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Entropy numbers of r-nuclear operators in Banach spaces of type q

by

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Abstract. It is shown that the entropy numbers of r-nuclear operators mapping a Banach space whose dual has type q into a Banach space of type p, belong to the Lorentz sequence space $l_{s,r}$, where $0 < r < 1$, $1 \leqslant p, q \leqslant 2$ and $1/s = 1 + 1/r - 1/p - 1/q$. This extends results of Carl and König.

0. Introduction. The entropy numbers describe the "degree of compactness" of bounded linear operators, but, moreover, they are also a powerful tool for the investigation of eigenvalue problems, cf. [6]. Therefore during the last few years a lot of results concerning entropy numbers of certain classes of operators has been established. We only remind of diagonal operators between Lorentz sequence spaces or embedding maps between Besov function spaces.

Recently Carl [4] considered operators S admitting a factorization through l_1:

$$\begin{tikzcd} l_q \arrow[rr, "S"] \arrow[dr, "D"'] & & X \\ & l_1 \arrow[ur, "B"'] & \end{tikzcd}$$

where B is a bounded and D a diagonal operator. Supposing that X is of some type p, he characterized these operators in terms of their entropy numbers.

In the present paper we deal with the "dual" situation of operators T factorizing through l_∞,

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "B"'] & & l_q \\ & l_\infty \arrow[ur, "D"'] & \end{tikzcd}$$

where again B is a bounded and D a diagonal operator. But now X^* (the dual of X) is assumed to have some type p. We shall estimate the asymptotic behaviour of the entropy numbers of these operators.

Then we apply our results to r-nuclear operators acting between Banach spaces of certain type, thus extending results of Carl [5] and König [7].

1. Preliminaries. We denote by $\mathscr{L}(X, Y)$ the class of all (bounded linear) operators from the Banach space X into the Banach space Y. The *n-th inner entropy number* $f_n(T)$ of an operator $T\in\mathscr{L}(X, Y)$ is defined as the supremum of all $\varepsilon > 0$ such that there are elements $x_1, \ldots, x_p \in U_X$, $p \geq 2^{n-1}$, with

$$\|Tx_i - Tx_j\| > 2\varepsilon \quad \text{whenever} \quad i \neq j,$$

see [12], 12.1.6. Here U_X is the closed unit ball of X. The *n-th outer entropy number* $e_n(T)$ is the infimum of all $\varepsilon > 0$ such that there are elements $y_1, \ldots, y_{2^{n-1}} \in Y$ with

$$T(U_X) \subseteq \bigcup_i \{y_i + \varepsilon U_Y\},$$

see [12], 12.1.2. For every $T\in\mathscr{L}(X, Y)$ one has

$$f_n(T) \leq e_n(T) \leq 2f_n(T),$$

[12], 12.1.10.

Now let $[l_{p,u}, \|\cdot\|_{p,u}]$ be the Lorentz sequence spaces, cf. e.g. [12], 13.9.1. Setting

$$\mathscr{L}_{p,u} = \{S\in\mathscr{L}\colon (e_n(S))\in l_{p,u}\}$$

and

$$L_{p,u}(S) = \|(e_n(S))\|_{p,u}$$

we get quasi-normed operator ideals (for the definition see [12], 14.3.5.) $[\mathscr{L}_{p,u}, L_{p,u}]$, $0 < p < \infty$, $0 < u \leq \infty$. The multiplicativity of entropy numbers yields the following useful product formula:

$$\mathscr{L}_{p_1,u_1} \circ \mathscr{L}_{p_2,u_2} \subseteq \mathscr{L}_{p,u},$$

$$0 < p_1, p_2 < \infty,\ 0 < u_1, u_2 \leq \infty,\ 1/p = 1/p_1 + 1/p_2,\ 1/u = 1/u_1 + 1/u_2.$$

An operator $S\in\mathscr{L}(X, Y)$ is said to be *r-nuclear*, $0 < r < 1$, if there are sequences $(a_i) \subseteq U_{X^*}$, $(y_i) \subseteq U_Y$ and $\sigma = (\sigma_i) \in l_r$ with

$$Sx = \sum_{i=1}^{\infty} \sigma_i \langle x, a_i\rangle y_i \quad \text{for all } x\in X,$$

see [12], 18.5. Finally we need some definitions from probability theory. A Banach space X is said to be *of (Rademacher) type p*, $1 \leq p \leq 2$, if there is a

constant $K > 0$ such that for all finite families $\{x_1, \ldots, x_n\} \subseteq X$ the inequality

$$\int_0^1 \Big\|\sum_{i=1}^n x_i r_i(t)\Big\|^p dt \leq K^p \sum_{i=1}^n \|x_i\|^p$$

holds, where (r_i) is the sequence of Rademacher functions defined over the interval $[0, 1]$. Detailed information on the notions of type and cotype can be found in [11].

A real random variable Θ is called *p-stable*, $0 < p \leq 2$, if there is a parameter $c > 0$ such that the characteristic function of Θ is

$$\hat{\Theta}(t) = \boldsymbol{E}\ \exp(it\Theta) = \exp(-c^p|t|^p), \qquad t\in\boldsymbol{R}.$$

(Here $\boldsymbol{E}$ is the symbol for the mathematical expectation). A random vector $(\Theta_1, \ldots, \Theta_n)$ in $\boldsymbol{R}_n$ is called *p-stable* if all linear combinations

$$\sum_{i=1}^n \alpha_i \Theta_i, \qquad \alpha_i\in\boldsymbol{R},$$

are p-stable real random variables.

In the sequel we shall always denote by $(\Theta_i^{(p)})_{i\geq 1}$ a sequence of independent p-stable random variables, each with parameter $c = 1$.

The space of all Bochner-integrable functions from a measure space $(\Omega, \mathscr{F}, P)$ into a Banach space X will be denoted by $L_1(\Omega, \mathscr{F}, P; X)$, or, when the underlying measure space is inessential, simply by $L_1(X)$.

2. l_∞-factorizable operators. We start with a lemma of Marcus and Pisier ([10], Corollary 2.7), which is a consequence of the deep comparison theorem for p-stable and Gaussian processes ([10], Theorem 2.5).

LEMMA 1. *For $1 < p \leq 2$ there are constants $a_p > 0$ such that for any p-stable random vector $(\Theta_1, \ldots, \Theta_n)$ in $\boldsymbol{R}_n$, $n\in\boldsymbol{N}$, the inequality*

$$\boldsymbol{E} \sup_{1\leq j\leq n} |\Theta_j| \geq a_p(\log_2 n)^{1/p'} \inf_{m\neq k} \boldsymbol{E}|\Theta_m - \Theta_k|, \qquad 1/p + 1/p' = 1,$$

holds.

This lemma enables us to prove the next one which is a reformulation of [10], Theorem 2.6, in the language of entropy ideals. The case $p = 2$ was already treated in [9].

LEMMA 2. *Let X be any Banach space and $T\in\mathscr{L}(X, l_p)$, $1 < p \leq 2$, an operator such that the series $\sum_{i=1}^{\infty} T^* e_i \Theta_i^{(p)}$ converges in $L_1(X^*)$. Then $T\in\mathscr{L}_{p',\infty}(X, l_p)$, $1/p + 1/p' = 1$.*

Proof. Fix $n\in\boldsymbol{N}$ and choose elements $x_1, \ldots, x_{2^n} \in U_X$ such that

$$\inf_{m\neq k} \|Tx_m - Tx_k\| \geq f_{n+1}(T) \geq \tfrac{1}{2} e_{n+1}(T).$$

Then consider the p-stable random vector $(\Theta_1, \ldots, \Theta_{2^n})$ in R^{2^n} with

$$\Theta_j = \sum_{i=1}^{\infty} \langle e_i, Tx_j\rangle \Theta_i^{(p)}, \qquad 1 \leqslant j \leqslant 2^n.$$

For $m \neq k$ we have

$$\begin{aligned} E|\Theta_m - \Theta_k| &= E\Big|\sum_{i=1}^{\infty} \langle e_i, Tx_m - Tx_k\rangle \Theta_i^{(p)}\Big| \\ &= \Big(\sum_{i=1}^{\infty} |\langle e_i, Tx_m - Tx_k\rangle|^p\Big)^{1/p} E|\Theta_1^{(p)}| \\ &= c_p\|Tx_m - Tx_k\| \geqslant \tfrac{1}{2}\cdot c_p \cdot e_{n+1}(T). \end{aligned}$$

Using this and Lemma 1 we get the following estimation

$$\begin{aligned} \infty > E\Big\|\sum_{i=1}^{\infty} T^* e_i \Theta_i^{(p)}\Big\| &= E \sup_{x\in U_X} \Big|\sum_{1}^{\infty} \langle T^* e_i, x\rangle \Theta_i^{(p)}\Big| \\ &\geqslant E \sup_{1\leqslant j\leqslant 2^n} \Big|\sum_{1}^{\infty} \langle e_i, Tx_j\rangle \Theta_i^{(p)}\Big| = E \sup_{1\leqslant j\leqslant 2^n} |\Theta_j| \\ &\geqslant a_p\cdot (\log_2 2^n)^{1/p'} \cdot \inf_{m\neq k} E|\Theta_m - \Theta_k| \\ &\geqslant \tfrac{1}{2}\cdot a_p \cdot c_p \cdot n^{1/p'} \cdot e_{n+1}(T). \end{aligned}$$

This means $T \in \mathscr{L}_{p',\infty}(X, l_p)$. ∎

LEMMA 3. *Let $1 \leqslant q < p \leqslant 2$, and let Z be a Banach space of type q. If $\sum_{i=1}^{\infty} \|z_i\|^q < \infty$ for a sequence $(z_i) \subseteq Z$, then the series $\sum_{i=1}^{\infty} z_i \Theta_i^{(p)}$ converges in $L_1(Z)$.*

Proof. It suffices to prove that there is a constant $c > 0$, depending only on p, q and Z but not on $n \in N$, such that for every finite family $\{z_1, \ldots, z_n\} \subseteq Z$ the inequality

$$E\Big\|\sum_{i=1}^{n} z_i \Theta_i^{(p)}\Big\| \leqslant c\Big(\sum_{i=1}^{n} \|z_i\|^q\Big)^{1/q}$$

holds. To this end fix $n \in N$, let K denote the type q constant of Z, and consider for every $t \in [0, 1]$ the Z-valued random variables

$$\sum_{i=1}^{n} z_i \Theta_i^{(p)} \quad \text{and} \quad \sum_{i=1}^{n} z_i r_i(t) \Theta_i^{(p)}.$$

(Here we assume that the $\Theta_i^{(p)}$'s are defined over a probability space $[\Omega, \mathscr{F}, P]$ and consider $r_i \Theta_i^{(p)}$ as random variables over $\Omega \times [0, 1]$.)

Since the Rademacher functions attain only the values $+1$ and -1, it follows from the symmetry of the random variables $\Theta_i^{(p)}$ that these two sums always have the same distribution, and consequently, for all $t \in [0, 1]$,

$$E\Big\|\sum_{i=1}^{n} z_i \Theta_i^{(p)}\Big\|^q = E\Big\|\sum_{i=1}^{n} z_i r_i(t) \Theta_i^{(p)}\Big\|^q.$$

Integrating this equation against t and applying Fubini's theorem, we obtain the desired estimate:

$$\begin{aligned} E\Big\|\sum_{i=1}^{n} z_i \Theta_i^{(p)}\Big\| &\leqslant \Big(E\Big\|\sum_{i=1}^{n} z_i \Theta_i^{(p)}\Big\|^q\Big)^{1/q} \\ &= \Big(\int_0^1 E\Big\|\sum_{i=1}^{n} z_i r_i(t) \Theta_i^{(p)}\Big\|^q dt\Big)^{1/q} \\ &\leqslant K\Big(E \sum_{i=1}^{n} \|z_i \Theta_i^{(p)}\|^q\Big)^{1/q} \\ &= K\cdot (E|\Theta_1^{(p)}|^q)^{1/q} \cdot \Big(\sum_{i=1}^{n} \|z_i\|^q\Big)^{1/q}. \end{aligned}$$

Since $q < p$, we have $E|\Theta_1^{(p)}|^q < \infty$, and the proof is finished. ∎

Remark. In the definition of type one can use instead of the Rademacher functions also standard Gaussian (i.e., 2-stable) random variables. Therefore Lemma 3 holds for $p = q = 2$, too.

THEOREM 4. *Let X be a Banach space whose dual is of type q and let $T \in \mathscr{L}(X, l_u)$ admit a factorization*

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "A"'] & & l_u \\ & l_\infty \arrow[ur, "D_\sigma"'] & \end{tikzcd}$$

with a diagonal operator $D_\sigma \in \mathscr{L}(l_\infty, l_u)$ and $A \in \mathscr{L}(X, l_\infty)$. If $\sigma = (\sigma_i) \in l_{r,t}$, then $T \in \mathscr{L}_{s,t}(X, l_u)$, provided that $1 \leqslant q \leqslant 2$, $1 \leqslant u \leqslant \infty$, $0 < t \leqslant \infty$, $0 < r < \min(q, u)$ and

$$1/s = 1/r + 1 - 1/q - 1/u.$$

Proof. We split the proof into three steps. First of all fix any operator $A \in \mathscr{L}(X, l_\infty)$.

Step 1. Next we prove that $\sigma \in l_q$ implies $T = D_\sigma A \in \mathscr{L}_{p',\infty}(X, l_p)$ whenever $1 \leqslant q < p \leqslant 2$ or $p = q = 2$. Because of

$$\sum_{1}^{\infty} \|T^* e_i\|^q \leqslant \|A^*\|^q \sum_{1}^{\infty} |\sigma_i|^q < \infty$$

we get from Lemma 3 the convergence of $\sum_1^\infty T^* e_i \Theta_i^{(p)}$ in $L_1(X^*)$. But this implies, by Lemma 2, $T \in \mathscr{L}_{p',\infty}(X, l_p)$.

Step 2. Now we show that

$$\sigma \in l_r \quad \text{implies} \quad T = AD_\sigma \in \mathscr{L}_{s,\infty}(X, l_u).$$

The assumptions on q, r, s, u guarantee that one can find real numbers p, v with the following properties:

$$p = q = 2 \text{ or } 1 \leqslant q < p \leqslant 2,\ 0 < v < \infty,\ 1/r = 1/q + 1/v,\ 1/v + 1/p > 1/u.$$

Then we split $\sigma = \mu \cdot \nu$ with $\nu \in l_q$ and $\mu \in l_v$. So we have the factorization diagram:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "A"'] & l_u \\ l_\infty \arrow[r, "D_\nu"'] \arrow[ur, "D_\sigma"] & l_p \arrow[u, "D_\mu"'] \end{tikzcd}$$

Now it follows from Step 1 that

$$D_\nu A \in \mathscr{L}_{p',\infty}(X, l_p)$$

and from [3], Theorem 3, that

$$D_\mu \in \mathscr{L}_{w,v}(l_p, l_u) \subseteq \mathscr{L}_{w,\infty}(l_p, l_u),$$

where $1/w = 1/p + 1/v - 1/u$. The multiplication formula for the entropy ideals yields

$$T = D_\mu D_\nu A \in \mathscr{L}_{w,\infty} \circ \mathscr{L}_{p',\infty}(X, l_u) \subseteq \mathscr{L}_{s,\infty}(X, l_u)$$

since $1/w + 1/p' = 1/p + 1/v - 1/u + 1/p' = 1 + 1/r - 1/q - 1/u = 1/s$.

Step 3. Improving the result of the preceding step by real interpolation we shall derive the desired assertion. Let $\mathscr{D}$ be the operator assigning to a sequence σ the composite operator $D_\sigma A$. By Step 2,

$$\mathscr{D}\colon l_r \to \mathscr{L}_{s,\infty}(X, l_u)$$

is a bounded linear operator whenever $0 < r < \min(q, u)$ and $1/s = 1 + 1/r - 1/q - 1/u$. For given r, $0 < r < \min(q, u)$, determine $0 < r_0 < r_1 < \min(q, u)$ and $0 < \Theta < 1$ with $1/r = (1-\Theta)/r_0 + \Theta/r_1$. Then clearly

$$\mathscr{D}\colon l_{r_i} \to \mathscr{L}_{s_i,\infty}(X, l_u), \qquad 1/s_i = 1 + 1/r_i - 1/q - 1/u,\ i = 0, 1,$$

are both bounded operators.

Now let us recall two results regarding the real interpolation method. The first one is classical and can be found, e.g., in [1], for the second see [4].

If $0 < p_0 \neq p_1 < \infty$, $0 < q_0, q_1, t \leqslant \infty$, $0 < \Theta < 1$ and $1/p = (1-\Theta)/p_0 + \Theta/p_1$, then

$$(l_{p_0,q_0}, l_{p_1,q_1})_{\Theta,t} = l_{p,t}$$

and

$$(\mathscr{L}_{p_0,q_0}(Y, Z), \mathscr{L}_{p_1,q_1}(Y, Z))_{\Theta,t} \subseteq \mathscr{L}_{p,t}(Y, Z)$$

for arbitrary Banach spaces Y and Z. Consequently, by the interpolation property,

$$\mathscr{D}\colon l_{r,t} \to \mathscr{L}_{s,t}(X, l_u)$$

is also bounded. The proof is finished. ■

3. *r*-nuclear operators. Before we can state a result on entropy of r-nuclear operators let us recall the result of Carl [4], Theorem 2, already mentioned in the introduction.

THEOREM 5. *Let Y be a Banach space of type p and let $S \in \mathscr{L}(l_v, Y)$ admit a factorization*

$$\begin{tikzcd} l_v \arrow[rr, "S"] \arrow[dr, "D_\sigma"'] & & Y \\ & l_1 \arrow[ur, "B"'] & \end{tikzcd}$$

with a diagonal operator $D_\sigma \in \mathscr{L}(l_v, l_1)$ and $B \in \mathscr{L}(l_1, Y)$. If $\sigma \in l_{r,t}$, then $S \in \mathscr{L}_{s,t}(l_v, Y)$ where $1 \leqslant p \leqslant 2$, $1 \leqslant v \leqslant \infty$, $0 < t \leqslant \infty$, $1/v + 1/r > 1$ and $1/s = 1/r + 1/v - 1/p$.

Combining the preceding theorem and Theorem 4 we obtain our main result, a description of r-nuclear operators in terms of their entropy numbers.

THEOREM 6. *Let X and Y be Banach spaces such that X^* is of type q and Y of type p. Then*

$$\mathscr{N}_r(X, Y) \subseteq \mathscr{L}_{s,r}(X, Y)$$

whenever $0 < r < 1$, $1 \leqslant p$, $q \leqslant 2$ and $1/s = 1 + 1/r - 1/p - 1/q$.

Proof. Every operator $T \in \mathscr{N}_r(X, Y)$ can be represented as $T = BD_\sigma A$ with bounded operators $A \in \mathscr{L}(X, l_\infty)$, $B \in \mathscr{L}(l_1, Y)$ and a diagonal operator $D_\sigma \in \mathscr{L}(l_\infty, l_1)$ generated by a sequence $\sigma \in l_r$. Choose now $0 < r_1 < 1$ and $0 < r_2 < \infty$ with $1/r = 1/r_1 + 1/r_2$ and split $\sigma = \mu \cdot \nu$ with $\nu \in l_{r_1}$ and $\mu \in l_{r_2}$. So we arrive at the following factorization diagram:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[d, "A"'] & & Y \\ l_\infty \arrow[rr, "D_\sigma"] \arrow[dr, "D_\nu"'] & & l_1 \arrow[u, "B"'] \\ & l_1 \arrow[ur, "D_\mu"'] & \end{tikzcd}$$

Applying Theorem 4 we obtain

$$D_\nu A \in \mathscr{L}_{s_1,r_1}(X, l_1), \qquad 1/s_1 = 1+1/r_1-1/q-1 = 1/r_1-1/q,$$

and from Theorem 5 we get

$$BD_\mu \in \mathscr{L}_{s_2,r_2}(l_1, Y), \qquad 1/s_2 = 1/r_2+1-1/p.$$

Using again the multiplication formula for the entropy ideals we see that

$$T = (BD_\mu)(D_\nu A) \in \mathscr{L}_{s_2,r_2} \circ \mathscr{L}_{s_1,r_1}(X, Y) \subseteq \mathscr{L}_{s,r}(X, Y),$$

since $1/r_1+1/r_2 = 1/r$ and

$$1/s_1+1/s_2 = 1/r_1-1/q+1/r_2+1-1/p = 1+1/r-1/p-1/q = 1/s. \ \blacksquare$$

This theorem extends recent results of Carl [5]. As corollaries we get once more two results of König [7], Theorem b, and [8], Proposition 5, on eigenvalue distributions of r-nuclear operators. However, König proved these results by completely different methods.

Since we are concerned with eigenvalue problems, all Banach spaces X are now assumed to be complex. The eigenvalues of compact operators $S \in \mathscr{L}(X, X)$ are denoted by $(\lambda_n(S))$, they are counted according to their algebraic multiplicities and are ordered so that $|\lambda_1(S)| \geqslant |\lambda_2(S)| \geqslant \ldots$ If S has less than n eigenvalues, we set $\lambda_n(S) = \lambda_{n+1}(S) = \ldots = 0$.

COROLLARY 7 [7]. *If S is an r-nuclear operator acting in some $L_p(\mu)$, then $(\lambda_n(S)) \in l_{s,r}$, where $0 < r < 1$, $1 < p < \infty$ and $1/s = 1/r-|1/p-1/2|$.*

Proof. Since $L_p(\mu)$ is of type $\min(p, 2)$, it follows from Theorem 6 that $S \in \mathscr{L}_{s,r}$, where

$$1/s = \left.\begin{cases} 1+1/r-1/p-1/2 & \text{if} \quad 1 < p \leqslant 2 \\ 1+1/r-1/2-1/p' & \text{if} \quad 2 \leqslant p < \infty \end{cases}\right\} = 1/r-|1/p-1/2|.$$

By [6], Corollary 2, we have

$$|\lambda_n(S)| \leqslant \sqrt{2}\, e_{n+1}(S), \qquad n = 1, 2, \ldots,$$

hence $(\lambda_n(S)) \in l_{s,r}$. $\blacksquare$

COROLLARY 8 [8]. *Let S be an r-nuclear operator, $0 < r < 1$, acting in a Banach space X of type p and cotype q, $1 < p \leqslant 2 \leqslant q < \infty$, $1/p-1/q < 1/2$. Then $(\lambda_n(S)) \in l_{s,r}$, where $1/s = 1/r-(1/p-1/q)$.*

Proof. By a recent result of Pisier [13], Corollary 2.5, X^* has type q', $1/q+1/q' = 1$. Again Theorem 6 and [6], Corollary 2, yield the assertion. $\blacksquare$

4. Final remarks.

Remark 1. Theorem 4 is optimal in the following sense: For every constellation of parameters q, r, t, u there are a Banach space X and an operator $T \in \mathscr{L}(X, l_u)$ possessing all described properties such that $T \in \mathscr{L}_{s,t}$ but $T \notin \mathscr{L}_{s,t_0}$ whenever $t_0 < t$. To see this it suffices to consider a diagonal operator $D_\sigma \in \mathscr{L}(l_q, l_u)$ generated by a sequence $\sigma \in l_{r,t}$ which does not belong to any l_{r,t_0}, $0 < t_0 < t$. Then by Carl [3], Theorem 2, D_σ belongs to the entropy ideal $\mathscr{L}_{s,t}$ but not to $\mathscr{L}_{s,t_0}$, $0 < t_0 < t$.

Remark 2. It seems very likely that the assumption $0 < r < \min(q, u)$ in Theorem 4 can be replaced by the weaker one $0 < r < u$. However, the methods of our proof do not apply to the case $q \leqslant r < u$.

Remark 3. The main open problem connected with entropy numbers is the question of complete symmetry of the entropy ideals, i.e., is it true that $S \in \mathscr{L}_{s,t}$ iff $S^* \in \mathscr{L}_{s,t}$? The comparison of Theorems 4 and 5 is one more hint that the answer to this question might be affirmative.

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