_\theta^w$. $\|T_\theta\| \leqslant M_0^{1-\theta} M_1^\theta$. Conversely, $[B_0, B_1]_\theta^w$ is the smallest space that contains $T_\theta(A_0 \cap A_1)$ for all such families $\{T_z\}$. The conclusions remain valid if $\mathcal{F}^w$ and $[B_0, B_1]_\theta^w$ are replaced by $\mathcal{F}^\sigma$ and $[B_0, B_1]_\theta^\sigma$ or $\mathcal{F}^\alpha$ and $[B_0, B_1]_\theta^\alpha = [B_0, B_1]_\theta$.

Proof. By multiplying T_z by a suitable scalar function, we may assume that $M_0 = M_1 = 1$. If $a \in A_0 \cap A_1$ and $\|a\|_{[A_0, A_1]_\theta} < 1$, there exist $a_1, \ldots, a_N \in A_0 \cap A_1$ and $\varphi_1, \ldots, \varphi_N \in A(\bar{S})$ such that $a = \sum_1^N \varphi_k(\theta) a_k$ and $\|\sum_1^N \varphi_k(z) a_k\|_{\mathcal{F}(A_0, A_1)} < 1$ [18]. Let $g(z)$ be $\sum_1^N \varphi_k(z) T_z a_k$. Then $g \in \mathcal{F}^w(B_0, B_1)$, whence $T_\theta a = g(\theta) \in [B_0, B_1]_\theta^w$ and $\|T_\theta a\|_{[B_0, B_1]_\theta^w} \leqslant \|g\|_{\mathcal{F}^w} < 1$. To prove the converse, let $f \in \mathcal{F}^w(B_0, B_1)$. Take a_0 in $A_0 \cap A_1$ and $a^* \in (A_0 + A_1)^*$ such that $\langle a^*, a_0\rangle = 1$. Define $\{T_z\}$ by $T_z a = \langle a^*, a\rangle f(z)$. $\{T_z\}$ satisfies the conditions and $T_\theta a_0 = f(\theta)$.

Remarks. (i) Theorem 1 remains valid for a weakened form of condition (B) allowing a moderate growth of $\|T_{j+it}\|_{A_j, B_j}$ as $t \to \pm\infty$. (Cf. Theorem 2 below where this is explicitly formulated.)

(ii) As stated in the theorem, the range space of T_θ cannot in general be smaller than $[B_0, B_1]_\theta^w$. Thus, for those couples (B_0, B_1) satisfying $[B_0, B_1]_\theta \neq [B_0, B_1]_\theta^w$ (see Section 2) we can construct an analytic family of operators $\{T_z\}$ exactly as in the proof above, which satisfies all the hypotheses of the theorem but such that $T_\theta(A_0 \cap A_1) \not\subset [B_0, B_1]_\theta$.

(iii) It is not reasonable to expect that T_θ extends uniquely to a bounded operator into $[B_0, B_1]_\theta^w$ of a space larger than $[A_0, A_1]_\theta$ such as $[A_0, A_1]_\theta^w$ or $[A_0, A_1]_\theta^\sigma$. This is simply because in general $A_0 \cap A_1$ is not dense in $[A_0, A_1]_\theta^\sigma$ or $[A_0, A_1]_\theta^w$. In fact (see below) the closed hull of $A_0 \cap A_1$ in either of these spaces coincides with $[A_0, A_1]_\theta$. (If one imposes additional stringent conditions on $\{T_z\}$ such as requiring that T_z: $A_0+A_1 \to B_0+B_1$ for each $z \in \bar{S}$, that T_z is an analytic $L(A_0+A_1, B_0+B_1)$ valued function on S and that it is continuous on $\bar{S}$ with respect to the $L(A_0+A_1, B_0+B_1)$ operator norm, then of course T_θ maps $[A_0, A_1]_\theta^\sigma$ into $[B_0, B_1]_\theta^\sigma$.)

Calderón [3] defined a different interpolation method as follows. Let $\mathscr{F}(B_0, B_1)$ be the space of all continuous functions $\bar{S} \to B_0+B_1$ that are analytic on S, bounded by $C(1+|z|)$ and satisfy

$$\sup\{\|f(j+it_1)-f(j+t_2)\|_{B_j}/|t_1-t_2|\} < \infty .$$

Then $[B_0, B_1]^\theta = \{f'(\theta)\colon f \in \mathscr{F}\}$.

It follows from Lemma 2 below that if the unit balls of B_0 and B_1 are closed in B_0+B_1, then $[B_0, B_1]_\theta^w \subset [B_0, B_1]^\theta$. (Strict inequality may occur; see the examples in the sequel.) We do not know whether this inclusion holds for an arbitrary Banach couple (B_0, B_1) although of course we will always have $[B_0, B_1]_\theta^w \subset [\tilde{B}_0, \tilde{B}_1]^\theta$, where $\tilde{B}_0$ and $\tilde{B}_1$ are the Gagliardo completions of B_0 and B_1 in B_0+B_1. Note also that $[\tilde{B}_0, \tilde{B}_1]_\theta = [B_0, B_1]_\theta$ [9], but it is not known whether $[\tilde{B}_0, \tilde{B}_1]^\theta$ and $[B_0, B_1]^\theta$ always coincide. Finally, since

$$[\tilde{B}_0, \tilde{B}_1]_\theta = [B_0, B_1]_\theta \subset [B_0,B_1]_\theta^\sigma \subset [B_0, B_1]_\theta^w \subset [\tilde{B}_0, \tilde{B}_1]^\theta,$$

all inclusions being continuous with norm one, we remark that, by a theorem of Bergh [1], the norms of all these spaces coincide for elements of $\tilde{B}_0 \cap \tilde{B}_1$ and thus also for elements of $[B_0,B_1]_\theta$. (Cf. also [9].)

2. A counter-example. Let $\mathrm{F}L^\infty$, FC, FBMO, FVMO, FQC denote the spaces of sequences which are the Fourier coefficients of functions on the unit circle belonging to L^∞, C, BMO, VMO, QC, respectively, These are Banach spaces with the norms induced by the Fourier transform. Here

$$\mathrm{VMO} = \{f \in \mathrm{BMO}\colon \|f(\cdot+h)-f(\cdot)\|_{\mathrm{BMO}} \to 0 \text{ as } h \to 0\}$$

and $\mathrm{QC} = L^\infty \cap \mathrm{VMO}$ [15]. Let

$$s_n = \begin{cases} 1, & n \geqslant 0, \\ -1, & n < 0. \end{cases}$$

Fefferman's characterisation of BMO may be formulated as follows

$$\{a_n\} \in \mathrm{FBMO} \Leftrightarrow \exists\, \{b_n\},\ \{c_n\} \in \mathrm{F}L^\infty \text{ such that } a_n = b_n + s_n c_n .$$

$C \subset \mathrm{QC} \subset L^\infty$ with strict inclusions; $\sin\log\log|4/t|$, $|t| \leqslant \pi$, is an example of a discontinuous function in QC.

Let $\mathrm{F}L_\alpha^\infty$ $(-\infty < \alpha < \infty)$ denote the space $\{\{a_n\}_{-\infty}^{\infty}\colon \{e^{\alpha n} a_n\} \in \mathrm{F}L^\infty\}$. Thus $\mathrm{F}L_0^\infty = \mathrm{F}L^\infty$. FBMO_α, etc., are defined similarly.

We will prove that, for $0 < \theta < 1$,

$$[\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta = \mathrm{FC}_\theta, \qquad [\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta^\sigma = [\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta^w = \mathrm{FQC}_\theta$$

and

$$[\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]^\theta = \mathrm{F}L_\theta^\infty .$$

We do this in several steps.

(i) Let $\{a_n\} \in [\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]^\theta$ with norm less than one. Then $\{a_n\} = g'(\theta)$ for an analytic function $g(z) = \{g_n(z)\}$ in $\mathscr{F}(\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty)$ with norm less than one. Let p be a trigonometric polynomial. Then, for $\varepsilon > 0$,

$$h(z) = \sum e^{nz} \frac{g_n(z+i\varepsilon)-g_n(z)}{i\varepsilon}\, \hat{p}(n)$$

is a bounded analytic function on S.

$$|h(it)| \leqslant \|g(it+i\varepsilon)-g(it)\|_{\mathrm{F}L^\infty} \|p\|_{L^1}/\varepsilon \leqslant \|p\|_{L^1}.$$

Similarly $|h(1+it)| \leqslant \|p\|_{L^1}$. By the maximum principle $|h(\theta)| \leqslant \|p\|_{L^1}$. Letting $\varepsilon \to 0$ we obtain $\left(\sum e^{n\theta} g_n'(\theta)\, \hat{p}(n)\right) \leqslant \|p\|_{L^1}$, which implies that $\{a_n\} = \{g_n'(\theta)\} \in \mathrm{F}L_\theta^\infty$.

Conversely, if $\{a_n\} \in \mathrm{F}L_\theta^\infty$, we may take $g(z) = \{g_n(z)\}$ with $g_n(z) = -e^{n(\theta-z)} a_n/n$, $n \neq 0$, and $g_0(z) = a_0 z$. Then $g \in \mathscr{F}$ and $\{a_n\} = g'(\theta) \in [\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]^\theta$. (The equality $[\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]^\theta = \mathrm{F}L_\theta^\infty$ also follows by duality from $[\mathrm{F}L_0^1, \mathrm{F}L_{-1}^1]_\theta = \mathrm{F}L_{-\theta}^1$, cf. [9].)

(ii) FC_θ is a closed subspace of $\mathrm{F}L_\theta^\infty$ and $\mathrm{F}L_0^\infty \cap \mathrm{F}L_1^\infty$ is a dense subspace of FC_θ. Thus $[\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta = \mathrm{FC}_\theta$ by (i) and Bergh's theorem [1].

(iii) Let $\{x_n\} \in \mathrm{FBMO}$. Then $\{x_n\} = \{b_n\} + \{s_n c_n\} = \{b_n + c_n\} + \{(s_n - 1) c_n\}$, with $\{b_n\}$, $\{c_n\} \in \mathrm{F}L^\infty$. Hence $\{b_n + c_n\} \in \mathrm{F}L_0^\infty$ and

$$\sum_{-\infty}^{\infty} |e^n (s_n - 1) c_n| = \sum_{-\infty}^{-1} 2|c_n| e^n \leqslant C \sum_{-\infty}^{-1} e^n < \infty,$$

whence $\{(s_n - 1) c_n\} \in \mathrm{F}L_1^\infty$. This proves that $\mathrm{FBMO} \subset \mathrm{F}L_0^\infty + \mathrm{F}L_1^\infty$. (The inclusion is continuous, e.g. by the closed graph theorem.)

Let $\{a_n\} \in \mathrm{FQC}_\theta$ and define $f(z) = \{a_n e^{n(\theta-z)}\}$. Let $h \in \mathrm{QC}$ be such that $\hat{h}(n) = e^{n\theta} a_n$. Then $\|f(j+it)\|_{\mathrm{F}L_j^\infty} = \|h\|_{L^\infty}$, $j = 0, 1$, $-\infty < t < \infty$.

$\{f(it)\}_{-\infty<t<\infty}$ are the Fourier transforms of translates of h. Since $h \in \mathrm{VMO}$, translation of h is continuous in BMO. Thus $t \to f(it)$ is continuous as a function into $\mathrm{FBMO} \subset \mathrm{F}L_0^\infty + \mathrm{F}L_1^\infty$. Similarly $t \to f(1+it)$ is continuous into $\mathrm{FBMO}_1 \subset \mathrm{F}L_0^\infty + \mathrm{F}L_1^\infty$. Since f equals the Poisson integral of its boundary values, $f \in \mathscr{F}^\sigma(\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty)$ and $\{a_n\} = f(\theta) \in [\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta^\sigma$.

(iv) If $f(z) = \{f_n(z)\} \in \mathscr{F}^w(\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty)$ and p is a trigonometric polynomial,

$$\left|\sum e^{nz} f_n(z)\hat{p}(n)\right| \leqslant \|f\|_{\mathscr{F}^w} \|p\|_{L^1}$$

on ∂S, and thus on $\bar{S}$. Hence $f(\theta) \in \mathrm{F}L_\theta^\infty$, and $[\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta^w \subset \mathrm{F}L_\theta^\infty$.

An argument similar to (i) and (ii) above shows that $[\mathrm{FBMO}_0, \mathrm{FBMO}_1]^\theta = \mathrm{FBMO}_\theta$ and $[\mathrm{FBMO}_0, \mathrm{FBMO}_1]_\theta = \mathrm{FVMO}_\theta$. (We use the duality of H^1 and BMO.) The mapping P: $\{a_n\} \to \{\frac{1}{2}(1+s_n)a_n\}$ maps $\mathrm{F}L^\infty$ into the subspace FBMO^+ of Fourier sequences of analytic BMO-functions. Since P maps $\mathrm{F}L_\alpha^\infty$ into FBMO_α^+, P maps $[\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta^w$ into

$$[\mathrm{FBMO}_0^+, \mathrm{FBMO}_1^+]_\theta^w = [\mathrm{FBMO}_0^+, \mathrm{FBMO}_1^+]_\theta \subset [\mathrm{FBMO}_0, \mathrm{FBMO}_1]_\theta = \mathrm{FVMO}_\theta,$$

where we have used $\mathrm{FBMO}_0^+ \supset \mathrm{FBMO}_1^+$. Similarly $I-P$ maps $[\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta^w$ into FVMO_θ. Thus, $[\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta^w \subset \mathrm{FVMO}_\theta$. Consequently,

$$[\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta^w \subset \mathrm{F}L_\theta^\infty \cap \mathrm{FVMO}_\theta = \mathrm{FQC}_\theta.$$

This completes the proof that

$$[\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta^\sigma = [\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta^w = \mathrm{FQC}_\theta \neq [\mathrm{F}L_0^\infty, \mathrm{F}L_1^\infty]_\theta.$$

3. Another interpolation theorem. Let φ be a 1-1 conformal map of the unit disc U onto S. We define $N^+(\bar{S})$ to be the algebra of all complex functions f on $\bar{S}$ such that $f \circ \varphi \in N^+(U)$ and $f(j+it) = \lim_{s \to j} f(s+it)$ for almost every t, $j = 0, 1$, cf. [6]. $H^\infty(\bar{S})$ is the algebra of all bounded complex functions on $\bar{S}$ which are analytic on S and satisfy $f(j+it) = \lim_{s \to j} f(s+it)$ for almost every $t, j = 0, 1$. Then $A(\bar{S}) \subset H^\infty(\bar{S}) \subset N^+(\bar{S})$.

LEMMA 2. *Assume that the Banach spaces B_0 and B_1 are continuously embedded in a Banach space B and that B^+ is a linear subspace of the dual space B^* that determines the norms B_0, B_1 and B, i.e., there are subsets $\Gamma_0, \Gamma_1, \Gamma$ of B^+ such that*

$$\sup\{|\langle b^+, b\rangle|,\ b^+ \in \Gamma_j\} = \|b\|_{B_j}, \quad j = 0, 1,\ b \in B$$

(where $\|b\|_{B_j} = \infty$ if $b \in B \setminus B_j$) and

$$\sup\{|\langle b^+, b\rangle|,\ b^+ \in \Gamma\} = \|b\|_B, \quad b \in B.$$

(By the bipolar theorem this is equivalent to the unit balls of B_0, B_1, B being $\sigma(B, B^+)$-closed.)

Let f be a function $\bar{S} \to B$ such that $\langle b^+, f(z)\rangle \in H^\infty(\bar{S})$ for $b^+ \in B^+$ and $\|f(j+it)\|_{B_j} \leqslant 1$ for a.e. t, $j = 0, 1$. Then $f(\theta) \in [B_0, B_1]^\theta$ and $\|f(\theta)\|_{[B_0, B_1]^\theta} \leqslant 1$.

Proof. Let $\varepsilon > 0$ and let

$$f_1(z) = e^{\varepsilon(z-\theta)^2} \frac{e^{\varepsilon(z-\theta)} - 1}{\varepsilon(z-\theta)} f(z).$$

Then $\langle b^+, f_1(z)\rangle \in H^\infty(\bar{S})$ for all $b^+ \in B^+$, and

$$\|f_1(j+it)\|_{B_j} \leqslant e^{2\varepsilon - \varepsilon t^2} \leqslant e^{2\varepsilon} \text{ a.e.}$$

Since the embeddings $B_j \to B$ are continuous, there exists a constant C such that $\|f_1(j+it)\|_B \leqslant C$ a.e., $j = 0, 1$. This and the maximum principle yield that

$$|\langle b^+, f_1(z)\rangle| \leqslant C\|b^+\|_{B^*}, \quad z \in S,\ b^+ \in B^+.$$

Since B^+ determines the norm on B, $\|f_1(z)\|_B \leqslant C$.

By uniform convergence, $\langle b^*, f_1(z)\rangle \in H^\infty(\bar{S})$ for any $b^* \in \bar{B}^+$, the closed hull of B^+ in B^*. Since we may regard B as a subspace of the dual of the Banach space $\bar{B}$, the uniform boundedness principle applies to show that $f_1(z)$ is an analytic, and thus continuous, map from S into B (cf. [8], p. 93).

Fix $z_0 \in S$ and define

$$g_1(z) = \int_{z_0}^{z} f_1(w)\,dw \quad \text{for each } z \in S.$$

Since f_1 is continuous on S, this integral, taken along any rectifiable curve in S, exists in B. It is path independent since

$$\left\langle b^+, \int_{z_0}^{z} f_1(w)\,dw \right\rangle = \int_{z_0}^{z} \langle b^+, f_1(w)\rangle\,dw$$

holds for any $b^+ \in B^+$, and B^+ separates points of B. Furthermore,

$$\|g_1(z_1) - g_1(z_2)\|_B = \left\|\int_{z_1}^{z_2} f_1(z)\,dz\right\|_B \leqslant \int_{z_1}^{z_2} \|f_1(z)\|_B\,d|z| \leqslant C|z_1 - z_2|.$$

Thus, $g_1(z)$ is uniformly continuous, and g_1 may be extended to a continuous function $\bar{S} \to B$.

Then, for $b^+ \in B^+$,

$$\langle b^+, g_1(it_2) - g_1(it_1)\rangle = \lim_{s \to 0} \langle b^+, g_1(s+it_2) - g_1(s+it_1)\rangle$$
$$= \lim_{s \to 0} \int_{t_1}^{t_2} \langle b^+, f_1(s+it)\rangle\,dt = \int_{t_1}^{t_2} \langle b^+, f_1(it)\rangle\,dt.$$

Thus, if $b^+ \in \Gamma_0$

$$|\langle b^+, g_1(it_2) - g_1(it_1)\rangle| \leqslant \int_{t_1}^{t_2} \|f_1(it)\|_{B_0}\, dt.$$

Since B^+ determines the norm of B_0,

$$\|g_1(it_2) - g_1(it_1)\|_{B_0} \leqslant \int_{t_1}^{t_2} \|f_1(it)\|_{B_0}\, dt \leqslant |t_2 - t_1|\, e^{2\varepsilon}.$$

Similarly,

$$\|g_1(1+it_2) - g_1(1+it_1)\|_{B_1} \leqslant |t_2 - t_1|\, e^{2\varepsilon}.$$

Now, let $g_2(z) = g_1(z) - g_1(z - 2\pi i/\varepsilon)$, $z \in \bar{S}$. Then, if $\operatorname{Re} z = j = 0, 1$, $g_2(z) \in B_j$ and $\|g_2(z)\|_{B_j} \leqslant 2\pi e^{2\varepsilon}/\varepsilon$. Furthermore, for any t, $|t|$ or $|t - 2\pi/\varepsilon|$ exceeds π/ε. Thus,

$$\|f_1(it)\|_{B_0} + \|f_1(it - 2\pi i/\varepsilon)\|_{B_0} \leqslant e^{2\varepsilon} + e^{2\varepsilon - \pi^2/\varepsilon} < e^{3\varepsilon},$$

whence

$$\|g_2(it_2) - g_2(it_1)\|_{B_0} \leqslant \int_{t_1}^{t_2} (\|f_1(it)\|_{B_0} + \|f_1(it - 2\pi i/\varepsilon)\|_{B_0})\, dt \leqslant |t_2 - t_1|\, e^{3\varepsilon}.$$

Similarly

$$\|g_2(1+it_2) - g_2(1+it_1)\|_{B_1} \leqslant |t_2 - t_1|\, e^{3\varepsilon}.$$

In particular, g_2 restricted to ∂S is a bounded continuous function into $B_0 + B_1$. Let g_3 be the Poisson integral of this restriction. Then g_3 is a bounded continuous function $\bar{S} \to B_0 + B_1$. On the boundary ∂S g_3 equals g_2. Also, for any $b^+ \in B^+ \subset (B_0 + B_1)^*$, $\langle b^+, g_3(z)\rangle$ is a bounded harmonic function, while $\langle b^+, g_2(z)\rangle$ is bounded and analytic. Consequently, $\langle b^+, g_3(x)\rangle = \langle b^+, g_2(z)\rangle$ in $\bar{S}$, and $g_3 = g_2$. Thus, $g_2 = g_3$ is a continuous function into $B_0 + B_1$. For any closed rectifiable curve in S, $\int_\gamma g_2(z)\, dz$ is defined in $B_0 + B_1$, but another application of $b^+ \in B^+$ shows that this integral vanishes and so $\int_\gamma \langle b^*, g_2(z)\rangle\, dz = 0$ for all $b^* \in (B_0 + B_1)^*$. By Morera's theorem g_2 is analytic $S \to B_0 + B_1$.

We have shown that $g_2 \in \mathscr{F}(B_0, B_1)$ with norm $\leqslant e^{3\varepsilon}$. Since

$$g_2'(\theta) = g_1'(\theta) - g_1'(\theta - 2\pi i/\varepsilon) = f_1(\theta) - f_1(\theta - 2\pi i/\varepsilon) = f(\theta),$$

$f(\theta) \in [B_0, B_1]^\theta$ and $\|f(\theta)\|_{[B_0, B_1]^\theta} \leqslant e^{3\varepsilon}$. Now, let $\varepsilon \to 0$.

Remark. In this lemma and the theorem below, the space B in which B_0 and B_1 are continuously embedded, may more generally be a locally convex topological vector space which is quasi-complete. (We recall that this means that each of its bounded closed subsets is complete.) B^+ should then be a subset of B^* which determines a fundamental set of seminorms on B (i.e. there exists a basis of neighborhoods of 0 which are $\sigma(B, B^+)$-closed). Of course, B^+ must also determine the norms of B_0 and B_1 as before.

The proof is the same except for the following modifications: Let p be a seminorm determined by B^+. Let $B_p = B/p^{-1}\{0\}$ and

$$B_p^+ = \{b^+ \in B^+ : |b^+(b)| \leqslant Cp(b),\ b \in B, \text{ for some } C < \infty\}.$$

The seminorm p becomes a norm on B_p and the completion $\tilde{B}_p$ is a Banach space. B_p^+ determines p on B_p and we may apply the same argument as above (cf. [8], pp. 93, 94) to $f_1(z)$ regarded as a $\tilde{B}_p$ valued function. Consequently, $f_1(z)$ is continuous on S with respect to p. It is also clear that $p(f_1(z))$ is bounded on S. This holds for all p from a fundamental set of seminorms on B, which is equivalent to f_1 being bounded and continuous $S \to B$.

Since B is quasicomplete, the integrals defining g_1 exist in B in the weak (Pettis) sense [8], p. 77, [16], p. 201. Again, g_1 is uniformly continuous ($p(g_1(z_1) - g_1(z_2)) \leqslant C_p|z_1 - z_2|$) and g_1 may be extended to $\bar{S}$, B being quasicomplete. The rest of the proof is the same.

Two examples of such B and B^+ when B_0 and B_1 are suitable function spaces are $B = L^1_{\text{loc}}$, $B^+ = \{$integrable simple functions$\}$ (cf. [19]) and $B = \mathscr{D}'$, $B^+ = C_0^\infty$.

THEOREM 2. *Assume that* (A_0, A_1) *is a compatible couple of Banach spaces, that* A *is a dense subspace of* $A_0 \cap A_1$ *and that* B_0, B_1, B, B^+ *are as above.*

Let $\{T_z\}_{z\in\bar{S}}$ *be a family of linear operators from* A *to* B *such that* $\langle b^+, T_z a\rangle \in N^+(\bar{S})$ *for any* $b^+ \in B^+$ *and* $a \in A$, *and that* $\|T_{j+it} a\|_{B_j} \leqslant M_j(t)\|a\|_{A_j}$ *a.e.,* $a \in A$, *where* M_0 *and* M_1 *are measurable functions satisfying*

$$m(\theta) = \int \log M_0(t)\, P_0(\theta, t)\, dt + \int \log M_1(t)\, P_1(\theta, t)\, dt < \infty,$$

where $P_j(\theta, t)$, $j = 0, 1$, *are the Poisson kernels for* S (cf. [2], p. 93). *Then* T_θ $(0 < \theta < 1)$ *has a unique extension to a bounded operator* $[A_0, A_1]_\theta \to [B_0, B_1]^\theta$ *and* $\|T_\theta\| \leqslant e^{m(\theta)}$.

Proof. There is an outer function $\psi \in N^+(\bar{S})$ such that $|\psi(j+it)| = M(j+it)^{-1}$ a.e. and $|\psi(\theta)| = e^{-m(\theta)}$. Multiplying T_z by $\psi(z)$, we may assume that $M_j(t) \equiv 1$. Then

$$|\langle b^+, T_z a\rangle| \leqslant C\|b^+\|_B \|a\|_{A_0\cap A_1} \quad \text{a.e. on } \partial S.$$

By the maximum principle, this holds for all $z \in \bar{S}$. Thus $\langle b^+, T_z a\rangle \in H(\bar{S})$. This also implies that $\|T_z a\|_B \leqslant C\|a\|_{A_0\cap A_1}$. Hence, each T_z may be extended to $A_0 \cap A_1$. These extensions will satisfy $\langle b^+, T_z a\rangle \in H^\infty(\bar{S})$ and

$\|T_{j+it}a\|_{B_j} \leqslant 1$ a.e. for every $a \in A_0 \cap A_1$ and $b^+ \in B^+$. Thus, we may assume that $A = A_0 \cap A_1$.

Now, let $a \in A_0 \cap A_1$ with $\|a\|_{[A_0, A_1]_\theta} < 1$. There exist $a_1, \ldots, a_N \in A_0 \cap A_1$ and $\varphi_1, \ldots, \varphi_N \in A(\bar{S})$ such that $a = \sum_{k=1}^{N} \varphi_k(\theta)\, a_k$ and $\|\sum \varphi_k(z)\, a_k\|_{\mathscr{F}(A_0, A_1)} < 1$. $\sum_1^N \varphi_k(z)\, T_z\, a_k$ satisfies all conditions of Lemma 2. Consequently,

$$T_\theta\, a = \sum \varphi_k(\theta)\, T_\theta\, a_k \in [B_0, B_1]^\theta \quad \text{and} \quad \|T_\theta\, a\|_{[B_0, B_1]^\theta} \leqslant 1.$$

Corollary. *Assume that (A_0, A_1) and (B_0, B_1) are two couples of Banach spaces and that $B_0 \cap B_1$ is dense in B_0 and B_1.*

Let $\{T_z\}_{z \in \bar{S}}$ be a family of linear operators from $A_0 \cap A_1$ into $(B_0 \cap B_1)^* = B_0^* + B_1^*$ such that $\langle b, T_z\, a\rangle \in N^+(\bar{S})$ for every $a \in A_0 \cap A_1$ and $b \in B_0 \cap B_1$, and that $\|T_{j+it}\, a\|_{B_j^*} \leqslant M_j(t)\, \|a\|_{A_j}$ a.e., $a \in A_0 \cap A_1$, where $M_j(t)$ are as above. Then $T_\theta (0 < \theta < 1)$ has a unique continuous extension $[A_0, A_1]_\theta \to [B_0^*, B_1^*]^\theta$ $(= [B_0, B_1]_\theta^*)$.

This corollary is also a special case of [5], Th. 4.2. The following example shows that in general T_θ does not map $A_0 \cap A_1$ into $[B_0, B_1]_\theta$ or $[B_0, B_1]_\theta^w$. Thus, weak-$*$-continuity is not sufficient in Theorem 1.

Example. We will use l^1, l^∞ and c_0 and their weighted counterparts $l_\alpha^1, l_\alpha^\infty, (c_0)_\alpha$. (The notation is as in Section 2.) Let $A_0 = A_1$ be one-dimensional and take $B_0 = l_0^1$, $B_1 = l_{-1}^1$. Then $B_0^* = l_0^\infty$, $B_1^* = l_1^\infty$, $[l_0^\infty, l_1^\infty]^\theta = l_\theta^\infty$ and $[l_0^\infty, l_1^\infty]_\theta = (c_0)_\theta$. To calculate $[l_0^\infty, l_1^\infty]_\theta^w$ observe first that if we deal with spaces of one-sided sequences, then either $l_1^\infty \subset l_0^\infty$ or $l_0^\infty \subset l_1^\infty$ so that $[l_0^\infty, l_1^\infty]_\theta^w = [l_0^\infty, l_1^\infty]_\theta$ (see Section 1). But the same result now follows for two-sided sequences by considering them as direct sums of one-sided sequence spaces.

Let $\{a_n\} \in l_\theta^\infty$ and define T_z by $T_z 1 = \{a_n e^{n(\theta - z)}\}$. If $b = \{b_n\} \in l_0^1 \cap l_{-1}^1$, then $\langle b, T_z\, a\rangle = a \sum a_n b_n e^{n(\theta - z)} \in A(\bar{S})$ since the series converges uniformly. $\|T_{j+it}\, a\|_{B_j} \leqslant |a|$. Thus, $\{T_z\}$ satisfies the conditions and $T_\theta 1 = \{a_n\}$, an arbitrary element of l_θ^∞.

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Received December 27, 1982 (1853)