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Nilpotent Lie groups and summability of eigenfunction expansions of Schrödinger operators *

by

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Abstract. Let A be an operator densely defined on $L^2(X)$, where X is a measure space, which is essentially self-adjoint on its domain and non-negative. Let

$$Af = \int_0^\infty \lambda dE(\lambda) f$$

be the spectral expansion of A. We study conditions on functions K on $\boldsymbol{R}^+$ such that

$$\lim_{t\to 0} \int_0^\infty K(t\lambda)\, dE(\lambda) f = f \quad \text{a.e.}$$

for functions $f \in L^p(X)$, $1 \leqslant p < \infty$, for operators A which are of the form $\pi(L)$, where

$$L = \sum_{j=1}^{k} (-1)^{n_j} X_j^{2n_j},$$

$X_1, \ldots, X_k$ are generators of the Lie algebra of a nilpotent Lie group G and π is a representation of G induced by a unitary character of a normal connected subgroup of G. As a corollary we obtain the following. Let

$$A = \sum_{j=0}^{k} (-1)^{n_j} \partial^{2n_j}/\partial x_j^{2n_j} + V,$$

where the potential V is a sum of squares of real polynomials on $\boldsymbol{R}^k$. Then there exists an N such that

$$\lim_{t\to 0} \int_0^\infty (1-\lambda)_+^N\, dE(\lambda) f = f \quad \text{a.e.}, \qquad f \in L^1(\boldsymbol{R}^k),$$

where E is the spectral measure of A on $L^2(\boldsymbol{R}^k)$.

Let A be an operator densely defined on $L^2(X)$, where X is a measure space, which is essentially self-adjoint on its domain and non-negative. Let

$$Af = \int_0^\infty \lambda dE(\lambda) f$$

be the spectral expansion of A.

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A bounded continuous function K on $\boldsymbol{R}^+$ such that $K(0)=1$ is called an L^p (a.e.) *summability kernel for* A if

$$\lim_{t\to 0}\int_0^\infty K(t\lambda)\,dE(\lambda)f = f \quad \text{in} \quad L^p(X) \text{ (a.e.)}.$$

Among the best known summability kernels are the Riesz kernels $K(\lambda) = (1-\lambda)_+^\alpha$ and the Abel kernel $K(\lambda) = e^{-\lambda}$.

In [7] the authors studied L^p and a.e. summability kernels for the sublaplacian on a stratified nilpotent group G and nilmanifolds G/Γ, where Γ is a discrete co-compact subgroup of G. These methods produced theorems on Riesz and Abel L^p and a.e. summability for Hermite expansions for f in $L^p(\boldsymbol{R}^n)$, $1 \leqslant p < \infty$, cf. [7], and similar for Laguerre expansions, cf. [3].

In the present paper we study L^p and a.e. summability kernels for operators of the form $\pi(L)$ on $L^p(G/H)$, where L is an operator on an arbitrary nilpotent Lie group of the form

$$L = \sum_{j=1}^{k} (-1)^{n_j} X_j^{2n_j},$$

where $X_1, \ldots, X_k$ are generators of the Lie algebra of G and π is a representation of G induced by a unitary character of a normal connected subgroup H of G.

Following an idea of W. Cupała [2], we obtain the following:

Let

$$A = \sum_{j=1}^{k} (-1)^{n_j}\, \partial^{2n_j}/\partial x_j^{2n_j} + V,$$

where the potential V is a sum of squares of real polynomials on $\boldsymbol{R}^k$. Then there exists an α such that the Riesz kernel $(1-\lambda)_+^\alpha$ and the Abel kernel $e^{-\lambda}$ are L^p and a.e. summability kernels for A for all functions f in $L^p(\boldsymbol{R}^k)$, $1 \leqslant p < \infty$.

1. Preliminaries. Let G be a homogenous group, cf. [4], with dilations $\{\delta_t\}_{t>0}$ and a homogeneous norm $|x|$. For the Lie algebra $\mathfrak{g}$ of G we write

$$\mathfrak{g} = \oplus \sum_{j=1}^{s} V_j,$$

where

$$\delta_t X = t^{d_j} X \quad \text{for} \quad X \in V_j \quad \text{and} \quad 0 < d_1 < \ldots < d_s.$$

Let

$$Q = d_1 + \ldots + d_s.$$

A left-invariant differential operator L on G is called a *Rockland operator*, cf. [4], if

$$\delta_t L = t^a L$$

and, for every irreducible non-trivial unitary representation π of G, $\pi(L)$ is injective on C^∞-vectors.

It is easy to verify, cf. [4], that if $X_1, \ldots, X_k$ generate $\mathfrak{g}$ as a Lie algebra and each of X_i's is homogeneous of degree d_{j_i}, i.e., $X_i \in V_{j_i}$, then

$$L = \sum_{j=1}^{k} (-1)^{n_i} X_i^{2n_i},$$

where $2n_i d_{j_i} = a$, is a Rockland operator since for every unitary non-trivial representation π we have $(\pi(L)\xi, \xi) > 0$ for every non-zero C^∞-vector ξ.

Let L be a fixed positive Rockland operator on G. In virtue of [5], L is hypoelliptic and as such it is essentially self-adjoint on $C_c^\infty(G)$ in $L^2(G)$. Let

$$Lf = \int_0^\infty \lambda\, dE(\lambda) f, \quad f \in C_c^\infty(G),$$

be the spectral expansion of L. For a bounded continuous function K on $\boldsymbol{R}^+$ let

$$\check{K} f = \int_0^\infty K(\lambda)\, dE(\lambda) f, \quad f \in L^2(G).$$

Consider the commutative C^*-algebra

$$B = \{\check{K}\colon\ K \in C_0(\boldsymbol{R}^+)\}$$

and let π be a unitary representation of G. Since G is amenable, for every k in $L^1(G)$ and

$$\pi(k) = \int_G k(x)\pi(x)\,dx,$$

the map

$$\pi\colon\ k \to \pi(k)$$

defines a $*$-representation π of B, and so there is a spectral measure P on $\boldsymbol{R}^+$ with the values in $B(\boldsymbol{H}_\pi)$ such that

$$\pi(\check{K})\xi = \int_0^\infty K(\lambda)\, dP(\lambda)\xi, \quad \xi \in \boldsymbol{H}_\pi,$$

cf. e.g. [8].

In other words, if $\check{K} f = f * k$ for some $k \in L^1(G)$, then

$$\pi(\check{K}) = \pi(k). \tag{1.1}$$

In [6] the following theorem is proved.

THEOREM 1.1 [6]. *There is an N such that if $K \in C^N(R^+)$ and*

$$\sup_{\lambda>0}(1+\lambda)^N \left|\frac{d^j}{d\lambda^j} K(\lambda)\right| < \infty, \quad j = 0, \ldots, N, \tag{1.2}$$

then

$$Kf = f * k, \tag{1.3}$$

where $k \in L^1(G)$ and $|k(x)| \leqslant C(1+|x|)^{-m}$ for an $m > Q$.

Since

$$\delta_t L = t^a L$$

for

$$K_t(\lambda) = K(t^a \lambda),$$

we have

$$(K_t)\check{}f = f * k_t, \quad \text{where} \quad k_t(x) = t^{-Q} k(\delta_{t^{-1}} x). \tag{1.4}$$

A topological space X equipped with a continuous function $\varrho(x, y)$, called a *distance function*, which satisfies

$$\begin{gathered} \varrho(x, y) = \varrho(y, x) \geqslant 0, \\ \varrho(x, y) = 0 \quad \text{iff} \quad x = y, \\ \varrho(x, z) \leqslant \gamma\big(\varrho(x, y)+\varrho(y, z)\big) \quad \text{for a constant } \gamma, \end{gathered} \tag{1.5}$$

is called a *space of homogenous type* if every ball

$$B_r(x) = \{y\colon \varrho(x, y) < r\}$$

of radius r contains no more than a fixed number of disjoint balls of radius $r/2$, cf. [1].

Suppose there is a measure μ on X such that

(1.6) there is a constant C such that

$$\mu(B_r(y)) \leqslant C\mu(B_{r/2}(x)) \text{ for all } r > 0 \text{ and } x, y \in X.$$

Then X is a space of homogenous type and the Hardy–Littlewood maximal function

$$m^* f(x) = \sup_{r>0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f(y)|\, d\mu(y)$$

is of weak type (1,1), cf. [1].

2. Theorems. Let G be a homogenous group and let H be a closed normal connected subgroup of G not necessarily stable under the dilations. The exponential map exp maps the corresponding Lie algebras $\boldsymbol{g}$ and $\boldsymbol{h}$, $\boldsymbol{h}$ being an ideal in $\boldsymbol{g}$, onto G and H respectively.

Let

$$G \ni x \to \dot{x} \in Hx \in G/H$$

be the canonical map of G onto G/H. Since H is normal, we have

$$(\exp X)^{\cdot} = \exp(X+\boldsymbol{h}) = \exp \dot{X},$$

where

$$\boldsymbol{g} \ni X \to \dot{X} = X+\boldsymbol{h} \in \boldsymbol{g}/\boldsymbol{h}.$$

For every $i = 1, \ldots, s$ we define

$$V_i^0 = \{X \in V_i\colon X+Y \in \boldsymbol{h} \text{ for a } Y \text{ in } \sum_{j>i} V_j\},$$

$$V_i^\infty = \{X \in V_i\colon X+Y \in \boldsymbol{h} \text{ for a } Y \text{ in } \sum_{j<i} V_j\}.$$

Let W_i^0 and W_i^∞ be linear complements of V_i^0 and V_i^∞ in V_i, respectively. Let

$$\boldsymbol{h}^0 = \sum_{i=1}^s V_i^0, \quad \boldsymbol{h}^\infty = \sum_{i=1}^s V_i^\infty,$$

and

$$\boldsymbol{k}^0 = \sum_{i=1}^s W_i^0, \quad \boldsymbol{k}^\infty = \sum_{i=1}^s W_i^\infty.$$

Thus we have

$$\boldsymbol{k}^0 \oplus \boldsymbol{h}^0 = \boldsymbol{g} = \boldsymbol{k}^\infty \oplus \boldsymbol{h}^\infty. \tag{2.1}$$

Moreover,

$$\lim_{t\to 0} \delta_t \boldsymbol{h} = \boldsymbol{h}^0 \quad \text{and} \quad \lim_{t\to\infty} \delta_t \boldsymbol{h} = \boldsymbol{h}^\infty. \tag{2.2}$$

It follows by an easy induction that also

$$\boldsymbol{k}^0 \oplus \boldsymbol{h} = \boldsymbol{g} = \boldsymbol{k}^\infty \oplus \boldsymbol{h}. \tag{2.3}$$

We define a distance function on G/H by

$$\varrho(\dot{x}, \dot{y}) = \inf\{|xy^{-1}z|\colon z \in H\}.$$

It is easy to see that ϱ satisfies (1.5). Let

$$\dot{B}_r(\dot{x}) = \{\dot{y}\colon \varrho(\dot{x}, \dot{y}) < r\}$$

and denote

$$\dot{B}_r = \dot{B}_r(\dot{e}).$$

If $|A|_{G/H}$ denotes the Haar measure of A in G/H, we have

$$|\dot{B}_r(\dot{x})|_{G/H} = |\dot{B}_r|_{G/H}, \quad \dot{x} \in G/H.$$

Let

$$\boldsymbol{b}_r = \{X \in \boldsymbol{g}\colon |X| < r\}.$$

LEMMA 2.1. *For every α, $\beta > 0$ there exist constants a, b, c such that*

(2.4) $$\boldsymbol{b}_r \subset \boldsymbol{b}_{ar} \cap \boldsymbol{k}^\infty \oplus \boldsymbol{b}_{br} \cap H \subset \boldsymbol{b}_{cr} \quad \text{for all } r \leqslant \alpha$$

and

(2.5) $$\boldsymbol{b}_r \subset \boldsymbol{b}_{ar} \cap \boldsymbol{k}^0 \oplus \boldsymbol{b}_{br} \cap H \subset \boldsymbol{b}_{cr} \quad \text{for all } r \geqslant \beta.$$

Proof. Since

(2.6) $$|X+Y| \leqslant d(|X|+|Y|)$$

for a constant d, it suffices to prove only the first inclusions in (2.4) and (2.5).

To prove (2.4) it suffices to show that if $|X+Y| < r$, $X \in \boldsymbol{k}^\infty$, $Y \in \boldsymbol{h}$, then $|X| < ar$ for all $r \leqslant \alpha$, since then $|Y| < d(a+1)r$.

For $0 < \alpha' \leqslant r \leqslant \alpha$, (2.4) holds trivially. Thus suppose that there exist $X_n \in \boldsymbol{k}^\infty$, $Y_n \in \boldsymbol{h}$ such that $|X_n + Y_n| = r_n \to 0$ and $|X_n| = t_n = a_n r_n$ with $a_n \to \infty$. Then

$$|\delta_{t_n^{-1}} X_n + \delta_{t_n^{-1}} Y_n| \to 0 \quad \text{and} \quad |\delta_{t_n^{-1}} X_n| = 1.$$

Passing to a subsequence, if necessary, $\delta_{t_n^{-1}} X_n \to X \in \boldsymbol{k}^\infty$ and $|X| = 1$, whence $\delta_{t_n^{-1}} Y_n \to -X$. Since $r_n \to 0$ implies $t_n \to 0$ and $t_n^{-1} \to \infty$, we obtain a contradiction to (2.1) and (2.2).

The proof of (2.5) is similar.

PROPOSITION 2.2. *There exists a constant C such that*

(2.7) $$|\dot{B}_r|_{G/H} \leqslant C |\dot{B}_{r/2}|_{G/H}.$$

Consequently, G/H with the distance function ϱ is a space of homogeneous type and the Hardy–Littlewood maximal function

$$m^* f(\dot{x}) = \sup_{r>0} |\dot{B}_r|_{G/H}^{-1} \int_{\dot{B}_r(\dot{x})} |f(\dot{y})|\, d\dot{y}$$

is of weak type (1,1).

Proof. Let

$$\boldsymbol{b}_r^0 = \boldsymbol{k}^0 \cap \boldsymbol{b}_r \quad \text{and} \quad \boldsymbol{b}_r^\infty = \boldsymbol{k}^\infty \cap \boldsymbol{b}_r.$$

Since both $\boldsymbol{k}^0$ and $\boldsymbol{k}^\infty$ are stable under dilations, for every $\varepsilon > 0$ there exists a constant c such that

(2.8) $$|\boldsymbol{b}_{\varepsilon r}^0|_{k^0} \geqslant c |\boldsymbol{b}_r^0|_{k^0} \quad \text{for all } r > 0,$$

and

(2.9) $$|\boldsymbol{b}_{\varepsilon r}^\infty|_{k^\infty} \geqslant c |\boldsymbol{b}_r^\infty|_{k^\infty} \quad \text{for all } r > 0.$$

Let $\dot{\boldsymbol{b}}_r = \log \dot{B}_r$. By (2.3) there are linear 1-1 mappings

$$T^0\colon \boldsymbol{g}/\boldsymbol{h} \xrightarrow{\text{onto}} \boldsymbol{k}^0, \qquad T^\infty\colon \boldsymbol{g}/\boldsymbol{h} \xrightarrow{\text{onto}} \boldsymbol{k}^\infty$$

and, by Lemma 2.1, there exist constants c, $C > 0$ such that

(2.10) $$\begin{aligned} \boldsymbol{b}_{cr}^0 &\subset T^0 \dot{\boldsymbol{b}}_r \subset \boldsymbol{b}_{Cr}^0 \quad \text{for all } r \geqslant 1, \\ \boldsymbol{b}_{cr}^\infty &\subset T^\infty \dot{\boldsymbol{b}}_r \subset \boldsymbol{b}_{Cr}^\infty \quad \text{for all } r \leqslant 2. \end{aligned}$$

This in virtue of (2.8) and (2.9) completes the proof of Proposition 2.2.

PROPOSITION 2.3. *There exists a constant C such that for $r > 0$*

$$\sup_{x \in G} |x^{-1} B_r \cap H|_H |\dot{B}_r|_{G/H} \leqslant C |B_r|_G,$$

where $B_r = \{x \in G\colon |x| < r\}$.

Proof. Let $r \leqslant 1$. In virtue of (2.3), $xH = \exp X \cdot H$, where $X \in \boldsymbol{k}^\infty$. Since H is normal, $\exp X \cdot H = \{\exp[X+Y]\colon Y \in \boldsymbol{h}\}$. By Lemma 2.1, $|X+Y| < r$ implies $|X| < ar$ for all $r \leqslant 1$. Hence

$$\exp[-X] B_r \subset B_{\gamma(1+a)r}$$

and the conclusion follows from (2.5) and (2.10).

For $r \geqslant 1$ the proof is similar.

PROPOSITION 2.4. *Let $k \in L^1(G)$ be such that for an $m > Q$*

$$|k(x)| \leqslant c(1+|x|)^{-m}.$$

If k_t is defined as in (1.4), then there is a constant C such that for $f \in L^1(G/H)$ we have

$$|f * k_t(\dot{x})| \leqslant C m^* f(\dot{x}) \quad \text{for all } t > 0.$$

Proof. Let $e_{2^n t}$ be the characteristic function of the set

$$\begin{array}{ll} B_{2^{n+1}t} \setminus B_{2^n t} & \text{if } n \geqslant 0, \\ B_t & \text{if } n = -1. \end{array}$$

We have

$$\begin{aligned} |f * k_t(\dot{x})| &= \Big|\int_G k_t(y) f(\dot{x}y)\, dy\Big| \\ &= \Big|\sum_{n=-1}^{\infty} \int_G e_{2^n t}(y) k_t(y) f(\dot{x}y)\, dy\Big| \\ &= \Big|\sum_{n=-1}^{\infty} \int_{G/H} \int_H e_{2^n t}(yh) k_t(yh)\, dh\, f(\dot{x}\dot{y})\, d\dot{y}\Big| \\ &\leqslant c \sum_{n=-1}^{\infty} t^{-Q} \int_{G/H} \int_H e_{2^n t}(yh)(1+|\delta_{t^{-1}}(yh)|)^{-m}\, dh\, |f(\dot{x}\dot{y})|\, d\dot{y} \end{aligned}$$

$$\leqslant c \sum_{n=-1}^{\infty} t^{-Q}(1+2^n)^{-m} \int_{G/H} \int_H e_{2^n t}(yh)\,dh\,|f(\dot{x}\dot{y})|\,d\dot{y}$$

$$\leqslant c \sum_{n=-1}^{\infty} t^{-Q}(1+2^n)^{-m} \sup_{y\in G} |y^{-1}B_{2^{n+1}t} \cap H|_H \int_{\dot{B}_{2^{n+1}t}} |f(\dot{x}\dot{y})|\,d\dot{y}.$$

Hence, by Proposition 2.3,

$$|f * k_t(\dot{x})| \leqslant c \sum_{n=-1}^{\infty} t^{-Q}(1+2^n)^{-m}(2^{n+1}t)^Q \sup_{r>0} |\dot{B}_r|^{-1} \int_{\dot{B}_r(\dot{x})} |f(\dot{y})|\,d\dot{y}$$
$$\leqslant Cm^* f(\dot{x}).$$

COROLLARY 2.5. *If for a function k in $L^1(G)$ we have $|k(x)| \leqslant c(1+|x|)^{-m}$ for an $m > Q$, then the operator K^* defined on $C_c(G/H)$ by*

$$K^* f(\dot{x}) = \sup_{t>0} |f * k_t(\dot{x})|$$

is of weak type (1,1).

Now let $\boldsymbol{G}$ be an arbitrary nilpotent Lie group and let π be a representation of $\boldsymbol{G}$ induced from a unitary character of a normal connected subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$. Then the operators π_x, $x \in \boldsymbol{G}$, act on functions on $\boldsymbol{G}/\boldsymbol{H}$ according to the formula

$$\pi_x f(\dot{y}) = a(x, \dot{y}) f(\dot{y}\dot{x}),$$

where the scalar function a is such that $|a(x, \dot{y})| = 1$ (and $\pi_x \pi_y = \pi_{xy}$).

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ be elements in the Lie algebra of $\boldsymbol{G}$ which generate it. Let

$$\boldsymbol{L} = \sum_{j=1}^{k} (-1)^{n_j} \boldsymbol{X}^{2n_j}.$$

THEOREM 2.6. *$\pi(\boldsymbol{L})$ is a positive self-adjoint operator on $L^2(\boldsymbol{G}/\boldsymbol{H})$ and there exists an N such that if $K \in C^N(\boldsymbol{R}^+)$, $K(0) = 1$ and K satisfies* (1.2), *then K is an L^p and a.e. summability kernel for $\pi(\boldsymbol{L})$ and all f in $L^p(\boldsymbol{G}/\boldsymbol{H})$, $1 \leqslant p < \infty$.*

Proof. Let G be the nilpotent free group of the same nilpotency class as $\boldsymbol{G}$ and let $X_1, \ldots, X_k$ be the free generators of the Lie algebra $\boldsymbol{g}$ of G. If we compose π with the homomorphism α of G onto $\boldsymbol{G}$ sending X_j onto $\boldsymbol{X}_j$, we obtain a representation π' of G which is induced by a unitary character of the normal connected subgroup $\alpha^{-1}(\boldsymbol{H}) = H$ of G. We define dilation δ_t, $t > 0$, of G by putting

$$\delta_t X_j = t^{1/2n_j} X_j, \quad j = 1, \ldots, k.$$

Then

$$L = \sum_{j=1}^{k} (-1)^{n_j} X_j^{2n_j}$$

is a Rockland operator on G, $\delta_t L = tL$, and $\pi'(L) = \pi(\boldsymbol{L})$.

By Theorem 1.1, there exists an N such that if $K \in C^N(\boldsymbol{R}^+)$, $K(0) = 1$ and K satisfies (1.2), then there is a k in $L^1(G)$ with $|k(x)| \leqslant c(1+|x|)^{-m}$ for an $m > Q$ such that

$$\int_0^{\infty} K(t\lambda)\,dE(\lambda) f = \pi'(k_t) f,$$

where E is the spectral measure of $\pi'(L)$. But since $\{k_t\}_{t\to 0}$ is an approximate identity in $L^1(G)$, $\pi'(k_t)f$ tends to f in the L^p norm for f in $L^p(\boldsymbol{G}/\boldsymbol{H})$, $1 \leqslant p < \infty$ and uniformly for f in $C_c(\boldsymbol{G}/\boldsymbol{H})$.

On the other hand,

$$\pi'(k_t) f(\dot{y}) = \int_G k_t(x) a(x, \dot{y}) f(\dot{y}\dot{x})\,dx,$$

whence

$$|\pi'(k_t) f(\dot{y})| \leqslant |f| * |k_t|(\dot{y})$$

and, consequently, by Corollary 2.5, the operator π^*

$$\pi^* f(\dot{y}) = \sup_{t>0} |\pi(k_t) f(\dot{y})|$$

is of weak type (1,1) and Theorem 2.6 follows.

Perhaps the most interesting application of Theorem 2.6 is one based on an idea of W. Cupała [2].

A function p on a nilpotent Lie group $\boldsymbol{G}$ is called a *polynomial* if $p \circ \exp$ is a polynomial on the real vector space, the Lie algebra of $\boldsymbol{G}$. Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ be generators of the Lie algebra of $\boldsymbol{G}$ and let

$$\boldsymbol{L} = \sum_{j=1}^{k} (-1)^{n_j} \boldsymbol{X}_j^{2n_j} + \sum_{i=1}^{s} p_i^2,$$

where $p_1, \ldots, p_s$ are polynomials on $\boldsymbol{G}$.

THEOREM 2.7 (W. Cupała [2]). *The Lie algebra $\boldsymbol{g}$ generated by the operators $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ and $V_1, \ldots, V_s$, $V_j f = ip_j f$, $j = 1, \ldots, s$, on $C_c^{\infty}(\boldsymbol{G})$ is nilpotent and finite-dimensional.*

Let $G = \exp \boldsymbol{g}$.

THEOREM 2.8 (W. Cupała [2]). *The representation of $\boldsymbol{g}$ as operators on $C_c^{\infty}(\boldsymbol{G})$ is the differential of a representation π of G which is induced by a unitary character of a normal connected subgroup of G.*

COROLLARY 2.9. *$\boldsymbol{L}$ is an essentially self-adjoint positive operator on $L^2(\boldsymbol{G})$ and there exists an N such that if $K \in C^N(\boldsymbol{R}^+)$, $K(0) = 1$ and K satisfies* (1.2), *then K is an L^p and a.e. summability kernel for $\boldsymbol{L}$ and all functions f in $L^p(\boldsymbol{G})$, $1 \leqslant p < \infty$.*

EXAMPLE. Consider the operator

$$L = (-1)^k \frac{d^{2k}}{dx^{2k}} + p(x),$$

where p is a positive polynomial on $\boldsymbol{R}$ and $k > 0$. Let $\varphi_0, \varphi_1, \ldots$ be the eigenfunctions of L with eigenvalues $\lambda_0, \lambda_1, \ldots$ By Corollary 2.9 there exists an α such that for every $f \in L^1(\boldsymbol{R})$

$$\lim_{n\to\infty} \sum_{\lambda_j < n} (1-j/n)^\alpha (f, \varphi_j)\varphi_j(x) = f(x) \qquad \text{a.e.}$$

In more dimensions our method gives a similar result only for the operators L with the potentials which are sums of squares of polynomials, since in several variables not every positive polynomial is a sum of squares of polynomials.

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Some classes of commuting n-tuples of operators

by

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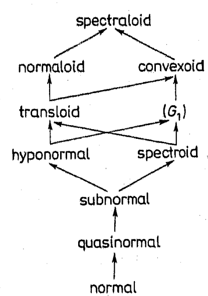
Abstract. In this paper we study the inclusion relations among some classes of operator-families and the topological properties of these classes.

1. Introduction. Throughout this paper, H will be a complex Hilbert space with the scalar product $(\ ,\ )$ and the norm $\|\cdot\|$, and all operators on H will be assumed to be linear and bounded. For an operator T on H, let $\sigma(T)$ denote its spectrum, $r(T)$ its spectral radius, $W(T)$ its numerical range and $w(T)$ its numerical radius.

In case of a single operator, the properties of a normal operator are well known because it has spectral resolution. Thus many authors have discussed some classes of operators which are close to being normal in some sense. It is well known that there exists an inclusion relation among these classes. We



shall indicate it by the diagram above (e. g. see [10]). Here, T is called *normaloid* iff $\|T\| = w(T)$, and *transloid* iff $T-\lambda I$ is normaloid for each $\lambda \in \boldsymbol{C}$. T is called *convexoid* iff $\overline{W(T)} = \operatorname{co}\sigma(T)$. ($\bar{X}$ denotes the closure of the set $X \subset \boldsymbol{C}$ and $\operatorname{co} X$ its convex hull.) T is called *spectraloid* iff $w(T) = r(T)$. T *belongs to* (G_1) iff $\|(T-\lambda I)^{-1}\| = 1/d(\lambda, \sigma(T))$ for each $\lambda \in \boldsymbol{C}-\sigma(T)$, where $d(\lambda, X)$ denotes the distance between λ and the set $X \subset \boldsymbol{C}$. And T is called *spectroid* iff $\sigma(T)$ is a spectral set for T in the sense of von Neumann.