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Received April 7, 1983
Revised version September 9, 1983 (1879)

STUDIA MATHEMATICA, T. LXXX. (1984)

Closed subgroups of nuclear spaces are weakly closed

by

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Abstract. A proof is given that a closed additive subgroup of a nuclear space is weakly closed. This generalizes the result obtained in [1].

It has been proved in [1] that if K is a discrete additive subgroup of a nuclear space E, then the quotient group E/K admits sufficiently many continuous characters, which means precisely that K is weakly closed in E. It appears, however, that it suffices to assume K to be closed. This result admits two equivalent formulations.

THEOREM A. *A closed additive subgroup of a nuclear space is weakly closed.*

THEOREM B. *If K is a closed additive subgroup of a nuclear space E, then the quotient group E/K admits sufficiently many continuous characters.*

We shall prove Theorem A. For the equivalence of A and B see Lemma 8 below. These theorems provide another illustration of the fact that nuclear spaces are more closely related to finite dimensional spaces than normed spaces are, since, as it has been proved in [2], they do not hold in any infinite dimensional normed space (see also Corollary 3 below). In fact, these theorems characterize nuclear spaces; more precisely, if they hold in a B_0^*-space E, then E is nuclear. The proof will be given elsewhere.

Let A be a subset of a topological vector space E. The symbols $\bar{A}$, $\bar{A}^w$, $\operatorname{span} A$ and $\operatorname{int} A$ will denote respectively the closure, the weak closure, the linear span and the interior of A. If E is a metric space, then $\operatorname{diam} A$ will denote the diameter of A, and $d(u, A)$ the distance of a point $u \in E$ to A. By $\operatorname{gp} A$ we shall denote the additive subgroup of E generated by A. Speaking of subgroups of vector spaces we shall omit the word "additive".

If E is a unitary space, then the scalar product of vectors $u, w \in E$ will be denoted by (u, w). By an *ellipsoid* in E we shall always mean an ellipsoid which is closed and convex. If T is a linear operator acting between normed spaces, then $d_n(T)$, $n = 1, 2, \ldots$, will denote the nth Kolmogorov number of T.

We shall obtain Theorem A as an easy consequence of the following proposition.

THEOREM C. *Let H_1, H_2 be unitary spaces, and let T: $H_1 \to H_2$ be a linear operator such that*

$$\sum_{n=1}^{\infty} n d_n(T) < \infty.$$

Then for each subset $A \subset H_1$ we have

$$T(\bar{A}^w) \subset \overline{\operatorname{gp} T(A)}.$$

We shall begin with some lemmas. Let E be an n-dimensional real unitary space, and let D be an n-dimensional ellipsoid in E. Let $u \in E \setminus \{0\}$, and let L be the set of all those straight lines l in E parallel to u for which $l \cap D$ is a segment with length $\geqslant 1$. Let N be the orthogonal complement of u in E. Then the set

$$D'_p = \bigcup_{l \in L} (l \cap N)$$

is an $(n-1)$-dimensional ellipsoid in N. If D is small, then D'_p can reduce to one point or can be empty. Obviously, D'_p is contained in the orthogonal projection of D onto N. In the described situation we shall say that D'_p is a *reduced projection* of D.

When the ellipsoid D is degenerate, i.e. when it is empty or reduces to one point, then by a reduced projection of D we shall mean the empty set.

Now, let M be an $(n-1)$-dimensional affine subspace in E such that the distance of M to the centre of D is equal to 1/2. Then the set $D'_s = D \cap M$ is an $(n-1)$-dimensional ellipsoid in M. If D is small, then D'_s can reduce to one point or can be empty. In this situation we shall say that D'_s is a *reduced section* of D. By a reduced section of a degenerate ellipsoid we shall mean the empty set.

There is a kind of duality between reduced sections and projections.

LEMMA 1. *Let E be an n-dimensional real unitary space and let D be an n-dimensional ellipsoid in E (degenerate or not). Then to each reduced projection of D there corresponds an isometric reduced section of D.*

Proof. We may assume that $E = \boldsymbol{R}^n$ and that

$$D = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n:\ x_1^2/\mu_1^2 + \ldots + x_n^2/\mu_n^2 \leqslant 1\}$$

for some $\mu_1, \ldots, \mu_n > 0$. Let $\xi = (\xi_1, \ldots, \xi_n)$ be an arbitrary vector belonging to the boundary of D, and let D'_p be the reduced projection of D determined by ξ. Let then

$$M' = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n:\ x_1 \xi_1/\mu_1 + \ldots + x_n \xi_n/\mu_n = 0\},$$

and let M be one of the two $(n-1)$-dimensional affine subspaces parallel to M' such that $d(0, M) = 1/2$. Then $D \cap M$ is a reduced section of D, and direct computations show that $D \cap M$ and D'_p are isometric. ■

LEMMA 2. *Let D be an n-dimensional ellipsoid with principal semiaxes $\lambda_1, \ldots, \lambda_n$, and let P be any n-dimensional rectangular parallelepiped circumscribed about D. Then*

$$\operatorname{diam} P = 2(\lambda_1^2 + \ldots + \lambda_n^2)^{1/2}.$$

The proof is standard.

LEMMA 3. *Let E be an n-dimensional real unitary space, and let D be an n-dimensional ellipsoid in E with principal semiaxes $\mu_1 \leqslant \ldots \leqslant \mu_n$ such that $\mu_1^{-2} + \ldots + \mu_n^{-2} < 4$. Let $D_n, \ldots, D_1$ be subsets of E such that $D_n = D$ and D_k is a reduced projection of D_{k+1} for each $k = 1, \ldots, n-1$. Then for each $k = 1, \ldots, n-1$, D_k is a non-degenerate k-dimensional ellipsoid, and if $\mu_{k1} \leqslant \ldots \leqslant \mu_{kk}$ are its principal semiaxes, then*

$$\mu_{ki} \geqslant \Big[1 - (1/4) \sum_{k=1}^{n} \mu_k^{-2}\Big]^{1/2} \mu_i, \quad 1 \leqslant i \leqslant k \leqslant n-1.$$

Proof. We may assume that the centre of D is zero. We may assume also, owing to Lemma 1, that D_k is a reduced section of D_{k+1} for $k = 1, \ldots, n-1$. Then there exist affine subspaces $M_1 \subset \ldots \subset M_{n-1}$ in E such that $\dim M_k = k$ and $D_k = M_k \cap D_{k+1}$ for $k = 1, \ldots, n-1$. Let B be the closed unit ball in E and let T be a linear operator in E such that $T(D) = B$. We can choose an orthonormal basis $e_1, \ldots, e_n$ in E such that

$$T(M_k) = \{u \in E:\ (u, e_i) = \gamma_i \text{ for } i > k\}, \quad k = 1, \ldots, n-1,$$

where $\gamma_2, \ldots, \gamma_n$ are some constants.

For each $k = 1, \ldots, n$ let o_k be the centre of D_k, and let

$$L_k = \{u \in E:\ (u, e_{k+1}) = \gamma_{k+1}\}, \quad k = 1, \ldots, n-1.$$

According to the definition of a reduced section, for each $k = 1, \ldots, n-1$ we have $d(o_{k+1}, M_k) = 1/2$, whence $M_k \cap (o_{k+1} + B/2) \neq \emptyset$. Therefore

$$T(M_k) \cap \big(To_{k+1} + T(B/2)\big) \neq \emptyset, \quad k = 1, \ldots, n-1.$$

But the vector To_{k+1} is parallel to L_k, and $T(M_k) \subset L_k$, which implies that

$$L_k \cap T(B/2) \neq \emptyset, \quad k = 1, \ldots, n-1. \tag{1}$$

Let P be the $(n-1)$-dimensional rectangular parallelepiped determined by the conditions $(u, e_1) = 0$ and $|(u, e_k)| \leqslant |\gamma_k|$ for $k = 2, \ldots, n$. Its one-dimensional edges are equal respectively to $2|\gamma_2|, \ldots, 2|\gamma_n|$, therefore

$$\operatorname{diam} P = 2(\gamma_2^2 + \ldots + \gamma_n^2)^{1/2}.$$

Now let P' be the n-dimensional rectangular parallelepiped circumscribed about the ellipsoid $T(B/2)$, with $(n-1)$-dimensional faces orthogonal respectively to $e_1, \ldots, e_n$. Let $\lambda_k = \mu_k^{-1}$ for $k = 1, \ldots, n$. The principal semiaxes of $T(B/2)$ are equal respectively to $\lambda_1/2, \ldots, \lambda_n/2$, therefore by Lemma 2 we have

$$\operatorname{diam} P' = (\lambda_1^2 + \ldots + \lambda_n^2)^{1/2}.$$

Now (1) implies that $P \subset P'$, whence $\operatorname{diam} P \leqslant \operatorname{diam} P'$, i.e.

$$\gamma_2^2 + \ldots + \gamma_n^2 \leqslant (1/4)(\lambda_1^2 + \ldots + \lambda_n^2). \tag{2}$$

We have $T(D_n) = B$ and $T(D_k) = T(D_{k+1}) \cap T(M_k)$ for $k = 1, \ldots, n-1$. Therefore $T(D_k)$ is a k-dimensional ball for $k = 1, \ldots, n$ — let r_k be its radius. We have $r_n = 1$ and, as is easily seen, $r_k^2 = r_{k+1}^2 - \gamma_{k+1}^2$ for $k = 1, \ldots, n-1$, therefore

$$r_k^2 = 1 - (\gamma_{k+1}^2 + \ldots + \gamma_n^2) \geqslant 1 - (\gamma_2^2 + \ldots + \gamma_n^2), \qquad k = 1, \ldots, n-1.$$

Hence by (2) we obtain

$$r_k^2 \geqslant 1 - (1/4)(\lambda_1^2 + \ldots + \lambda_n^2), \qquad k = 1, \ldots, n-1. \tag{3}$$

For each $k = 1, \ldots, n-1$ let N_k be the k-dimensional linear subspace parallel to $T(M_k)$, and let $B_k = B \cap N_k$. Then $T(D_k) = To_k + r_k B_k$, and

$$D_k = o_k + r_k\big(T^{-1}(B) \cap T^{-1}(N_k)\big) = o_k + r_k\big(D \cap T^{-1}(N_k)\big)$$

for $k = 1, \ldots, n-1$. Let $\mu'_{k1} \leqslant \ldots \leqslant \mu'_{kk}$ be the principal semiaxes of the ellipsoid $D \cap T^{-1}(N_k)$, $k = 1, \ldots, n-1$. We have $\mu'_{ki} \geqslant \mu_i$ for $i = 1, \ldots, k$ — this is a simple geometrical fact. To complete the proof it is enough now to observe that for each $k = 1, \ldots, n-1$ the principal semiaxes of D_k are equal to the corresponding principal semiaxes of $D \cap T^{-1}(N_k)$ multiplied by r_k, and use (3). ■

A subgroup K of a normed space is called *1-discrete*, if $\|u\| \geqslant 1$ for any $u \in K$, $u \neq 0$.

LEMMA 4. *Let E be an n-dimensional real unitary space, and let D be an n-dimensional ellipsoid in E with centre a and principal semiaxes $\mu_1, \ldots, \mu_n$ such that $\mu_1^{-2} + \ldots + \mu_n^{-2} \leqslant 1$. Let K be a subgroup of E such that $K \cap \operatorname{int} D = \emptyset$. Then there exists an orthogonal projection P: $E \to E$ such that $P(K)$ is 1-discrete, and $d(Pa, P(K)) \geqslant 1/2$.*

Proof. The condition $\mu_1^{-2} + \ldots + \mu_n^{-2} \leqslant 1$ implies that

$$\mu_k \geqslant 1 \quad \text{for} \quad k = 1, \ldots, n; \tag{4}$$

$$[1 - (1/4)(\mu_1^{-2} + \ldots + \mu_n^{-2})]^{1/2} \geqslant 1/2. \tag{5}$$

Let B be the closed unit ball in E. If K is 1-discrete, then we take $P = \mathrm{id}_E$. Then (4) gives $D \supset a + B$, whence

$$K \cap (a + \operatorname{int} B) \subset K \cap \operatorname{int} D = \emptyset,$$

and

$$d\big(Pa, P(K)\big) = d(a, K) \geqslant 1 \geqslant 1/2.$$

So let us assume that K is not 1-discrete, i.e. that there is an $u \in K$ with $0 < \|u\| < 1$. Let N be the orthogonal complement of u in E and let P_n be the orthogonal projection onto N. Let D_{n-1} be the reduced projection of D determined by u. Then, as is easily seen, we have $P_n(K) \cap \operatorname{int} D_{n-1} = \emptyset$. Moreover, the centre of D_{n-1} is $P_n a$. If $P_n(K)$ is 1-discrete, then we take $P = P_n$. Then (4), (5) and Lemma 3 give $D_{n-1} \supset P_n a + B_{n-1}/2$, where B_{n-1} is the closed unit ball in $P_n(E)$. Hence

$$P_n(K) \cap (P_n a + \operatorname{int} B_{n-1}/2) \subset P_n(K) \cap \operatorname{int} D_{n-1} = \emptyset,$$

and

$$d\big(Pa, P(K)\big) = d\big(P_n a, P_n(K)\big) \geqslant 1/2.$$

If, on the other hand, $P_n(K)$ is not 1-discrete, then we can repeat the above procedure to obtain an orthogonal projection P_{n-1} in $P_n(E)$, and so on. Thus we shall obtain orthogonal projections $P_n, P_{n-1}, \ldots, P_{k+1}$ and ellipsoids $D, D_{n-1}, \ldots, D_k$ such that D_i is a reduced projection of D_{i+1} for $i = k, \ldots, n-1$ (we define $D_n = D$). The described process can stop only in the following two cases:

(i) $k = 2, \ldots, n-2$ and $P_{k+1} \ldots P_{n-1} P_n(K)$ is 1-discrete. Then we take $P = P_{k+1} \ldots P_{n-1} P_n$. The proof that $d\big(Pa, P(K)\big) \geqslant 1/2$ is now the same as above for $k = n-1$.

(ii) $k = 1$; then, by (4), (5) and Lemma 3, D_1 is a segment with length $\geqslant 1$ and centre $P_2 \ldots P_n a$, so that we can take $P = P_2 \ldots P_n$. ■

LEMMA 5. *Let E, F be n-dimensional real unitary spaces, and let T: $E \to F$ be an invertible linear operator. Let $a \in E$ and let K be a subgroup of E such that $T(K)$ is 1-discrete and $d(a, K) \geqslant 1/4$. Then there exists an $f \in E^*$ such that $f(K) \subset \boldsymbol{Z}$, $f(a) \in [1/4, 3/4] + \boldsymbol{Z}$, and*

$$\|f\| \leqslant 1 + \Big[\sum_{k=1}^{n} k^2 \big(d_1(T) \ldots d_k(T)\big)^{2/k}\Big]^{1/2}.$$

The proof can be obtained by repeating the proofs of Lemmas 3 and 4 from [1], with slightly modified constants.

LEMMA 6. *If $a_1, a_2, a_3, \ldots$ is a sequence of non-negative real numbers, not all equal to zero, then*

$$\sum_{n=1}^{\infty} (a_1 a_2 a_3 \ldots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n.$$

The proof can be found, for instance, in [4], chap. XVI, 4.

LEMMA 7. *Let E, F be n-dimensional real unitary spaces, and let $T\colon E \to F$ be a linear operator such that*

$$\sum_{k=1}^{n} k d_k(T) \leqslant 1.$$

Let $a \in E$ and let K be a subgroup of E such that $d(Ta,\ T(K)) \geqslant 1$. Then there exists an $f \in E^*$ such that $\|f\| < 6$, $f(K) \subset \mathbf{Z}$, and $f(a) \in [1/4, 3/4] + \mathbf{Z}$.

Proof. Let $E' = E/\ker T$ and let $\psi\colon E \to E'$ be the canonical mapping. The operator $T'\colon E' \to F$ determined by the condition $T'\psi = T$ is invertible, and $d_k(T') = d_k(T)$ for $k = 1, 2, \ldots$ As easily seen, there exist invertible linear operators $S\colon E' \to E'$ and $R\colon E' \to F$ such that $RS = T'$,

$$d_k(R) = \big(k d_k(T)\big)^{1/2} \quad \text{for} \quad k = 1, 2, \ldots,$$
$$d_k(S) = \big(k^{-1} d_k(T)\big)^{1/2} \quad \text{for} \quad k = 1, 2, \ldots$$

Let B be the closed unit ball in F and let $D = R^{-1}(B)$. Since $d(Ta,\ T(K)) \geqslant 1$ and $T = T'\psi = RS\psi$, we have

$$d\big(RS\psi(a),\ RS\psi(K)\big) \geqslant 1,$$

i.e.

$$[RS\psi(a) + \operatorname{int} B] \cap RS\psi(K) = \emptyset \quad \text{or} \quad [S\psi(a) + \operatorname{int} D] \cap S\psi(K) = \emptyset.$$

The principal semiaxes of D are equal respectively to $(d_1(R))^{-1}, \ldots, (d_{n'}(R))^{-1}$, where $n' = \dim E'$; moreover,

$$(d_1(R))^2 + \ldots + (d_{n'}(R))^2 = \sum_{k=1}^{n'} k d_k(T) \leqslant 1.$$

Therefore we can apply Lemma 4. We obtain an orthogonal projection $P\colon E' \to E'$ such that $PS\psi(K)$ is 1-discrete and $d\big(PS\psi(a),\ PS\psi(K)\big) \geqslant 1/2$.

Let $E'' = E'/\ker PS$, let $\varphi\colon E' \to E''$ be the canonical mapping, and let $S'\colon E'' \to P(E')$ be the operator for which $S'\varphi = PS$. We have the following commutative diagram:

$$\begin{tikzcd}
E \arrow[rr, "T"] \arrow[d, "\psi"'] & & F \arrow[d, "\mathrm{id}"] \\
E' \arrow[r, "S"] \arrow[rru, "T'"] \arrow[d, "\varphi"'] & E' \arrow[r, "R"] \arrow[d, "P"] & F \\
E'' \arrow[r, "S'"] & P(E') &
\end{tikzcd}$$

According to Lemma 5, there is an $f' \in (E'')^*$ such that $f'(\varphi\psi(K)) \subset \mathbf{Z}$, $f'(\varphi\psi(a)) \in [1/4, 3/4] + \mathbf{Z}$, and

$$\|f'\| \leqslant 1 + \Big[\sum_{k=1}^{n''} k^2 \big(d_1(S') \ldots d_k(S')\big)^{2/k}\Big]^{1/2},$$

where $n'' = \dim E''$. For each $k = 1, 2, \ldots$ we have

$$d_k(S') = d_k(PS) \leqslant \|P\|\, d_k(S) = d_k(S).$$

Therefore, owing to Lemma 6, we obtain

$$\begin{aligned}
\|f'\| &\leqslant 1 + \Big[\sum_{k=1}^{n''} k^2 \big(d_1(S) \ldots d_k(S)\big)^{2/k}\Big]^{1/2} \\
&\leqslant 1 + \left[\sum_{k=1}^{\infty} k^2 \left(\frac{d_1(T)}{1}\frac{d_2(T)}{2} \ldots \frac{d_k(T)}{k}\right)^{1/k}\right]^{1/2} \\
&= 1 + \Big[\sum_{k=1}^{\infty} k^2 (k!)^{-2/k} \big(d_1(T) \cdot 2d_2(T) \cdot \ldots \cdot k d_k(T)\big)^{1/k}\Big]^{1/2} \\
&< 1 + \Big[e^3 \sum_{k=1}^{\infty} k d_k(T)\Big]^{1/2} \leqslant 1 + e^{3/2} < 6,
\end{aligned}$$

because $k! > k^k e^{-k}$. Thus the functional $f = f'\varphi\psi$ satisfies the desired conditions. ∎

Proof of Theorem C. We may assume that H_1 and H_2 are real. Let us suppose the contrary, that there exists an $A \subset H_1$ such that $T(\bar{A}^w) \not\subset \overline{\operatorname{gp} T(A)}$. We may assume that $A = \operatorname{gp} A$; then $\operatorname{gp} T(A) = T(A)$. Thus there is an $a \in \bar{A}^w$ such that $Ta \notin \overline{T(A)}$. Finally we may assume that $d(Ta,\ T(A)) \geqslant 1$ and that $\sum_{k=1}^{\infty} k d_k(T) \leqslant 1$.

To obtain a contradiction, we have to show that $a \notin \bar{A}^w$, i.e. that there is an $f \in H_1^*$ such that $f(A) \subset \mathbf{Z}$ and $f(a) \notin \mathbf{Z}$. Owing to the weak compactness of closed balls in H_1^*, it suffices to show that for each finite subset $J \subset A$ there is an $f \in H_1^*$ such that $\|f\| \leqslant 6$, $f(J) \subset \mathbf{Z}$ and $f(a) \in [1/4, 3/4] + \mathbf{Z}$.

So let J be an arbitrary finite subset of A. Let $M = \operatorname{span}(J \cup \{a\})$ and let $T' = T|_M$. Then

$$\sum_{k=1}^{\infty} k d_k(T') \leqslant \sum_{k=1}^{\infty} k d_k(T) \leqslant 1.$$

Let $K = \operatorname{gp} J$; then

$$d(T'a,\ T'(K)) \geqslant d(Ta,\ T(K)) \geqslant 1,$$

and Lemma 7 implies the existence of an $f' \in M^*$ such that $\|f'\| < 6$,

$f'(K) \subset \boldsymbol{Z}$ and $f'(a) \in [1/4, 3/4] + \boldsymbol{Z}$. It is enough now to extend f' to an $f \in H_1^*$ with $\|f\| = \|f'\|$. ∎

Proof of Theorem A. Let K be a closed subgroup of a nuclear space E. We have to show that K is weakly closed. The topology of E can be defined by a family $\{p_i\}_{i \in I}$ of seminorms, such that for each $i \in I$ the space $E_i = E/\ker p_i$ with the canonical quotient norm is a unitary space. Moreover, for each $i \in I$ there is a $j \in I$ such that $p_j \geqslant p_i$ and the canonical operator $T_{ji}\colon E_j \to E_i$ is a Hilbert–Schmidt operator, i.e. $\sum_{n=1}^{\infty} (d_n(T_{ji}))^2 < \infty$. For each $i \in I$ let $\psi_i\colon E \to E_i$ be the canonical mapping.

Let us choose now an arbitrary $a \in E \setminus K$. We have to show that $a \notin \bar{K}^w$. Since K is closed, there is a neighbourhood of a in E disjoint with K. Hence for a certain $i \in I$ the point $\psi_i(a)$ does not belong to the closure of $\psi_i(K)$ in E_i. Then there is a $j \in I$ such that $p_j \geqslant p_i$, and the operator $T_{ji}\colon E_j \to E_i$ satisfies the condition

$$\sum_{n=1}^{\infty} n d_n(T_{ji}) < \infty.$$

It suffices here to take a composition of four Hilbert–Schmidt operators. The point $\psi_j(a)$ cannot belong to the weak closure of $\psi_j(K)$ in E_j, for, by Theorem C, we would then have

$$\psi_i(a) = T_{ji}\psi_j(a) \in \overline{T_{ji}\psi_j(K)} = \overline{\psi_i(K)}$$

(the closure in E_i), which is impossible. Hence $a \notin \bar{K}^w$. ∎

Lemma 8. *If K is a subgroup of a real topological vector space E, then the group E/K admits sufficiently many continuous characters iff K is weakly closed.*

Proof. This is a direct consequence of the following simple observation: if $\psi\colon E \to E/K$ is the canonical homomorphism, then the formula $\chi\psi(u) = \exp(2\pi i f(u))$ defines a one-to-one correspondence between continuous characters χ of E/K and functionals $f \in E^*$ such that $f(K) \subset \boldsymbol{Z}$. ∎

As an easy consequence of Theorem A we can obtain the following proposition which has been proved in [3].

Corollary 1. *If K is a closed connected subgroup of a real nuclear space E, then K is a linear subspace of E.*

Proof. Let $F = \overline{\operatorname{span} K}$. Being connected, K is easily seen to be weakly dense in F. Then, by Theorem A, K is strongly dense in F; hence $K = F$. ∎

Corollary 2. *Let E, F be normed spaces and let $T\colon E \to F$ be a linear operator which can be represented as a composition of five nuclear operators or nine absolutely summing operators. Then for each subset $A \subset E$ we have*

$$T(\bar{A}^w) \subset \overline{\operatorname{gp} T(A)}.$$

Proof. Let T be a composition of five nuclear operators. We have the following factorization:

$$\begin{tikzcd}
E \arrow[rr, "T_1"] \arrow[dr] & & E_1 \arrow[rr, "T_2"] & & E_2 \arrow[rr, "T_3"] \arrow[dr] & & E_3 \arrow[rr, "T_4"] & & E_4 \arrow[rr, "T_5"] \arrow[dr] & & F \\
& l^2 \arrow[ur] \arrow[rrrr, "S_1"] & & & & l^2 \arrow[ur] \arrow[rrrr, "S_2"] & & & & l^2 \arrow[ur] &
\end{tikzcd}$$

where all the operators are continuous, $T_1, \ldots, T_5$ are nuclear, and $T = T_5 \ldots T_1$. The operators S_1, S_2 are thus nuclear, which implies that

$$\sum_{k=1}^{\infty} d_k(S_i) < \infty \quad \text{for} \quad i = 1, 2.$$

Hence also

$$\sum_{k=1}^{\infty} k d_k(S_2 S_1) < \infty,$$

and it suffices now to apply Theorem C. For absolutely summing operators the proof is analogical. ∎

If K is a subgroup of a real topological vector space E, then we define $K^* = \{f \in E^*\colon f(K) \subset \boldsymbol{Z}\}$. Thus K^* is a weakly closed subgroup of E^*. When E is semi-reflexive, i.e. when the canonical imbedding $\alpha\colon E \to E^{**}$ is onto, then clearly $K^{**} = \alpha(\bar{K}^w)$. In particular, if K is closed and E – finite dimensional, then $K^{**} = \alpha(K)$. On the other hand, each infinite dimensional normed space contains a discrete subgroup K with $K^* = \{0\}$ (see [2]). In nuclear spaces the situation is similar to the finite dimensional case – the following proposition is an immediate consequence of Theorem A.

Corollary 3. *Let K be a closed subgroup of a semi-reflexive nuclear space E, and let $\alpha\colon E \to E^{**}$ be the canonical mapping. Then $K^{**} = \alpha(K)$.*

Acknowledgements. The author is indebted to W. Wojtyński and J. Grabowski for valuable remarks and suggestions. The author also wishes to thank the referee for revealing an error in the proof of Theorem C and suggesting the way of correction.

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Received April 12, 1983
Revised version October 5, 1983 (1880)

STUDIA MATHEMATICA, T. LXXX. (1984)

On the ratio maximal function for an ergodic flow

by

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Abstract. In this paper an integrability problem is investigated for the supremum of ergodic ratios defined by means of a conservative and ergodic measurable flow of measure preserving transformations on a σ-finite measure space. The results obtained below include, as a special case, the continuous parameter versions of Davis's recent results concerning the supremum of ergodic averages defined by means of an invertible and ergodic measure preserving transformation on a probability measure space.

1. Introduction. Let $(\Omega, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\{T_t\}_{t\in R}$ a conservative and ergodic measurable flow of measure preserving transformations on $(\Omega, \mathfrak{F}, \mu)$. In what follows we shall assume that μ is nonatomic and complete. As is easily seen, this is done without loss of generality.

Fix any $0 < e \in L_1(\mu)$ such that $\int e\,d\mu = 1$. If $f \in L_1(\mu)$, the ratio maximal function $M_e(f)(\omega)$ with respect to e is defined by

$$M_e(f)(\omega) = \sup_{b>0} \Big|\int_0^b f(T_t\omega)\,dt \Big/ \int_0^b e(T_t\omega)\,dt\Big| \qquad (\omega \in \Omega).$$

Let $\hat{f}_e$ denote the decreasing function on the interval $[0, 1)$ which is equidistributed with f/e $(\in L_1(e\,d\mu))$ with respect to the measure $e\,d\mu$. Extending $\hat{f}_e$ to the real line R by $\hat{f}_e(t+1) = \hat{f}_e(t)$ for $t \in R$, we define

$$H_e(f) = \int_0^{1/2} \frac{1}{t}\Big|\int_{-t}^{t} \hat{f}_e(s)\,ds\Big|\,dt.$$

Clearly, $H_e(f) = 0$ if and only if f/e and $-f/e$ are equidistributed with respect to $e\,d\mu$. Further it is known (cf. [8]) that if $f \geqslant 0$ then $H_e(f) < \infty$ if and only if $\int f \log^+(f/e)\,d\mu < \infty$, where $\log^+ a = \log(\max\{a, 1\})$ for $a \geqslant 0$. This, together with Theorem 2 in [8], shows that if $f \geqslant 0$ then $H_e(f) < \infty$ if and only if $\int M_e(f)\cdot e\,d\mu < \infty$. However, if the nonnegativity of f is not assumed, then, as is easily seen by a simple example, $H_e(f) < \infty$ does not necessarily imply $\int M_e(f)\cdot e\,d\mu < \infty$. (It will be proved below that $\int M_e(f)\cdot e\,d\mu < \infty$ implies $H_e(f) < \infty$.) Therefore it would be of interest to know what condition on the ratio maximal function with respect to e is necessary (and sufficient) for the condition $H_e(f) < \infty$. This is the starting