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The Hölder continuity of the Bergman projection and proper holomorphic mappings

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Abstract. The integral formulae are used to prove the Hölder continuity of the Bergman projection for bounded strictly pseudoconvex domains with C^k-smooth boundaries. It gives us a tool to study the boundary behaviour of proper holomorphic mappings onto such domains.

1. Introduction. We begin with the statement of our main result:

Let $\Lambda_\alpha(D)$ denote the Hölder space defined as follows: If $0 < \alpha < 2$ then

$$\Lambda_\alpha(D) = \{f \in L^\infty(D)\colon \|f(x+t)+f(x-t)-2f(x)\|_\infty \leqslant A|t|^\alpha\},$$

$$\|f\|_\alpha = \|f\|_\infty + \sup_{|t|>0} \frac{\|f(x+t)+f(x-t)-2f(x)\|_\infty}{|t|^\alpha}.$$

If $\alpha > 1$ then $\Lambda_\alpha = \{f\colon f \in L^\infty(D);\ \partial f/\partial x_i \in \Lambda_{\alpha-1},\ i = 1, \ldots, 2n\}$. Note that if $0 < \alpha < 1$ then Λ_α is the space of functions satisfying the Hölder condition $|f(x)-f(y)| \leqslant A|x-y|^\alpha$. The space $L_s^\infty(D)$ is a Sobolev space of functions of bounded derivatives up to order s. L_s^∞ is isomorphic to the space $\mathrm{Lip}_{s-1}(D)$ of functions with derivatives of order $s-1$ satisfying Lipschitz condition. We shall denote by P the orthogonal projection from $L^2(D)$ onto the space $L^2H(D)$ of square integrable holomorphic functions. P is called the *Bergman projection*. P is an integral operator $Pf(z) = \int_D K_D(z,t)f(t)\,dt$. The kernel $K_D(z,t)$ is the Bergman function of D.

THEOREM 1. *If D is a bounded strictly pseudoconvex domain in C^n with the boundary of class C^{k+4} then*

(1) *P is a continuous operator from $\Lambda_{k+\beta}(D)$ into $\Lambda_{k+\beta/2}(D)$ for $0 < \beta < 1$,*

(2) *P maps continuously $L_{k+1}^\infty(D)$ into $\Lambda_{k+1/2}(D)$,*

(3) *P is a continuous mapping from $\Lambda_\alpha(D)$ into $\Lambda_\alpha(D)$ if $\alpha \leqslant k$.*

The proof of Theorem 1 will be based on the methods developed in Kerzman–Stein paper [9] to study the singularities of the Szegö kernel for smooth strictly pseudoconvex domains. Their approach is the following. Let S be a Szegö projector from $L^2(\partial D)$ onto $H^2(\partial D)$. They constructed an integral formula which gives another projection H from $L^2(\partial D)$ onto $H^2(\partial D)$

and proved that $S = H(I-(H^*-H))^{-1} = (I+(H^*-H))H^*$. It enables us to write down the singularity of the Szegö kernel in terms of the kernel of H given explicitly.

We shall use the same procedure to study the Bergman projector P. We shall use the Kerzman-Stein integral formula to get an explicit projection

$$G\colon L^2(D) \to L^2 H(D).$$

We shall next show that G, G^* and $[I-(G^*-G)]^{-1}$ are continuous in the Λ_α norms. This implies that $P = [I+(G^*-G)]^{-1} G^*$ is continuous in the Λ_α norms.

In studying the arising singular integral operators we shall use only the standard methods, originated by Henkin ([6], [7]) and refined by Kranz [10]. We shall also use the very convenient method of integration by parts originated by Elgueta [5] and used by Ahern and Schneider [1].

It should be mentioned that Ahern and Schneider proved the Hölder estimates for Bergman projection in the case of strictly pseudoconvex domains with C^∞-boundary. However, their proof is based on the Boutet de Monvel expression of the Bergman function. This expression is very difficult to prove even for C^∞-smooth domains (its proof is not elementary and uses such things as pseudodifferential operators with complex phase and asymptotic expansions) and probably is not valid for domains with C^k-smooth boundary. Therefore we prefer more elementary methods.

In our proof we shall use only Hörmander's L^2 estimates for $\bar\partial$-problem to construct the kernel G. The rest of the proof consists of standard elementary estimates. In the special case of a strictly convex domain, the kernel G can be written globally without using any $\bar\partial$-problem and the proof in this case is entirely elementary.

We shall apply Theorem 1 to study the boundary behaviour of biholomorphic and proper holomorphic mappings. We shall prove

THEOREM 2. *If D is a bounded strictly pseudoconvex domain with boundary of class C^{k+4} then*

(A) *For each $t_0 \in D$ the Bergman kernel function $K_D(z, t_0)$ belongs to $\Lambda_{k+1/2}$,*

(B) *If $k \geqslant 2$, then for each $z \in \partial D$ there exist $n+1$ points $t_0, \ldots, t_n \in D$ such that*

$$\det \begin{vmatrix} K_D(z, t_j) \\ \dfrac{\partial K_D(z, t_j)}{\partial z_i} \end{vmatrix} \neq 0.$$

In [14] and [11] (compare also [12]) it was shown how these conditions can be used in the study of biholomorphic mappings. The transformation rule for the Bergman function with respect to proper mappings was proved in [2]. It yields, in particular:

COROLLARY 1. *Let D be a bounded strictly pseudoconvex domain with boundary of class C^m, $m \geqslant 6$ and let G be a strictly pseudoconvex domain with boundary of class C^n, $n \geqslant 6$. Then every proper holomorphic mapping from D onto G extends to the mapping of class $\Lambda_{k+1/2}$ from $\bar D$ onto $\bar G$, where $k = \min(n, m)-4$.*

COROLLARY 2. *Let D be a bounded strictly pseudoconvex domain with boundary of class C^{k+4}, $k \geqslant 2$. Let G be a bounded pseudoconvex domain with real analytic boundary or a bounded circular complete domain such that $\lambda \bar G \subset G$ for $\lambda < 1$. Then a biholomorphic mapping between D and G extends to a $\Lambda_{k+1/2}$ diffeomorphism between $\bar D$ and $\bar G$. This implies that if such a mapping exists then D must be also strictly pseudoconvex with boundary of class at least C^k.*

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2. The proof of Theorem 1.

(a) *The kernel $G(w, z)$*. Let ϱ denote a defining function for a domain D. We can always assume that ϱ is strictly plurisubharmonic in a neighbourhood of $\bar D$ and of class C^{k+4} on C^n.

Denote by $L\varrho(z)$ the Levi form of the function ϱ. We can find ε_0 and δ_0 such that $L\varrho(z)(z-w) \geqslant c|z-w|^2$ if $\varrho < \delta_0$ and $|z-w| < \varepsilon_0$. Put

$$F_1(w, z) = \sum_i \frac{\partial\varrho}{\partial z_i}(z)(z_i-w_i) + \tfrac{1}{2}\sum_{ij} \frac{\partial^2\varrho}{\partial z_i\,\partial z_j}(z)(w_i-z_i)(z_j-w_j).$$

It follows from the Taylor formula that

$$\operatorname{Re} F_1(w, z) - \varrho(z) \geqslant -\frac{\varrho(z)+\varrho(w)}{2} + \frac{c}{2}|z-w|^2 \quad \text{if} \quad \varrho(z) < \delta_0 \quad \text{and} \quad |z-w| < \varepsilon_0.$$

Let $\psi(t)$ be a cutting function such that $\psi(t) = 1$ if $t < \varepsilon_0/4$ and $\psi(t) = 0$ if $t > \varepsilon_0/2$. We put $t = |z-w|$ and take

$$F(w, z) = \psi(t) F_1(w, z) + (1-\psi(t))|w-z|^2,$$

$$g_i(w, z) = \psi(t)\left(\frac{\partial\varrho}{\partial z_i}(z) + \tfrac{1}{2}\sum_j \frac{\partial^2\varrho}{\partial z_i\,\partial z_j}(w_j-z_j)\right) + (1-\psi(t))(\bar z_i - \bar w_i).$$

Thus we have

$$\operatorname{Re} F(w, z) - \varrho(z) \geqslant -c_1(\varrho(z)+\varrho(w)) + c_2|z-w|^2 \geqslant c(-\varrho(z)-\varrho(w)+|z-w|^2)$$

for $w \in D_{\delta_0} = \{p \in C^n\colon \varrho(p) < \delta_0\}$ and $z \in \bar D$. The functions $F(w, z)$ and $g_i(w, z)$ are of class C^∞ in w and of class C^{k+2} in z.

Consider now the $(n, n-1)$-form

$$N(w, z) = c\sum_i (-1)^{i-1} \frac{g_i(w, z)}{(F(w, z)-\varrho(z))^n}\, \bar\partial_z g_1 \wedge \ldots \wedge \widehat{\bar\partial_z g_i} \wedge \ldots \wedge \bar\partial_z g_n \wedge dz.$$

The coefficients of the form $N(w, z)$ are of class $C^\infty \times C^{k+1}$ on $D \times \bar{D}$.

Since $N(w, z)$ is a Cauchy–Fantappié form for $z \in \partial D$, for each holomorphic $f \in C^1(\bar{D})$ we have

$$f(w) = \int_{\partial D} N(w, z) f(z), \quad w \in D.$$

Thus, by the Stokes theorem $f(w) = \int_D \bar{\partial}_z N(w, z) f(z)$.

Note that the form $\bar{\partial}_z N(w, z)$ is also of class $C^\infty \times C^{k+1}$ on $D \times \bar{D}$, because only the functions g_i and $F - \varrho$ have been differentiated.

We shall now proceed in the same manner as Kerzman and Stein did in [9].

Let $\delta < \delta_0$ be so small that the form $\bar{\partial}_w \bar{\partial}_z N(w, z)$ is of class $C^\infty \times C^{k+1}$ on $\bar{D}_\delta \times \bar{D}$ and let P_w be a Hörmander operator which solves $\bar{\partial}$-problem on D_δ. We define

$$Q(w, z) = -P_w(\bar{\partial}_w \bar{\partial}_z N(w, z)) \quad \text{and} \quad G(w, z) = Q(w, z) + \bar{\partial}_z N(w, z).$$

The form $G(w, z)$ is holomorphic in w. We shall now show the reproductive property of $G(w, z)$, i.e., that for every function f holomorphic on D_δ and each $w \in D$

$$f(w) = \int_D G(w, z) f(z).$$

It suffices to show that

$$\int_D f(z) P_w(\bar{\partial}_w \bar{\partial}_z (N(w, z))) = 0.$$

We have

$$\int_D P_w(\bar{\partial}_w \bar{\partial}_z N(w, z)) f(z) = P_w \int_D \bar{\partial}_w \bar{\partial}_z N(w, z) f(z) = P_w \int_D \bar{\partial}_z \bar{\partial}_w N(w, z) f(z)$$
$$= P_w \int_{\partial D} \bar{\partial}_w N(w, z) f(z) \text{ (by the Stokes theorem).}$$

It was proved by Kerzman and Stein in [9] that the last term vanishes.

Since $\bar{\partial}$ is an elliptic operator, the form $Q(w, z)$ is of class $C^\infty \times C^{k+1}$ on $\bar{D} \times \bar{D}$ and thus is a nonsingular kernel. We can now write down a singularity of $G(w, z)$. We have

$$\bar{\partial}_z N(w, z) = \frac{\dfrac{(-1)^p}{\pi^n} \det \begin{vmatrix} \varrho(z) & g_i(z, w) \\ \dfrac{\partial \varrho}{\partial z_j} & \dfrac{\partial g_i(z, w)}{\partial z_j} \end{vmatrix}}{(F(z, w) - \varrho(z))^{n+1}}.$$

It can easily be checked that

$$\bar{\partial}_z N(w, z) = \frac{\dfrac{(-1)^p}{\pi^n} \det \begin{vmatrix} \varrho(z) & \dfrac{\partial \varrho}{\partial z_i}(z) \\ \dfrac{\partial \varrho}{\partial z_i}(z) & \dfrac{\partial^2 \varrho}{\partial z_i \partial \bar{z}_j}(z) \end{vmatrix} + O(|z - w|)}{(F(z, w) - \varrho(z))^{n+1}}.$$

The kernel $\bar{\partial}_z N$ is of class $C^\infty \times C^{k+1}$ on $\bar{D} \times D$.

(b) *The properties of the projection* G. We shall now prove that the integral operator

$$Gu = \int_D G(w, z) u(z) \, dV_z$$

is a well-defined projection from $L^2(D)$ onto $L^2 H(D)$. By now we know that it is defined on holomorphic functions on D_δ and on functions from $C_0^\infty(D)$.

The formal adjoint of G is the operator

$$G^* u = \int_D \overline{G(z, w)} u(z) \, dV_z.$$

Let $B = G^* - G$ be the integral operator with the kernel $B(w, z) = \overline{G(z, w)} - G(w, z)$. It was proved by Kerzman and Stein that

$$\overline{F(z, w)} - F(w, z) = \varrho(w) - \varrho(z) + O(|z - w|^3),$$

and hence

$$[F(w, z) - \varrho(z)] - [\overline{F(z, w)} - \varrho(w)] = O(|z - w|^3).$$

This implies that the kernel B is dominated by

$$C \frac{|z - w|}{((-\varrho(z) - \varrho(w))/2 + C_1 |w - z|^2)^{n+1}}$$

and therefore, by the Kranz estimates [10], it is of weak type $(2n+2)/(2n+1)$. In particular, it extends to a continuous operator from $L^2(D)$ into $L^2(D)$.

We have now for $u \in C_0^\infty(D)$ or $u \in H(D_\delta)$

$$(*) \qquad \|Gu\|^2 = \langle Gu, Gu \rangle = \langle u, (G + B) Gu \rangle \leqslant \|u\| (\|G^2 u\| + \|B\| \|Gu\|)$$
$$= \|u\| \|Gu\| (1 + \|B\|) \quad \text{because } G^2 u = Gu \text{ if } Gu \in H(D_\delta).$$

Thus $\|Gu\| \leqslant \|u\| (1 + \|B\|)$ (compare Kerzman–Stein [9], Theorem 13.1). It can easily be proved, by using Hörmander's L^2-estimates for $\bar{\partial}$ problem, that functions holomorphic on D_δ are dense in $L^2 H(D)$. It follows from $(*)$ that G is continuous on $L^2 H(D)$ so it must be equal to identity on $L^2 H(D)$. Thus $G^2 u = Gu$ for every $u \in C_0^\infty(D)$. Hence, G is a well-defined continuous projector from $L^2(D)$ onto $L^2 H(D)$.

Remark. In a special case where D is a *strictly convex domain*, the construction of the kernel $G(w,z)$ and of the projector P is very simple. If ϱ is a strictly convex defining function for D, it is sufficient to take

$$N(w,z) = \sum_{i=1}^{n} (-1)^{i-1} \frac{g_i}{(F(w,z)-\varrho(z))} \bar{\partial} g_1 \ldots \widehat{\bar{\partial} g_i} \ldots \bar{\partial} g_n \, dz$$

where

$$g_i(w,z) = \sum_{i=1}^{n} \frac{\partial\varrho}{\partial z_i}(z) + \frac{1}{2}\sum_{ij} \frac{\partial^2\varrho}{\partial z_i \partial z_j}(z)(w_j - z_j),$$

$$F(w,z) = \sum_{i=1}^{n} \frac{\partial\varrho}{\partial z_i}(z)(z_i - w_i) + \frac{1}{2}\sum_{ij} \frac{\partial^2\varrho}{\partial z_i \partial z_j}(w_j - z_j)(w_i - z_i)$$

to get $G(w,z) = \bar{\partial} N(w,z)$.

The density of $H(D_\delta)$ in $L^2 H(D)$ is obvious in this case. Thus no $\bar{\partial}$-problem is needed.

Note that, usually, in literature one can find another formula for strictly convex domains. In this formula

$$g_i = \frac{\partial\varrho}{\partial z_i}(z) \quad \text{and} \quad F(w,z) = \sum_{i=1}^{n} \frac{\partial\varrho}{\partial z_i}(z)(z_i - w_i).$$

However, this formula is of no use for our purposes, because it gives only the estimate

$$[F(w,z)-\varrho(z)] - [\overline{F(z,w)} - \varrho(w)] = O(|z-w|^2),$$

which is too weak.

Now, let P denote the Bergman projector. We have $PG = G$, $GP = P$, $G^*P = G^*$, $PG^* = P$. Thus $P(I-B) = G$ and $(I+B)P = G^*$ $(B = G^* - G)$. The operators $I+B$ and $I-B$ are invertible in $L^2(D)$, and hence

$$P = G(I-B)^{-1} = (I+B)^{-1} G^*.$$

This implies that the proof of Theorem 1 will be complete if we prove the following four facts:

I. If $\alpha \leqslant k+1/2$ then $I-B$ and $I+B$ are isomorphisms of $\Lambda_\alpha(D)$,

II. If $k < \alpha < k+1$ then G and G^* map continuously $\Lambda_\alpha(D)$ into $\Lambda_{k+(\alpha-k)/2}(D)$,

III. The operators G and G^* are continuous from $\mathrm{Lip}_k(D)$ into $\Lambda_{k+1/2}(D)$,

IV. If $\alpha \leqslant k$ then G and G^* map continuously $\Lambda_\alpha(D)$ into $\Lambda_\alpha(D)$.

The proof of these facts will need the gradient estimates which were done by Kranz [10] and an integration by parts lemma originated by Elguetá [5] and used by Ahern and Schneider [1]. We begin with the

gradient estimates and the proof of I, II and III for $k = 0$. If $k = 0$ then no integration by parts lemma is needed.

(d) *The gradient estimates*. At first we formulate the following simple estimates for $F(w,z) - \varrho(z)$ and $\overline{F(z,w)} - \varrho(z)$:

$$(**) \quad 1) \quad \frac{\partial}{\partial w_j}(F(w,z)-\varrho(z)) = -\frac{\partial\varrho}{\partial z_j}(z) + O(|w-z|),$$

$$2) \quad \frac{\partial}{\partial \bar{w}_j}(F(w,z)-\varrho(z)) = 0 \quad \text{if} \quad |w-z| < \frac{\varepsilon}{2},$$

$$3) \quad \frac{\partial}{\partial z_i}(F(w,z)-\varrho(z)) = -\frac{\partial\varrho}{\partial z_i}(z) + O(|z-w|),$$

$$4) \quad \frac{\partial}{\partial \bar{z}_i}(F(w,z)-\varrho(z)) = O(|z-w|),$$

$$5) \quad \frac{\partial}{\partial w_j}(\overline{F(z,w)}-\varrho(w)) = -\frac{\partial\varrho}{\partial w_j}(w),$$

$$6) \quad \frac{\partial}{\partial \bar{w}_j}(\overline{F(z,w)}-\varrho(w)) = O(|z-w|),$$

$$7) \quad \frac{\partial}{\partial \bar{z}_j}(\overline{F(z,w)}-\varrho(w)) = -\frac{\partial\varrho}{\partial \bar{w}_i}(w) + O(|z-w|),$$

$$8) \quad \frac{\partial}{\partial z_j}(\overline{F(z,w)}-\varrho(w)) = O(|z-w|).$$

Since $[F(w,z)-\varrho(z)] - [\overline{F(z,w)} - \varrho(w)] = O(|z-w|^3)$, the kernel $B(w,z) = \overline{G(z,w)} - G(w,z)$ is dominated by

$$\frac{|z-w|}{[(-\varrho(z)-\varrho(w))/2 + c|z-w|^2]^{n+1}} \quad \text{if} \quad |z-w| < \frac{\varepsilon_0}{2}.$$

We shall now proceed in the same manner as Kranz did in [10] and find a suitable σ, $0 < \sigma < \varepsilon_0/6$ and a $C^\infty(C^n \times C^n)$ cutting function $h(w,z)$ such that $h(w,z) = 1$ if $-\varrho(z) - \varrho(w) + |w-z| < \sigma/2$, $h(w,z) = 0$ if $-\varrho(z) - \varrho(w) + |z-w| > \sigma$.

We can now consider integral operators B_1 with the kernel $(1 - h(w,z))B(w,z)$ and B_2 with the kernel $h(w,z)B(w,z)$. The first operator is of class C^{k+1} on $\bar{D} \times \bar{D}$ and therefore is nonsingular. It follows from $(**)$ that $|\mathrm{grad}_w B_2(w,z)|$ is dominated by

$$\frac{|w-z|}{\left(\dfrac{-\varrho(w)-\varrho(z)}{2} + c|w-z|^2\right)^{n+2}} \leqslant \frac{c_1}{|w-z|\left(\dfrac{-\varrho(w)-\varrho(z)}{2} + c|w-z|^2\right)^{n+1}}.$$

This implies that we can use Kranz gradient estimates (see [10]; our singularity is the same as in the case $\alpha = 2$, $k = 1$ in this paper). Thus for every bounded function $f \in L^\infty(D)$

$$\left|\operatorname{grad}_w \int B_2(w, z) f(z)\, dV_z\right| \leqslant \|f\|_\infty \int\limits_{|z-w|<\sigma} \frac{C_1}{|w-z|\left(\frac{-\varrho(w)-\varrho(z)}{2}+c|w-z|^2\right)^{n+1}} dV_z \leqslant c_2 \|f\|_\infty \cdot \frac{1}{(-\varrho(w))^{1/2}}.$$

Hence B_2 is continuous from L^∞ into $\Lambda_{1/2}$. This implies that B is continuous from L^∞ into $\Lambda_{1/2}$. From the Ascoli theorem it follows that $I+B$ and $I-B$ are invertible Fredholm operators from L^∞ onto L^∞. Let $0 < \alpha < 1/2$.

If $(I+B)f = g \in \Lambda_\alpha$ then $f \in \Lambda_\alpha$. The same is true for $I-B$. Hence $I+B$ and $I-B$ are isomorphisms of Λ_α.

To prove II and III for $k = 0$, we need the reproducing property of G. Let $f \in \Lambda_\alpha$, $0 < \alpha < 1$ or $f \in \operatorname{Lip}_1(D)$. We have

$$(I-G)f(w) = \int_D (f(w)-f(z)) \cdot G(w, z)\, dV_z$$

and

$$\frac{\partial f}{\partial w_i}(w) - \frac{\partial Gf}{\partial w_i} = \int_D \frac{\partial f}{\partial w_i}(w)\, G(w, z)\, dV_z - \int_D (f(w)-f(z)) \frac{\partial G(w, z)}{\partial w_i} dV_z.$$

Thus $\operatorname{grad} Gf = \int_D (f(w)-f(z)) \operatorname{grad}_w G(w, z)\, dV_z$.

If w is near z then the kernel on the right-hand side is dominated by

$$\frac{|z-w|^\alpha}{((-\varrho(z)-\varrho(w))/2+c|z-w|^2)^{n+2}}.$$

Take now the cutting function $h(w, z)$ as before and define G_1 to be the operator with kernel $(1-h(w, z))G(w, z)$ and G_2 to be the operator with kernel $h(w, z) \cdot G(w, z)$. Since G_1 is nonsingular, we have to estimate only G_2. Thus

$$f(w) h(w, w) - G_2 f(w) = \int_D [h(w, w) f(w) - h(w, z) f(z)]\, G(w, z)\, dV_z$$

and

$$\frac{\partial G_2 f}{\partial w_i} = \int_D [h(w, w) f(w) - h(w, z) f(z)] \frac{\partial G}{\partial w_i}(w, z)\, dV_z,$$

$$\frac{\partial G_2 f}{\partial \bar{w}_i} = \int_D f(z) \cdot \frac{\partial h}{\partial \bar{w}_j} G(w, z)\, dV_z.$$

The first expression is dominated by

$$\int_D \frac{c_1 |z-w|^\alpha \|f\|_{\Lambda_\alpha}}{[(-\varrho(w)-\varrho(z))/2+c|z-w|^2]^{n+2}} dV_z$$

and the second by

$$\int_D \frac{c_2 \|f\|_{\Lambda_\alpha}}{(-\varrho(z)-\varrho(w)+c|z-w|^2)^{n+1}}.$$

Thus $|\operatorname{grad}_w G_2 f| \leqslant \|f\|_{\Lambda_\alpha} \left(\frac{c_1}{(-\varrho)^{1-\alpha/2}(w)} + c_2 |\ln^2 \varrho(w)|\right)$. This implies that G_2 maps Λ_α into $\Lambda_{\alpha/2}$ and Lip_1 into $\Lambda_{1/2}$. We have proven I, II and III for $k = 0$.

To deal with the highest order derivatives we shall need the following:

(e) *Integration by parts lemma.* Suppose that the cutting function $h(w, z)$ is chosen such that

$$\left|\sum_i \frac{\partial(F(w, z)-\varrho(z))}{\partial \bar{z}_i} \cdot \frac{\partial \varrho}{\partial z_i}\right| > c > 0 \quad \text{and} \quad \left|\sum_i \frac{\partial(\overline{F(z, w)}-\varrho(w))}{\partial \bar{z}_i} \cdot \frac{\partial \varrho}{\partial z_i}\right| > c > 0$$

on the support of h. This is possible to achieve because of (**).

Let Φ denote $F(w, z) - \varrho(z)$ or $\overline{F(z, w)} - \varrho(z)$ and let $Q(w, z)$ be equal to $\sum_i \frac{\partial}{\partial \bar{z}_i} \Phi \cdot \frac{\partial \varrho}{\partial z_i}$. Then it follows from the Stokes theorem that for every $u(w, z)$ which is $C^1(\bar{D})$ in z,

$$(1) \quad \int_D \frac{u \cdot h}{\Phi^{m+1}} dV_z = c_1 \int_{\partial D} \frac{u \cdot h}{Q \cdot \Phi^m} \sum_i (-1)^i \frac{\partial \varrho}{\partial z_i} d\bar{z}_i \wedge dz_i + c_2 \int_D \sum_i \frac{\partial}{\partial \bar{z}_i} \left(\frac{u \cdot h \cdot \partial \varrho / \partial z_i}{Q}\right) \cdot \frac{1}{\Phi^m} dV_z$$

and for each $v(w, z)$ of class $C^1(\bar D)$ in z

$$(2)\quad \int_{\partial D} \frac{v\cdot h}{\Phi^m}\, d\sigma = \sum_i \int_{\partial D} \frac{\partial \varrho}{\partial z_i}\left(\frac{v\cdot h}{Q_1}\right)\cdot\frac{\partial\varrho/\partial\bar z_i}{|\nabla\varrho|}\cdot\frac{1}{\Phi^{m-1}}\, d\sigma + \sum_i \int_{\partial D}\left(\frac{v\cdot h}{Q_1}\right)\cdot\frac{\partial\varrho/\partial\bar z_i}{|\nabla\varrho|}\cdot\frac{1}{\Phi^{m-1}}\, d\sigma$$

where

$$Q_1 = \frac{\sum_i \frac{\partial\Phi}{\partial\bar z_i}\cdot\frac{\partial\varrho}{\partial z_i}+\frac{\partial\Phi}{\partial z_i}\frac{\partial\varrho}{\partial\bar z_i}}{|\operatorname{grad}\varrho|}.$$

Thus to prove I, II and III for $k > 0$ it suffices to differentiate $B_2 f$ k times in w and apply the above given integration by parts lemma k times. For B_2 the gradients of the arising area kernels will be dominated by

$$\int_D \frac{|z-w|^{\alpha-k}}{|\Phi|^{n+2}} \quad\text{or}\quad \int_D \frac{1}{|\Phi|^{n+1}}$$

and the gradients of boundary terms by

$$\int_{\partial D} \frac{|z-w|^{\alpha-k}}{|\Phi|^{n+1}} \quad\text{or}\quad \int_{\partial D} \frac{1}{|\Phi|^{n}}.$$

To estimate $G_2(w, z)$ and $B_2(w, z)$ we shall use both the reproducing property (over the domain) and the reproducing property of Kerzman–Stein kernel (over the boundary) and proceed as in the case of $k = 0$.

The details of calculations are the same as in the paper by Ahern–Schneider for Henkin kernel and the Boutet de Monvel expression of the Bergman kernel function. To make the whole process understandable we shall give more details in the case $k = 1$.

Denote by $l(z)$ the function

$$\frac{(-1)^p}{\pi^n}\det\begin{vmatrix} \varrho(z) & \frac{\partial\varrho}{\partial z_i}(z) \\ \frac{\partial\varrho}{\partial\bar z_j} & \frac{\partial^2\varrho}{\partial z_i\,\partial\bar z_j}\end{vmatrix}.$$

Let $f\in\Lambda_\alpha$, $\alpha > 1$ or $f\in\operatorname{Lip}(D) = L_1^\infty$, and

$$B_2 f = \int_D h(w, z) f(z)\left[\overline{G(z, w)} - G(w, z)\right]$$
$$= \int_D h(w, z) f(z)\left[\frac{l(w)+O(|z-w|)}{(\overline{F(z, w)}-\varrho(w))^{n+1}} - \frac{l(z)+O(|z-w|)}{(F(w, z)-\varrho(z))^{n+1}}\right] + \text{nonsingular terms.}$$

Thus

$$\frac{\partial B_2 f}{\partial w_j} = \int_D \frac{\partial h}{\partial w_j}(w, z) f(z)[\ldots]\, dV_z +$$
$$+\int_D h(w, z) f(z)\left|\frac{\frac{\partial}{\partial w_j}(l(w)+O(|z-w|))}{(\overline{F(z, w)}-\varrho(w))^{n+1}} - \frac{\frac{\partial}{\partial w_j}(l(z)+O(|z-w|))}{(F(w, z)-\varrho(z))^{n+1}}\right| dV_z +$$
$$+c\int_D h(w, z) f(z)\left|\frac{\frac{\partial}{\partial w_j}(\overline{F(z, w)}-\varrho(w))(l(w)+O(|z-w|))}{(\overline{F(z, w)}-\varrho(w))^{n+2}} - \frac{\frac{\partial}{\partial w_j}(F(w, z)-\varrho(z))(l(z)+O|z-w|)}{(F(w, z)-\varrho(z))^{n+2}}\right| dV_z.$$

The expression for $\partial B_2/\partial\bar w_j$ has the same form.

We must observe now that the first term on the right-hand side is nonsingular since $h(z, w) = 1$ if $|z-w| < \sigma/2$. If we apply the integration by parts lemma to the second term on the right-hand side, then we get area kernels which are dominated by $\|f\|_{L_1^\infty}/\Phi^n$ with gradients in w dominated by $\|f\|_{L_1^\infty}/\Phi^{n+1}$ and boundary kernels which are dominated by $\|f\|_{L_1^\infty}/\Phi^{n-1}$ with gradients in w dominated by $\|f\|_{L_1^\infty}/\Phi^n$.

It is shown in [6], [7] and [10] that

$$\left|\int_D \frac{c}{\Phi^{n+1}}\right| < c_1\cdot c\left|\ln^2(-\varrho(w))\right| \quad\text{and}\quad \left|\int_{\partial D} \frac{c}{\Phi^{n+1}}\right| < c_1\cdot c\left|\ln^2(-\varrho(w))\right|$$

and thus we have the desired estimates.

It remains to estimate the last term.

We apply the integration by parts lemma (1) to it and consider the resulting area kernel. It is equal to

$$S(w, z)$$

$$= c_1 \sum_i \frac{\partial}{\partial \bar{z}_i} \left(\frac{f(z) h(w, z) \cdot \frac{\partial \varrho}{\partial z_i} \cdot \frac{\partial}{\partial w_j} \overline{(F(z, w) - \varrho(w))} \cdot (\bar{l}(w) + O(|z-w|))}{Q} \right) \times$$

$$\times \frac{1}{\overline{(F(z, w) - \varrho(w))^{n+1}}} -$$

$$- \frac{\partial}{\partial \bar{z}_i} \left(\frac{f(z) h(w, z) \cdot \frac{\partial \varrho}{\partial z_i} \cdot \frac{\partial}{\partial w_j} (F(w, z) - \varrho(z)) \cdot (l(z) + O(|z-w|))}{Q} \right) \times$$

$$\times \frac{1}{(F(w, z) - \varrho(z))^{n+1}}.$$

Denote by $\tilde{S}(w, z)$ the sum of all terms of $S(w, z)$ where $f(z)$ is not differentiated. Since $f(z)$ is in $L_1^\infty(D) = \mathrm{Lip}(D)$ and other functions are in $C^1(\bar{D})$, $S(w, z)$ can be written as

$$\frac{u(w, z)}{\overline{(F(z, w) - \varrho(w))^{n+1}}} - \frac{v(w, z)}{(F(w, z) - \varrho(z))^{n+1}} \quad \text{where}$$

$$|u(w, z) - u(w, w)| < c|z-w| \quad \text{and} \quad |v(w, z) - v(w, w)| < c|z-w|.$$

The kernel $\tilde{S}(w, z)$ can be written as

$$\tilde{S}(w, z) = \frac{u(w, z) - u(w, w)}{\overline{(F(z, w) - \varrho(w))^{n+1}}} + \frac{v(w, w) - v(w, z)}{(F(w, z) - \varrho(z))^{n+1}} +$$

$$+ u(w, w) \left(\frac{1}{\overline{(F(z, w) - \varrho(w))^{n+1}}} - \frac{1}{(F(w, z) - \varrho(z))^{n+1}} \right) + \frac{u(w, w) - v(w, w)}{(F(w, z) - \varrho(z))^{n+1}}.$$

We must now estimate $\operatorname{grad}_w \int_D \tilde{S}(w, z)\, dV_z$.

The gradients of the first three terms on the right-hand side can be estimated by

$$c\|f\|_{L_1^\infty} \int_D \left(\frac{|z-w|}{|\Phi|^{n+2}} + \frac{c_1}{|\Phi|^{n+1}} \right) dV_z$$

which is exactly what we need.

To estimate $\operatorname{grad}_w \int_D \frac{u(w, w) - v(w, w)}{(F(w, z_j - \varrho(z))^{n+1}}$ we shall use the reproducing property of

$$\bar{\partial}_z N(w, z) = \frac{L(w, z)}{(F(w, z) - \varrho(z))^{n+1}} = \frac{l(z) + O|z-w|}{(F(w, z) - \varrho(z))^{n+1}}.$$

Denote by $q(w)$ the function $(u(w, w) - v(w, w))/l(w)$. (The function $l(w)$ is bounded away from zero if w is near ∂D.) Then

$$\operatorname{grad}_w \int_D \frac{u(w, w) - v(w, w)}{(F(w, z) - \varrho(z))^{n+1}} dV_z = \operatorname{grad}_w \int_D \frac{q(w)\, l(w)}{(F(w, z) - \varrho(z))^{n+1}} dV_z$$

$$= \operatorname{grad}_w \int_D \frac{q(w)[l(w) - L(w, z)]}{(F(w, z) - \varrho(z))^{n+1}} dV_z + \operatorname{grad} q(w)\, m.$$

Since $q(w)(l(w) - L(w, z)) = O|z-w|$, our gradient can be estimated by

$$c\|f\|_{L_1^\infty} \left(\int_D \left[\frac{|z-w|}{|\Phi|^{n+2}} + \frac{c_1}{|\Phi|^{n+1}} \right] dV_z + c_2 \right).$$

It remains now to estimate those terms of $S(w, z)$ in which $f(z)$ is differentiated. Denote by $\tilde{\tilde{S}}(w, z)$ the kernel consisting of these terms. It is easy to check that $\tilde{\tilde{S}}(w, z)$ can be written as

$$\sum_i \frac{\partial f}{\partial \bar{z}_i}(z) \left(\frac{u_0(w, z)}{\overline{(F(z, w) - \varrho(w))^{n+1}}} - \frac{v_0(w, z)}{(F(w, z) - \varrho(z))^{n+1}} \right)$$

where $u_0(w, w) = v_0(w, w)$. Thus we can proceed as before, writing

$$\tilde{\tilde{S}}(w, z) = \sum_i \left| \frac{u_0(w, z) - u_0(w, w)}{\overline{(F(z, w) - \varrho(w))^{n+1}}} + \frac{v_0(w, w) - v(w, z)}{(F(w, z) - \varrho(z))^{n+1}} + \right.$$

$$\left. + u_0(w, w) \left(\frac{1}{\overline{(F(z, w) - \varrho(w))^{n+1}}} - \frac{1}{(F(w, z) - \varrho(z))^{n+1}} \right) \right| \cdot \frac{\partial f}{\partial \bar{z}_i},$$

and get the desired estimates.

To estimate $\operatorname{grad}_w(\partial B_2/\partial w_j)$ we must deal also with the boundary terms which arose from the integration by parts formula (1). We can at first observe that the form $N(w, z)$ has a reproducing property as the Cauchy–Fantappié form. We can always write

$$N(w, z) = \frac{K(w, z)}{\Phi^n} = \frac{K(w, z)}{(F(w, z) - \varrho(z))^n} = \frac{K(w) + O(|w-z|)}{(F(w, z) - \varrho(z))^n}$$

where $K(w) = K(w, w)$. Thus after using the integration by parts formula (2) we can repeat word by word the whole proof of our estimates given for area kernels. The only difference is that the integration will be over boundary (in z, of course).

It is clear from (**) that the estimates for $\partial B/\partial \bar{w}_j$ are the same. We have thus proved that the operator B maps continuously $\mathrm{Lip}(D)$ into $\Lambda_{1+1/2}$. Hence the operators $I-B$ and $I+B$ are isomorphisms of Λ_α for each α, $1 < \alpha \leqslant 1+1/2$.

To end the proof of our theorem we must estimate

$$\mathrm{grad}_w \frac{\partial}{\partial w_j} Gf = \mathrm{grad}_w \frac{\partial}{\partial w_j} \int_D G(w, z) f(z)\, dV_z$$

for $f \in \Lambda_\alpha$, $1 < \alpha \leqslant 3/2$ or $f \in \mathrm{Lip}_1(D)$. This means that we must, as before, estimate the gradient of the following kernel:

$$\int_D \frac{\frac{\partial}{\partial w_j}(F(w, z) - \varrho(z)) \cdot L(w, z)}{(F(w, z) - \varrho(z))^{n+2}} h(w, z) f(z).$$

Integrate by parts as before and consider two area kernels. The first of them consisting of terms in which $f(z)$ is not differentiated can be written as

$$\int_D \frac{u(w, z)}{\Phi(w, z)^{n+1}} \cdot f(z)\, dV_z$$

$$= \int_D \frac{u(w, z) f(z) - u(w, w) f(w)}{\Phi(w, z)^{n+1}} dV_z + \int_D \frac{u(w, w) f(w)}{\Phi(w, z)^{n+1}} dV_z$$

$$= \int_D \frac{u(w, z) f(z) - u(w, w) f(w)}{\Phi(w, z)^{n+1}} dV_z - \int_D \frac{\frac{u(w, w)}{l(w)} f(w)(L(w, z) - l(w))}{\Phi(w, z)^{n+1}} dV_z + $$

$$+ f(w) \frac{u(w, w)}{l(w)}.$$

Thus the gradient estimates are here the same as before. The second area kernel is a sum of integrals of the type

$$\int_D \frac{v(w, z) L(w, z)}{\Phi(w, z)^{n+1}} \cdot \frac{\partial f}{\partial \bar{z}_i}(z) = v(w, w) \cdot \frac{\partial f}{\partial \bar{w}_i}(w) +$$

$$+ \int_D \frac{\frac{\partial f}{\partial z_i}(z) V(w, z) \cdot \frac{\partial f}{\partial \bar{w}_i}(w) V(w, w) \cdot L}{\Phi(w, z)^{n+1}}.$$

Since $f(z) \in \Lambda_\alpha$ or Lip_1, the gradient of the last term is dominated by

$$c\|f\| \int_D \left(\frac{|z-w|^{\alpha-1}}{|\Phi|^{n+2}} + \frac{c_1}{|\Phi|^{n+1}} \right) dV_z.$$

The same procedure can be applied to the boundary terms. Thus G maps continuously Λ_α into $\Lambda_{1+(\alpha-1)/2}$ and Lip_1 into $\Lambda_{3/2}$. This ends the proof of I, II and III for $k = 1$.

The part IV can be proved by exactly the same method as used in the paper of Phong–Stein [13] for the Boutet de Monvel expression for the Bergman kernel function.

Thus we have the desired estimates for Bergman projection.

3. The proof of Theorem 2. Condition (A) is valid for each $k \geqslant 0$, because $K_D(\cdot, t_0) \in P(C_0^\infty(D))$ (see [3], [4]). It remains to prove condition (B). The proof of it will be actually the same as that given in [4] although it needs some modification, because $C_0^\infty(D)$ is not dense in Λ_α. Thus we shall use the following Sobolev imbedding theorem: Let W_m^s denote the Sobolev space of functions D with derivatives up to order s belonging to $L^m(D)$ and let $\mathring{W}_m^s$ denote the closure of $C_0^\infty(D)$ in W_m^s. As in the case $m = 2$ we have

$$\mathring{W}_m^s = \left\{ f \in W_m^s\colon \frac{\partial^\alpha f}{\partial z^\beta\, \partial \bar{z}^{\alpha-\beta}} = 0 \text{ on } \partial D \text{ if } |\alpha| \leqslant s-1 \right\}.$$

If $m > 2n$ then $\mathring{W}_m^s$ is compactly imbedded in $\Lambda_{s-1+(m-2n)/2}$. This implies, in particular, that the Bergman projection P maps continuously $\mathring{W}_{2n}^2$ into $\Lambda_{1+(m-2n)/4n} = \Lambda_{1+\beta}$. Now we shall prove that each holomorphic function which is C^∞-smooth on $\bar{D}$ belongs to the closure of the linear span of the set $\{K_D(\cdot, t)\colon t \in D\}$ in $\Lambda_{1+\beta}$.

Let $f \in C^\infty(\bar{D}) \cap H(D)$. Since $k \geqslant 2$ and ∂D is of class at least C^6, we can use the construction described in [4] (proof of Lemma 1) to find the function u such that: (a) $u \in C^2(\bar{D})$, (b) u vanishes on ∂D with its first derivatives, (c) $P(u) = f$. It is clear that $u \in \mathring{W}_m^2$.

Now for $t \in D$ let $\Phi \in C_0^\infty$ be radially symmetric about t with $\int_C \Phi\, dV = 1$.

Then $P(\Phi) = K_D(\cdot, t)$.

The linear span of such functions Φ is dense in $\mathring{W}_m^2$ (see [4], proof of Lemma 1 for details). This implies that $\mathrm{span}\{K_D(\cdot, t)\}$ is dense in $P(\mathring{W}_m^2) \subset \Lambda_{1+\beta}$. Thus the function $f = P(u)$ belongs to the closure of $\mathrm{span}\{K_D(\cdot, t), t \in D\}$ in $\Lambda_{1+\beta}$.

Now, let $z_0 \in \partial D$. The function

$$F_{z_0}(t_0, \ldots, t_n) = \det \begin{vmatrix} K_D(z_0, t_i) \\ \dfrac{\partial K_D(z_0, t_i)}{\partial z_j} \end{vmatrix}.$$

cannot be identically zero because if it were, then we would get

$$\det \begin{vmatrix} g_i(z_0) \\ \dfrac{\partial g_i}{\partial z_j}(z_0) \end{vmatrix} = 0 \quad \text{for each } (g_1, \ldots, g_n) \subset C^\infty(\bar{D}) \cap H(D)$$

and get a contradiction for $g_0 = 1$, $g_i = z_i$.

Proof of Corollary 1. Let D and G be bounded strictly pseudoconvex domains with boundaries of class C^{k+4}, $k \geq 2$ and let h be a proper mapping from D onto G. Pinchuk in [14] and [15] proved that h extends to a continuous mapping from $\bar{D}$ onto $\bar{G}$ and that the Jacobian Jh is bounded away from zero on $\bar{D}$. S. Bell in [2] proved the following transformation rule for the Bergman function under proper mappings:

Let $w \in G$ and $h^{-1}(w) = \{w_1, \ldots, w_m\}$ and let $U_i(w)$ denote the Jacobian of the ith local inverse of h. Then

$$\sum_{i=1}^{m} K_D(z, w_i)\overline{U_i(w)} = K_G(h(z), w)\cdot Jh(z).$$

It follows from condition (B) that for every $z_0 \in \partial D$ we can find $n+1$ points from G such that the quotients $v_k(w) = K_G(w, a_k)/K_G(w, a_0)$ are local coordinates in the neighbourhood of $h(z_0)$ (see [11]). For each k, v_k is of class $\Lambda_{k+1/2}$ on some neighbourhood of $h(z_0)$ in $\bar{G}$. The transformation rule implies that

$$v_k(h(w)) = \frac{\sum_{i=1}^{m} K_D(z, a_{ki})\overline{U(a_{ki})}}{\sum_{i=1}^{m} K_D(z, a_{0i})\overline{U(a_{0i})}}.$$

Since $Jh(z)$ is bounded away from zero on D, the function on the right-hand side is of class $\Lambda_{k+1/2}$ in some neighbourhood of z_0 in $\bar{D}$. It is obvious that $h(z)$ extends to a mapping of class C^k on $\bar{D}$.

Since $k \geq 2$, we have that $h(z) \in C^2(\bar{D})$ and is a Lipschitz mapping between $\bar{D}$ and $\bar{G}$. Thus it is also of class $\Lambda_{k+1/2}$ on $\bar{D}$.

The proof of Corollary 2 is the same as that given in [11] and thus it can be omitted.

4. Remark on extension of proper holomorphic mappings. The transformation rule for the Bergman functions under proper mappings due to S. Bell [2] (see the proof of Corollary 1) enables us to prove the following fact:

Proposition. *Let D and G be bounded domains such that*

(A) *The Bergman functions $K_D(z, \cdot)$ and $K_G(w, \cdot)$ extend to functions of class C^k (or Λ_α, $\alpha > 1$) on $\bar{D}$ and $\bar{G}$, respectively.*

Assume also that condition (B) *from Theorem* 2 *holds for $K_G(w, \cdot)$. Let h be a proper holomorphic mapping from D onto G. Then*

(1) *the complex Jacobian $Jh(z)$ and the mapping $Jh\cdot h$ extend continuously to $\bar{D}$,*

(2) *the proper mapping h extends to the set $\bar{D}\setminus\{z: z \in \partial D,\ Jh(z) = 0\}$ as a mapping of class C^k (locally of class Λ_α, $\alpha > 1$)*

Proof. Similarly as in [4] and [11] we prove (1) by showing that the partial derivatives $\partial Jh/\partial z_i$ and $\partial(Jh\cdot h_k)/\partial z_i$ are bounded functions on D. If it is not true, then we can find a sequence $z_n \to z_0 \in \partial D$ such that $s_n = h(z_n) \to s_0 \in \partial G$ and $\operatorname*{Max}_{i,k}\left(\left|\dfrac{\partial Jh}{\partial z_i}(z_n)\right|, \left|\dfrac{\partial(Jh\cdot h_k)}{\partial z_i}\right|\right) \to \infty$. By Condition (B) there exist $b_0, b_1, \ldots, b_n \in G$ such that

$$\det \begin{vmatrix} K_G(s_0, b_j) \\ \dfrac{\partial K_G(s_0, b_j)}{\partial s_i} \end{vmatrix} \neq 0.$$

Let $\{a_{j1}, \ldots, a_{jm}\} = h^{-1}(b_j)$ and let $u(a_{jr})$ be the Jacobian of the rth inverse of h at b_j. The Bell transformation rule implies that

$$u_j(z) = \sum_{r=1}^{m} K_D(z, a_{jr})\overline{U(a_{jr})} = Jh(z)\cdot K_G(h(z), b_j)$$

and

$$\frac{\partial u_j(z)}{\partial z_i} = \frac{\partial Jh(z)}{\partial z_i} K_G(h(z), b_j) + \sum_{k=1}^{n} Jh(z)\cdot\frac{\partial K_G(h(z), b_j)}{\partial s_k}\cdot\frac{\partial h_k}{\partial z_i}.$$

Thus we get the following matrix equality:

$$\begin{vmatrix} u_j \\ \dfrac{\partial u_j}{\partial z_i} \end{vmatrix} = \begin{vmatrix} K_G(h(z), b_j) \\ \dfrac{\partial K_G}{\partial s_k}(h(z), b_j) \end{vmatrix} \cdot \begin{vmatrix} Jh(z) & 0 \\ \dfrac{\partial Jh_j}{\partial z_i} & \dfrac{\partial h_k}{\partial z_i}\cdot Jh(z) \end{vmatrix}.$$

There exists $N > 0$ such that the entries of the matrix on the left-hand side are bounded on the set $\{z_n\}_{n>N}$ and the entries of the inverse of the first matrix on the right-hand side are bounded on the set $\{s_n\}_{n>N}$. Thus the derivatives $\partial Jh/\partial z_i$ and $\partial(Jh\cdot h)/\partial z_i$ are bounded on the set $\{z_n\}_{n>N}$. A contradiction.

Note that (1) implies that h extends continuously to the set $\bar{D}\setminus\{z \in \partial D: Jh(z) = 0\}$. The smoothness of this extension ((2)) is proved in the same way as in the proof of Corollary 1, using the equality

$$\frac{\sum_{i=1}^{m} K_D(z, a_{jr})\overline{U(a_{jr})}}{\sum_{i=1}^{m} K_D(z, a_{0r})\overline{U(a_{0r})}} = \frac{K_G(h(z), b_j)}{K_G(h(z), b_0)}$$

and the fact that if $z_0 \in \overline{\partial D} \setminus \{Jh(z) = 0\}$ then $Jh(z)$ is bounded away from zero on some neighbourhood of z_0 in $\bar{D}$.

Remark. In the case where D and G are pseudoconvex bounded domains with C^∞-boundary, S. Bell in [2] got more precise description of the boundary behaviour of proper holomorphic mappings. Nevertheless, his methods, based on the transformation rule for Bergman projection and estimates in the negative Sobolev norms, cannot be used if the boundaries of D and G are *not* C^∞-smooth.

Note that there exist at least three classes of domains with boundaries not C^∞-smooth but satisfying conditions (A) and (B) (see [11]):

1) complete circular domains such that $\lambda D \subset D$, $\lambda < 1$,
2) Cartesian products of domains with properties (A) and (B),
3) strictly pseudoconvex domains with boundary of class C^m, $m \geqslant 6$.

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