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On the asymptotic behaviour of the solutions of an n-th order difference equation

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Abstract. In this paper asymptotic formulas for the solution of certain classes of difference equations of n-th order are presented.

In this note we consider the non-linear difference equation of the form

$$\Delta^m y_n + F(n, y_n, y_{n+1}, \ldots, y_{n+m-1}) = b_n, \tag{1}$$

where $F\colon N \times \boldsymbol{R}^m \to \boldsymbol{R}$ is continuous on $\boldsymbol{R}^m$ for each fixed $n \in N$, $b\colon N \to \boldsymbol{R}$, $N = \{n_0, n_0+1, \ldots\}$, where $n_0 > 0$ is a given integer. The results presented here are generalizations of those contained in [4]. Similar problem for differential equations was investigated by Hallam [2] and Marušiak [3].

Here $y_n = y(n)$, $b_n = b(n)$; by Δy_n we denote the difference $\Delta y_n = y_{n+1} - y_n$ and we write $\Delta^k y_n = \Delta(\Delta^{k-1} y_n) = \Delta^{k-1} y_{n+1} - \Delta^{k-1} y_n$, $k = 1, 2, \ldots, m$, $\Delta^0 y_n = y_n$. Throughout we assume that the function F satisfies the inequality

$$|F(n, z_1, z_2, \ldots, z_m)| \leqslant B(n, |z_1|, |z_2|, \ldots, |z_m|), \tag{2}$$

where $B(n, z_1, z_2, \ldots, z_m)$ is continuous on $\boldsymbol{R}^m$ for each fixed $n \in N$ and such that for $n \in N$ and $z_k \geqslant 0$ $(k = 1, \ldots, m)$ we have

(i) $\quad 0 \leqslant B(n, z_1, z_2, \ldots, z_m) \leqslant B(n, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_m) \quad$ for $z_k \leqslant \bar{z}_k$, $k = 1, \ldots, m$,

(ii) $\quad B(n, a_n z_1, a_n z_2, \ldots, a_n z_m) \leqslant A(a_n) B(n, z_1, z_2, \ldots, z_m) \quad$ for $a_n \geqslant \varepsilon > 0$,

where $A\colon \langle \varepsilon, \infty) \to \boldsymbol{R}_+$ is non-decreasing and $\displaystyle\int_{0<\varepsilon}^{\infty} \frac{ds}{A(s)} = \infty$.

We first recall some useful lemmas.

LEMMA 1. *Let* $\{F_n\}$, $\{Q_n\}$ *are non-negative sequences,* $\{Q_n\}$ *is non-decreasing,* $Q_n > 0$, $n \in N$, $\lim_{n\to\infty} Q_n = \infty$ *and* $\sum_{n_0}^{\infty} \frac{F_n}{Q_n} < \infty$. *If there exists a se-*

quence $\{\beta_n\}$, $0 < \beta_n \leqslant n$ *such that* $\lim_{n\to\infty} \beta_n = \infty$ *and* $\lim_{n\to\infty} Q_{[\beta_n]}(Q_n)^{-1} = 0$, *then*

$$\lim_{n\to\infty} \frac{1}{Q_n} \sum_{k=n_0}^{n} F_k = 0.$$

Here the symbol $[\beta_n]$ denotes the greatest integer not exceeding β_n. For the proof see [4].

LEMMA 2. *Let* $v_n \geqslant 0$ *be a sequence such that, for some* $n \geqslant n_0$, $\Delta v_n > 0$ *and* $\lim_{n\to\infty} v_n = \infty$. *If there exists* $\lim_{n\to\infty} \frac{\Delta u_n}{\Delta v_n} = L$, *then* $\lim_{n\to\infty} \frac{u_n}{v_n} = L$.

For the proof see [1].

The following lemma is the discrete version of Gronwall's lemma.

LEMMA 3. *Let the function B satisfy* (i), (ii) *for* $m = k$ *and suppose that the non-negative sequences* $\{u_n\}$, $\{v_n\}$ *satisfy the following conditions*:

(a) $u_{n+k} \leqslant v_{n+k}[c + \sum_{j=n_0}^{n} B(j, u_j, \ldots, u_{j+k-1})]$, $0 < c = \text{const}$,

(b) $\sum_{j=n_0}^{\infty} B(j, v_j, \ldots, v_{j+k-1}) < \infty$, *where k is a given natural number.*

Then there exists a constant $M > 0$ *such that* $u_n \leqslant M v_n$ *for* $n \geqslant n_0 + k$.

Proof. Denote

$$d_n = c + \sum_{j=n_0}^{n} B(j, u_j, \ldots, u_{j+k-1}), \qquad n \in N.$$

Then (a) can be restated as follows:

$$u_{n+k} \leqslant v_{n+k} d_n, \qquad n \in N.$$

From the conditions imposed on B we obtain

$$\begin{aligned}\Delta d_n = B(n+1, u_{n+1}, \ldots, u_{n+k}) &\leqslant B(n+1, v_{n+1}d_{n+1-k}, \ldots, v_{n+k}d_n)\\ &\leqslant B(n+1, v_{n+1}d_n, \ldots, v_{n+k}d_n) \leqslant A(d_n) B(n+1, v_{n+1}, \ldots, v_{n+k}),\\ &\qquad\qquad n \geqslant n_0 + k - 1.\end{aligned}$$

Dividing the last inequality by $A(d_n)$ and using the monotonicity property of the integral operator we have

$$\int_{d_n}^{d_{n+1}} \frac{ds}{A(s)} \leqslant \frac{\Delta d_n}{A(d_n)} \leqslant B(n+1, v_{n+1}, \ldots, v_{n+k}).$$

Summing the last inequality from $n = n_0 + k - 1$ to $n = n - 1$ we obtain

$$\int_{d_{n_0+k-1}}^{d_n} \frac{ds}{A(s)} \leqslant \sum_{j=n_0+k-1}^{n-1} B(j+1, v_{j+1}, \ldots, v_{j+k}) = \sum_{j=n_0+k}^{n} B(j, v_j, \ldots, v_{j+k-1}).$$

Define $G(z) = \int\limits_{\varepsilon}^{z} \frac{ds}{A(s)}$ $(0 < \varepsilon \leqslant d_{n_0+k-1}$, ε is any constant). Hence

$$\int\limits_{d_{n_0+k-1}}^{d_n} \frac{ds}{A(s)} = G(d_n) - G(d_{n_0+k-1}).$$

Therefore

$$d_n \leqslant G^{-1}\big(G(d_{n_0+k-1}) + \sum_{j=n_0+k}^{n} B(j, v_j, \ldots, v_{j+k-1})\big), \qquad n \geqslant n_0 + k.$$

Since G^{-1}, as well as G, is strictly increasing, the above inequality together with condition (b) and

$$d_{n_0+k-1} = c + \sum_{j=n_0}^{n_0+k-1} B(j, u_j, \ldots, u_{j+k-1}) \leqslant K$$

give for $n \geqslant n_0 + k - 1$

$$d_n \leqslant G^{-1}\big(G(K) + \sum_{j=n_0+k}^{n} B(j, v_j, \ldots, v_{j+k-1})\big) \leqslant M.$$

Applying this estimate for d_n in (a), we see that the inequality

$$u_{n+k} \leqslant M v_{n+k}$$

is satisfied for $n \geqslant n_0 + k$.

Since u_n, v_n are finite for $n \in N$, we can find a constant M such that the last inequality is also valid for $n = n_0 - k, \ldots, n_0, \ldots, n_0 + k - 1$ and the proof of the lemma is complete.

THEOREM 1. *Let*

$$Q\colon N \to (0, \infty), \quad \Delta Q\colon N \to \langle 0, \infty), \quad \lim_{n\to\infty} Q_n = \infty \tag{3}$$

and

$$\lim_{n\to\infty} \frac{Q_{[\beta_n]}}{Q_n} = 0 \quad \textit{for some function } \beta\colon N \to (0, \infty)$$

such that $\beta_n \leqslant n$ *and* $\lim_{n\to\infty} \beta_n = \infty$.

If

$$\lim_{n\to\infty} \frac{b_n}{\Delta Q_n} = L \neq 0, \qquad L = \text{const}, \tag{4}$$

$$\sum_{j=n_0+m}^{\infty} \frac{1}{Q_j} B\big(j, \prod_{k=1}^{m-1} (j+k-m-n_0) Q_j, \ldots, \prod_{k=1}^{m-1} (j+k-1-n_0) Q_{j+m-1}\big) < \infty, \tag{5}$$

then every solution y of (1) *has the property*

$$\lim_{n\to\infty} \frac{\Delta^{m-1} y_n}{Q_n} = L. \tag{6}$$

Proof. Summing (1) m times from n_0 to n, we obtain the following equality:

$$y_{n+m} = \sum_{j=0}^{m-1} \binom{n-n_0+j}{j} \Delta^j y_{n_0+m-j-1} + \sum_{j=n_0}^{n} \binom{n-j+m-1}{m-1} b_j - \tag{7}$$

$$- \sum_{j=n_0}^{n} \binom{n+m-j-1}{m-1} F(j, y_j, \ldots, y_{j+m-1}) \quad \text{for } n \in N.$$

Dividing (7) by $\prod_{k=1}^{m-1} (n+k-n_0) Q_{n+m}$ we get

$$\frac{y_{n+m}}{\prod_{k=1}^{m-1} (n+k-n_0) Q_{n+m}}$$

$$= \frac{1}{Q_{n+m}} \left\{ \sum_{j=0}^{m-1} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+j-n_0)!}{j!(n-n_0)!} \Delta^j y_{n_0+m-j-1} \right\} +$$

$$+ \frac{1}{Q_{n+m}} \left\{ \sum_{j=n_0}^{n} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+m-j-1)!}{(m-1)!(n-j)!} b_j \right\} -$$

$$- \frac{1}{Q_{n+m}} \left\{ \sum_{j=n_0}^{n} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+m-j-1)!}{(m-1)!(n-j)!} F(j, y_j, \ldots, y_{j+m-1}) \right\}.$$

Hence, for $n \in N$,

$$\frac{|y_{n+m}|}{\prod_{k=1}^{m-1} (n+m-n_0) Q_{n+m}}$$

$$\leqslant \frac{1}{Q_{n+m}} \left\{ \sum_{j=0}^{m-1} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+j-n_0)!}{j!(n-n_0)!} |\Delta^j y_{m+n_0-j-1}| \right\} +$$

$$+ \frac{1}{Q_{n+m}} \left\{ \sum_{j=n_0}^{n} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+m-j-1)!}{(m-1)!(n-j)!} |b_j| \right\} +$$

$$+ \frac{1}{Q_{n+m}} \left\{ \sum_{j=n_0}^{n} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+m-j-1)!}{(m-1)!(n-j)!} |F(j, y_j, \ldots, y_{j+m-1})| \right\}.$$

We now show that the first term of the above sum is bounded. Indeed, since Q is non-decreasing and positive, we have

$$(Q_n)^{-1} \leqslant A - \text{const}, \qquad n \in N.$$

One can readily see that for $n \in N$, $0 \leqslant j \leqslant m-1$,

$$\prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+j-n_0)!}{j!(n-n_0)!}$$

$$\leqslant \frac{1}{j!} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \cdot (n-n_0+1) \cdot \ldots \cdot (n-n_0+j) \cdot \ldots \cdot (n-n_0+m-1) = \frac{1}{j!}.$$

Therefore, it follows by the finiteness of $\Delta^j y_{n_0+m-j-1}$, $j = 0,\ldots,m-1$ that

$$\frac{1}{Q_{n+m}} \left\{ \sum_{j=0}^{m-1} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \binom{n-n_0+j}{j} |\Delta^j y_{n_0+m-j-1}| \right\}$$

$$\leqslant A \left\{ \sum_{j=0}^{m-1} \frac{1}{j!} |\Delta^j y_{n_0+m-j-1}| \right\} = C_1.$$

Now, we estimate the second term. Observe that for $n_0 \leqslant j \leqslant n$

$$\prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+m-j-1)!}{(n-j)!} \leqslant 1.$$

Hence

$$\frac{1}{Q_{n+m}} \sum_{j=n_0}^{n} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+m-j-1)!}{(n-j)!} \frac{1}{(m-1)!} |b_j|$$

$$\leqslant \frac{1}{(m-1)!} \frac{1}{Q_{n+m}} \sum_{j=n_0}^{n} |b_j| \leqslant \frac{1}{(m-1)!} \frac{1}{Q_{n+1}} \sum_{j=n_0}^{n} |b_j|.$$

From Lemma 2 it follows that

$$\lim_{n\to\infty} \frac{\sum_{j=n_0}^{n} |b_j|}{Q_{n+1}} = \lim_{n\to\infty} \frac{|b_{n+1}|}{\Delta Q_{n+1}},$$

and by (4) we obtain the boundedness of the second term. Applying condition (2) we have

$$\frac{|y_{n+m}|}{\prod_{k=1}^{m-1} (n+k-n_0) Q_{n+m}}$$

$$\leqslant C + \frac{1}{Q_{n+m}} \left\{ \sum_{j=n_0}^{n} \prod_{k=1}^{m-1} \frac{(n+k-j)}{(n+k-n_0)(m-1)!} \frac{1}{(m-1)!} |F(j, y_j, \ldots, y_{j+m-1})| \right\}$$

$$\leqslant C + \frac{1}{Q_{n+m}} \left\{ \sum_{j=n_0}^{n} \frac{1}{(m-1)!} B(j, |y_j|, \ldots, |y_{j+m-1}|) \right\}$$

$$\leqslant C+\sum_{j=n_0}^{n}\frac{1}{(m-1)!}\frac{1}{Q_{j+m}}B(j,|y_j|,\ldots,|y_{j+m-1}|)$$
$$\leqslant C+\sum_{j=n_0}^{n}\frac{1}{(m-1)!}\frac{1}{Q_j}B(j,|y_j|,\ldots,|y_{j+m-1}|).$$

From Lemma 3 it follows that for $n\in N$

$$|y_{n+m}|\leqslant M\prod_{k=1}^{m-1}(n+k-n_0)Q_{n+m}. \tag{8}$$

Summing (1) from n_0 to n and dividing by Q_{n+1} we obtain

$$\frac{\Delta^{m-1}y_{n+1}}{Q_{n+1}}=\frac{1}{Q_{n+1}}\{\Delta^{m-1}y_{n_0}+\sum_{j=n_0}^{n}b_j-\sum_{j=n_0}^{n}F(j,y_j,\ldots,y_{j+m-1})\},\qquad n\in N. \tag{9}$$

It is obvious that $\lim_{n\to\infty}\frac{\Delta^{m-1}y_{n_0}}{Q_{n+1}}=0$. By Lemma 2 we have

$$\lim_{n\to\infty}\frac{\sum_{j=n_0}^{n}b_j}{Q_{n+1}}=L. \tag{10}$$

We show that the third term on the right-hand side of (9) approaches zero as $n\to\infty$. Using estimations (8) we obtain for $n\in N$

$$\frac{1}{Q_{n+m}}\sum_{j=n_0}^{n}|F(j,y_j,\ldots,y_{j+m-1})|$$
$$\leqslant\frac{1}{Q_n}\sum_{j=n_0}^{n_0+m-1}|F(j,y_j,\ldots,y_{j+m-1})|+\frac{1}{Q_n}\sum_{j=n_0+m}^{n}B(j,|y_j|,\ldots,|y_{j+m-1}|)$$
$$\leqslant\frac{1}{Q_n}\sum_{j=n_0}^{n_0+m-1}|F(j,y_j,\ldots,y_{j+m-1})|+$$
$$+\frac{1}{Q_n}\sum_{j=n_0+m}^{n}B\Big(j,M\prod_{k=1}^{m-1}(j-m+k-n_0)Q_j,\ldots,M\prod_{k=1}^{m-1}(j+k-1-n_0)Q_{j+m-1}\Big)$$
$$\leqslant\frac{1}{Q_n}\sum_{j=n_0}^{n_0+m-1}|F(j,y_j,\ldots,y_{j+m-1})|+$$
$$+\frac{A(M)}{Q_n}\sum_{j=n_0+m}^{n}B\Big(j,\prod_{k=1}^{m-1}(j+k-m-n_0)Q_j,\ldots,\prod_{k=1}^{m-1}(j+k-1-n_0)Q_{j+m-1}\Big).$$

The first term approaches zero, since the sum is finite and $\lim_{n\to\infty}Q_n=\infty$. So does the second one, by (5) and Lemma 1.

This together with (10) completes the proof.

Remark. Observe that

$$\prod_{k=1}^{m-1}(j+k-m-n_0)\leqslant (j-1-n_0)^{m-1}\leqslant j^{m-1},$$

and so we can use the following stronger form of assumption (5):

$$\sum_{j=n_0}^{\infty}\frac{1}{Q_j}B\big(j,j^{m-1}Q_j,\ldots,(j+m-1)^{m-1}Q_{j+m-1}\big)<\infty.$$

Let us specify certain types of sequences for which assumption (3) holds.

Corollary. *If, under the assumptions of Theorem* 1, $Q_n = n^\alpha v_n$, *where* $\alpha > 0$,

$$\lim_{n\to\infty} n\frac{\Delta v_{n-1}}{v_n} = c\ -\ \text{const},\qquad c\neq -\alpha-k,\ k=1,2,\ldots,m,$$

then

$$\lim_{n\to\infty}\frac{\Delta^{m-k}y_n}{n^{\alpha+k-1}v_{n-k+1}} = \frac{L}{\prod_{i=1}^{k-1}(\alpha+i+c)},\qquad k=1,2,\ldots,m. \tag{11}$$

Proof. For $k = 1$ we obtain the thesis of the theorem. Let (11) be fulfilled for $k = l$, $1 \leqslant l \leqslant m-1$. We prove that it is also true for $l+1$. Observe that by the assumption on v we have

$$\lim_{n\to\infty}\frac{\Delta v_n}{v_{n+1}} = 0.$$

Hence

$$\lim_{n\to\infty} n\frac{\Delta v_{n-l}}{v_{n-l+1}} = \lim_{n\to\infty}\left\{(n-l)\frac{\Delta v_{n-l}}{v_{n-l+1}}+l\frac{\Delta v_{n-l}}{v_{n-l+1}}\right\} = c. \tag{12}$$

We now show that the sequence $a_n = n^{\alpha+l}v_{n-l}$ fulfils the conditions of Lemma 2.

By (3), $v_{n-l} > 0$, and since n is a natural number, $a_n > 0$. By the assumption of Stolz' Theorem there exists a number $N_Q \geqslant n_0$ such that for every $n \geqslant N_Q$, $\Delta Q_n > 0$. To prove the monotonicity of a_n, let us rewrite a_n in the following form:

$$a_n = \frac{n^{\alpha+l}}{(n-l)^\alpha}(n-l)^\alpha v_{n-l} = \frac{n^{\alpha+l}}{(n-l)^\alpha}Q_{n-l}.$$

The function $f(x) = x^{\alpha+l}/(x-l)^\alpha$ is strictly increasing for $x > l \geqslant 1$, and $\lim_{x\to\infty} f(x) = \infty$. The sequence a_n, $n > \max[N_Q, l]$, is strictly increasing and tends to infinity, hence it can be used in Lemma 2. Assume that (11) holds for $k = l$, i.e.,

$$\lim_{n\to\infty} \frac{\Delta^{m-l} y_n}{n^{\alpha+l-1} v_{n-l+1}} = \frac{L}{\prod_{j=1}^{l-1} (\alpha+j+c)}.$$

Furthermore $\lim_{n\to\infty} \frac{\Delta n^{\alpha+l}}{n^{\alpha+l-1}} = \alpha+l$.

According to the above and (12), we obtain

$$\lim_{n\to\infty} \frac{\Delta^{m-(l+1)} y_n}{n^{\alpha+l} v_{n-l}} = \lim_{n\to\infty} \frac{\Delta^{m-l} y_n}{v_{n-l+1} \Delta n^{\alpha+l} + n^{\alpha+l} \Delta v_{n-l}}$$

$$= \lim_{n\to\infty} \frac{\dfrac{\Delta^{m-l} y_n}{n^{\alpha+l-1} v_{n-l+1}}}{\dfrac{\Delta n^{\alpha+l}}{n^{\alpha+l-1}} + n \dfrac{\Delta v_{n-l}}{v_{n-l+1}}} = \frac{\dfrac{L}{\prod_{j=1}^{l-1} (\alpha+j+c)}}{\alpha+l+c} = \frac{L}{\prod_{j=1}^{l} (\alpha+j+c)},$$

i.e., (11) holds for $k = l+1$, which completes the proof.

Let now $Q_n = e^{\alpha n} v_n$, where

$$\lim_{n\to\infty} \frac{\Delta v_{n+1}}{v_n} = c - \text{const}, \quad c \neq e^{\alpha} - 1;$$

then

$$\lim_{n\to\infty} \frac{\Delta^{m-k} y_n}{e^{\alpha n} v_{n-k+1}} = \frac{L}{(e^{\alpha}-1+c)^{k-1}}, \quad k = 1, 2, \ldots, m. \tag{13}$$

Proof. Analogously to the preceding example, we see that is (13) valid for $k = 1$, and assuming its validity for $k = l \leqslant m-1$, we prove that it is true for $k = l+1$. Using the same argument as before, we have

$$\lim_{n\to\infty} \frac{\Delta^{m-(l+1)} y_n}{e^{\alpha n} v_{n-l}} = \lim_{n\to\infty} \frac{\Delta^{m-l} y_n}{v_{n-l+1} \Delta e^{\alpha n} + e^{\alpha n} \Delta v_{n-l}}$$

$$= \lim_{n\to\infty} \frac{\dfrac{\Delta^{m-l} y_n}{v_{n-l+1} e^{\alpha n}}}{\dfrac{\Delta e^{\alpha n}}{e^{\alpha n}} + \dfrac{\Delta v_{n-l}}{v_{n-l+1}}} = \frac{\dfrac{L}{(e^{\alpha}-1+c)^{l-1}}}{e^{\alpha}-1+c} = \frac{L}{(e^{\alpha}-1+c)^{l}},$$

since $\lim_{n\to\infty} \frac{\Delta e^{\alpha n}}{e^{\alpha n}} = e^{\alpha}-1$ and the sequence

$$v_n = e^{\alpha n} v_{n-l} = e^{\alpha l} e^{\alpha(n-l)} v_{n-l} = e^{\alpha l} Q_{n-l}$$

fulfils conditions of Lemma 2.

THEOREM 2. *Suppose that the conditions*

$$\sum_{j=n_0}^{\infty} |b_j| < \infty, \tag{14}$$

$$\sum_{j=n_0}^{\infty} B\big(j, j^{m-1}, \ldots, (j+m-1)^{m-1}\big) < \infty \tag{15}$$

are satisfied. Then every solution y of (1) *has the property*

$$\lim_{n\to\infty} \frac{\Delta^{m-l} y_n}{\prod_{k=0}^{l-2} (n-k)} = \frac{L}{(l-1)!}, \quad m \geqslant l \geqslant 1, \ L = \text{const}. \tag{16}$$

Proof. By the same calculations as in the preceding theorem we obtain

$$\frac{|y_{n+m}|}{\prod_{k=1}^{m-1} (n+k-n_0)} \leqslant \sum_{j=0}^{m-1} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+j-n_0)!}{j!\,(n-n_0)!} |\Delta^j y_{n_0+m-j-1}| +$$

$$+ \sum_{j=n_0}^{n} \sum_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+m-j-1)!}{(m-1)!\,(n-j)!} |b_j| +$$

$$+ \sum_{j=n_0}^{n} \prod_{k=1}^{m-1} (n+k-n_0)^{-1} \frac{(n+m-j-1)!}{(m-1)!\,(n-j)!} |F(j, y_j, \ldots, y_{m+j-1})|$$

$$\leqslant C + \sum_{j=n_0}^{n} \frac{1}{(m-1)!} B(j, |y_j|, \ldots, |y_{j+m-1}|),$$

or, equivalently,

$$|y_{n+m}| \leqslant \prod_{k=1}^{m-1} (n+k-n_0) \left\{ C + \sum_{j=n_0}^{n} \frac{1}{(m-1)!} B(j, |y_j|, \ldots, |y_{j+m-1}|) \right\},$$

$$n \in N.$$

We may now apply Lemma 3, from which by (15) we get

$$|y_n| \leqslant C_1 \prod_{j=1}^{m-1} (n+j-m-n_0) \leqslant C_1 n^{m-1} \quad \text{for } n \geqslant m+n_0. \tag{17}$$

Summing from n_0 to n we obtain

$$\Delta^{m-1} y_{n+1} = \Delta^{m-1} y_{n_0} + \sum_{j=n_0}^{n} b_j - \sum_{j=n_0}^{n} F(j, y_j, \ldots, y_{j+m-1}). \tag{18}$$

By (14) we have

$$\lim_{n\to\infty} \sum_{j=n_0}^{n} b_j = L_1. \tag{19}$$

We show that there exists a finite limit

$$\lim_{n\to\infty} \sum_{j=n_0}^{n} F(j, y_j, \ldots, y_{j+m-1}).$$

Observe that by (17) and by the preliminary conditions on the functions F and B we have

$$(20)\quad \sum_{j=n_0}^{n} |F(j, y_j, \ldots, y_{j+m-1})| \leqslant \sum_{j=n_0}^{n_0+m-1} |F(j, y_j, \ldots, y_{j+m-1})| + $$
$$+ \sum_{j=n_0+m}^{n} B(j, |y_j|, \ldots, |y_{j+m-1}|)$$
$$\leqslant \sum_{j=n_0}^{n_0+m-1} |F(j, y_j, \ldots, y_{j+m-1})| +$$
$$+A(C_1) \sum_{j=n_0+m}^{n} B\Big(j, \prod_{k=1}^{m-1} (j+k-m-n_0), \ldots, \prod_{k=1}^{m-1} (j+k-1-n_0)\Big)$$
$$\leqslant L_2.$$

Therefore, combining (18), (19) and (20), we obtain $\lim_{n\to\infty} \Delta^{m-1} y_n = L$, i.e., (16) valid for $l = 1$.

An inductive argument will be used to show that (16) is valid for $1 < l \leqslant m$. We assume that the statement is true for $l \leqslant m-1$ and consider the statement with the integer $l+1$.

First observe that

$$\Delta \prod_{k=0}^{l-1} (n-k) = \prod_{k=0}^{l-1} (n+1-k) - \prod_{k=0}^{l-1} (n-k)$$
$$= (n+1) \prod_{k=1}^{l-1} (n+1-k) - (n-l+1) \prod_{k=0}^{l-2} (n-k)$$
$$= (n+1) \prod_{k=0}^{l-2} (n-k) - (n-l+1) \prod_{k=0}^{l-2} (n-k) = l \prod_{k=0}^{l-2} (n-k),$$

and that the sequence $\prod_{k=0}^{l-1} (n-k)$ is strictly increasing for $n > l-1$ and $\lim_{n\to\infty} \prod_{k=0}^{l-1} (n-k) = \infty$. Hence, applying Lemma 2,

$$\lim_{n\to\infty} \frac{\Delta^{m-(l+1)} y_n}{\prod_{k=0}^{l-1} (n-k)} = \frac{1}{l} \lim_{n\to\infty} \frac{\Delta^{m-l} y_n}{\prod_{k=0}^{l-2} (n-k)} = \frac{1}{l} \frac{L}{(l-1)!} = \frac{L}{l!}$$

and the proof of the theorem is complete.

THEOREM 3. *Let*

$$\sum_{j=n_0}^{\infty} j^{m-1}|b_j| < \infty, \tag{21}$$

$$\sum_{j=n_0}^{\infty} j^{m-1}B\big(j, j^{m-1},\dots,(j+m-1)^{m-1}\big) < \infty. \tag{22}$$

Then every solution y of equation (1) *has the property*

$$\Delta^{m-k}y_n = \sum_{i=1}^{k-1}\Big(\sum_{j=1}^{i} a_j^{i,k}L_j\Big)n^{k-i} + L_k + o(1), \tag{23}$$

$$L_i = \text{const},\ i = 1,\dots,k-1,\ k = 1,\dots,m,$$

where $a_q^{p,k}$ is given by the recurrence formula:

$$a_q^{p,k} = \frac{1}{(k-p)!}\sum_{i=p}^{k-1}(i-p+1)!\sum_{j=q}^{p-1} a_q^{j,i}b_{i-p+1}^{i-j}, \qquad 1 \leqslant q < p < k,$$

$$a_p^{p,k} = \frac{1}{(k-p)!}, \qquad p \leqslant k,$$

$$b_{p-r}^{p} = \frac{(p-r-1)!\,A_{p-r}^{p}}{(p+1)!}, \qquad -1 \leqslant r < p,$$

where

$$A_{p+1}^{p} \equiv 1,$$

$$A_{p-r}^{p} = \begin{vmatrix} (2^1-1)\binom{p+1}{1} & 0 & 0 & \dots & \binom{p}{0} \\ (2^2-1)\binom{p+1}{2} & (2^1-1)\binom{p}{1} & 0 & \dots & \binom{p}{1} \\ \dots & \dots & \dots & \dots & \dots \\ (2^{r+2}-1)\binom{p+1}{r+2} & (2^{r+1}-1)\binom{p}{r+1} & (2^r-1)\binom{p-1}{r} & \dots & \binom{p}{r+1} \end{vmatrix},$$

$$0 \leqslant r < p,$$

$$b_0^p = b_0^p(n_0, k, p) = (n_0+k-1)^p - \sum_{l=1}^{p+1} b_l^p(n_0+k)^l.$$

Proof. Observe that (21) and (22) yield (14) and (15), respectively. Hence Theorem 2 implies that every solution of (1) has a finite limit of sequence of the $(n-1)$-th differences, i.e., $\lim_{n\to\infty}\Delta^{m-1}y_n = L_1$. By a similar argument as in the proof of Theorem 2 we see that there exists a number C such that for $n\in N$ we have $|y_n| \leqslant Cn^{m-1}$.

Summing equation (1) from s to n we obtain

$$\Delta^{m-1}y_{n+1} = \Delta^{m-1}y_s + \sum_{j=s}^{n} b_j - \sum_{j=s}^{n} F(j, y_j, \ldots, y_{j+m-1}),$$

and hence

$$\Delta^{m-1}y_s = \Delta^{m-1}y_{n+1} - \sum_{j=s}^{n} b_j + \sum_{j=s}^{n} F(j, y_j, \ldots, y_{j+m-1}).$$

Passing with m to infinity we have

$$\Delta^{m-1}y_s = L_1 - \sum_{j=s}^{\infty} b_j + \sum_{j=s}^{\infty} F(j, y_j, \ldots, y_{j+m-1}), \qquad n \in N. \tag{24}$$

The convergence of series (14) and (15) implies

$$\lim_{s\to\infty} \sum_{j=s}^{\infty} b_j = 0, \qquad \lim_{s\to\infty} \sum_{j=s}^{\infty} F(j, y_j, \ldots, y_{j+m-1}) = 0,$$

since

$$\begin{aligned}
\Big|\sum_{j=s}^{\infty} F(j, y_j, \ldots, y_{j+m-1})\Big| &\leqslant \sum_{j=s}^{\infty} |F(j, y_j, \ldots, y_{j+m-1})| \\
&\leqslant \sum_{j=s}^{\infty} B(j, |y_j|, \ldots, |y_{j+m-1}|) \\
&\leqslant \sum_{j=s}^{\infty} B\big(j, Cj^{m-1}, \ldots, C(j+m-1)^{m-1}\big) \\
&\leqslant A(C) \sum_{j=s}^{\infty} B\big(j, j^{m-1}, \ldots, (j+m-1)^{m-1}\big) < \infty,
\end{aligned}$$

and hence

$$\Delta^{m-1}y_n = L_1 + o(1).$$

We see that the solution has property (23) for $k = 1$.

Now we prove the following formula:

$$\begin{aligned}
\Delta^{m-k}y_n = \sum_{i=1}^{k-1} \Big(\sum_{j=1}^{i} a_j^{i,k} L_j\Big) n^{k-i} + L_k + (-1)^k \sum_{j=n}^{\infty} \binom{j-n+(k-1)}{k-1} b_j + \\
+(-1)^{k+1} \sum_{j=n}^{\infty} \binom{j-n+(k-1)}{k-1} F(j, y_j, \ldots, y_{j+m-1}),
\end{aligned} \tag{25}$$

where $n \geqslant n_0+k-1$, $k \geqslant 1$.

For $k = 1$ formula (25) reduces to (24). Assume that (25) holds for $k < m$; we prove that it is also true for $k+1$. Summing (25) from n_0+k-1 to n we obtain

$$(26)\quad \Delta^{m-(k+1)}y_{n+1}-\Delta^{m-(k+1)}y_{n_0+k-1} = \sum_{l=n_0+k-1}^{n}\Big(\sum_{i=1}^{k-1}\sum_{j=1}^{i} a_j^{i,k}L_j\Big)l^{k-i}+ + \sum_{l=n_0+k-1}^{n} L_k+(-1)^k\sum_{l=n_0+k-1}^{n}\sum_{j=l}^{\infty}\binom{j-l+k-1}{k-1}b_j+ +(-1)^{k+1}\sum_{l=n_0+k-1}^{n}\sum_{j=l}^{\infty}\binom{j-l+k-1}{k-1}F(j,y_j,\ldots,y_{j+m-1}),\quad n\geqslant n_0+k-1.$$

To begin with, we examine the first term of the above sum:

$$(27)\quad \sum_{l=n_0+k-1}^{n}\sum_{i=1}^{k-1}\Big(\sum_{j=1}^{i} a_j^{i,k}L_j\Big)l^{k-i}+\sum_{l=n_0+k-1}^{n} L_k,\quad n\geqslant n_0+k-1,\ k\geqslant 2.$$

At first we observe that

$$\sum_{l=n_0+k-1}^{n} l^p = \sum_{r=0}^{p+1} b_r^p(n+1)^r.$$

Let us write

$$d_i^k = \sum_{j=1}^{i} a_j^{i,k}L_j \quad \text{for } 1\leqslant i\leqslant k-1<n,$$
$$d_k^k = L_k \quad \text{for } 1\leqslant k\leqslant n.$$

Then we can rewrite (27) in the form

$$(28)\quad \sum_{l=n_0+k-1}^{n}\sum_{i=1}^{k-1} d_i^k l^{k-i}+d_k^k(n+1)-L_k(n_0+k-1)$$
$$=\sum_{i=1}^{k-1} d_i^k\sum_{l=n_0+k-1}^{n} l^{k-i}+d_k^k(n+1)-L_k(n_0+k-1)$$
$$=\sum_{i=1}^{k-1} d_i^k\Big(\sum_{r=0}^{k-i+1} b_r^{k-i}(n+1)^r\Big)+d_k^k(n+1)-L_k(n_0+k-1)$$
$$=\sum_{i=1}^{k-1} d_i^k\Big(\sum_{r=1}^{k-i+1} b_r^{k-i}(n+1)^r\Big)+d_k^k(n+1)+\sum_{i=1}^{k-1} d_i^k b_0^{k-i}-L_k(n_0+k-1)$$
$$=\sum_{i=1}^{k}\Big(\sum_{j=1}^{i} d_j^k b_{k+1-i}^{k-j}\Big)(n+1)^{k+1-i}+\sum_{i=1}^{k-1} d_i^k b_0^{k-1}-L_k(n_0+k-1).$$

Now we show that the following formula holds:

$$(29)\quad \sum_{j=1}^{i} d_j^k b_{k+1-i}^{k-j} = d_i^{k+1} \quad \text{for } i=1,\ldots,k,\ k=1,\ldots,n-2.$$

For $k = 1$, $i = 1$ we obtain

$$d_1^2 = \sum_{j=1}^{1} a_j^{1,2} L_j = a_1^{1,2} L_1 = L_1 = d_1^1 b_1^0 = \sum_{j=1}^{1} d_j^1 b_{2-i}^{1-j}.$$

Now we prove (29) for $i = k$. By definition we have

$$\begin{aligned}
d_k^{k+1} &= \sum_{j=1}^{k} a_j^{k,k+1} L_j \\
&= L_1 \sum_{j=1}^{k-1} a_1^{j,k} \frac{1}{(k-j+1)!} + L_2 \sum_{j=2}^{k-1} a_2^{j,k} \frac{1}{(k-j+1)!} + \\
&\qquad + \ldots + L_{k-1} \sum_{j=k-1}^{k-1} a_{k-1}^{j,k} \frac{1}{(k-j+1)!} + L_k a_k^{k,k+1} \\
&= \frac{1}{k!} \sum_{j=1}^{1} a_j^{1,k} L_j + \frac{1}{(k-1)!} \sum_{j=1}^{2} a_j^{2,k} L_j + \ldots + \frac{1}{2!} \sum_{j=1}^{k-1} a_j^{k-1,k} L_j + L_k \\
&= \sum_{j=1}^{k-1} d_j^k \frac{1}{(k-j+1)!} + d_k^k b_1^0 = \sum_{j=1}^{k} d_j^k b_1^{k-j}.
\end{aligned}$$

Let now $i \leqslant k-1$. Observe that (29) is equivalent to the following equality:

$$(30) \qquad (k+1-i)!\, d_i^{k+1} - (k-i)!\, d_i^k = (k+1-i)! \sum_{j=1}^{i-1} d_j^k b_{k+1-i}^{k-j}, \qquad i \geqslant 1.$$

Assume that (30) holds for some $k = s-1$ and $k < s-1$. We prove that this equality is true for $k = s$. By definitions we have

$$\begin{aligned}
d_i^{s+1} &= \sum_{j=1}^{i} a_j^{i,s+1} L_j = \frac{1}{(s+1-i)!} L_i + \sum_{j=1}^{i-1} a_j^{i,s+1} L_j \\
&= \frac{1}{(s+1-i)!} L_i + \sum_{j=1}^{i-1} \left[\frac{1}{(s+1-i)!} \sum_{p=i}^{s} (p-i+1)! \sum_{r=j}^{i-1} a_j^{r,p} b_{p-i+1}^{p-r} \right] L_j \\
&= \frac{1}{(s+1-i)!} L_i + \frac{1}{(s+1-i)!} \sum_{p=i}^{s} (p+1-i)! \sum_{j=1}^{i-1} \Big(\sum_{r=1}^{j} a_r^{j,p} L_r \Big) b_{p+1-i}^{p-j} \\
&= \frac{1}{(s+1-i)!} d_i^i + \frac{1}{(s+1-i)!} \sum_{p=i}^{s} (p+1-i)! \sum_{j=1}^{i-1} d_j^p b_{p+1-i}^{p-j} \\
&= \frac{1}{(s+1-i)!} d_i^i + \sum_{j=1}^{i-1} d_j^s b_{s+1-i}^{s-j} + \frac{1}{(s+1-i)!} \sum_{p=i}^{s-1} (p+1-i)! \sum_{j=1}^{i-1} d_j^p b_{p+1-i}^{p-j} \\
&= \frac{1}{(s+1-i)!} d_i^i + \sum_{j=1}^{i-1} d_j^s b_{s+1-i}^{s-j} + \frac{1}{(s+1-i)!} \sum_{p=i}^{s-1} \Delta (p-i)!\, d_i^p \\
&= \frac{1}{(s+1-i)!} d_i^i + \sum_{j=1}^{i-1} d_j^s b_{s+1-i}^{s-j} + \frac{(s-i)!\, d_i^s}{(s+1-i)!} - \frac{d_i^i}{(s+1-i)!} \\
&= \frac{1}{(s+1-i)} d_i^s + \sum_{j=1}^{i-1} d_j^s b_{s+1-i}^{s-j} = \sum_{j=1}^{i} d_j^s b_{s+1-i}^{s-j}.
\end{aligned}$$

We shall prove that the series occurring in formula (26) as the third term on the right-hand side converges. It easy to see that

$$\binom{j-n_0+1}{k} \leqslant \frac{k+1}{k!} \max_{i=0,\dots,k} \binom{k}{i} j^k;$$

consequently,

$$\left| \sum_{j=n_0+k-1}^{\infty} \binom{j-n_0+1}{k} b_j \right| \leqslant \sum_{j=n_0+k-1}^{\infty} \binom{j-n_0+1}{k} |b_j|$$
$$\leqslant \frac{k+1}{k!} \max_{i=0,\dots,k} \binom{k}{i} \sum_{j=n_0+k-1}^{\infty} j^k |b_j| < \infty.$$

Analogously, for $j \geqslant n$,

$$\binom{k+j-n}{k} \leqslant \frac{1}{k!}(j+k)^k = \frac{k+1}{k!} \max_{i=0,\dots,k} \binom{k}{i} k^{k-i} j^k,$$

from which, applying (21), we see that the sum $\sum_{j=n}^{\infty} \binom{k+j-n}{k} b_j$ tends to zero, as the remainder of a convergent series. To conclude this part of the proof it suffices to observe that

$$\sum_{j=n_0+k-1}^{n} \sum_{i=j}^{\infty} \binom{k-1+i-j}{k-1} b_i$$
$$= \sum_{i=n_0+k-1}^{\infty} \left[\sum_{r=0}^{i-n_0+1-k} \binom{k-1+r}{k-1} \right] b_j - \sum_{i=n+1}^{\infty} \left[\sum_{r=0}^{i-n-1} \binom{k-1+r}{k-1} \right] b_j$$
$$= \sum_{i=n_0+k-1}^{\infty} \binom{k+i-n_0-k+1}{k} b_j - \sum_{i=n+1}^{\infty} \binom{k+i-(n+1)}{k} b_j.$$

Similarly, we get

$$\sum_{p=n_0+k-1}^{n} \sum_{j=p}^{\infty} \binom{k-1+j-p}{k-1} F(j, y_j, \dots, y_{j+m-1})$$
$$= \sum_{j=n_0+k-1}^{\infty} \binom{j-n_0+1}{k} F(j, y_j, \dots, y_{j+m-1}) - \sum_{j=n+1}^{\infty} \binom{k+j-(n+1)}{k} F(j, y_j, \dots, y_{j+m-1}),$$

and by (2), according to the previous remark,

$$\left| \sum_{j=n_0+k-1}^{\infty} \binom{j-n_0+1}{k} F(j, y_j, \dots, y_{j+m-1}) \right|$$
$$\leqslant \frac{k+1}{k!} \max_{i=0,\dots,k} \binom{k}{i} \sum_{j=n_0+k-1}^{\infty} j^k B(j, |y_j|, \dots, |y_{j+m-1}|)$$
$$\leqslant A(C) \frac{k+1}{k!} \max_{i=0,\dots,k} \binom{k}{i} \sum_{j=n_0+k-1}^{\infty} j^k B(j, j^{m-1}, \dots, (j+m-1)^{m-1}) < \infty.$$

We remark that the sum of the series $\sum_{j=n}^{\infty}\binom{k+j-n}{k}F(j, y_j,\ldots,y_{j+m-1})$ tends to zero with $n \to \infty$.

We can write (26) in the form

$$\begin{aligned}
\Delta^{m-(k+1)}y_{n+1} = {} & \sum_{i=1}^{k}\Big(\sum_{j=1}^{i} a_j^{i,k+1}L_j\Big)(n+1)^{k+1-i}+\Delta^{m-(k+1)}y_{n_0+k-1}+\\
&+\sum_{i=1}^{k-1}\Big(\sum_{j=1}^{i} a_j^{i,k}L_j\Big)b_0^{k-i}-L_k(n_0+k-1)+(-1)^k\sum_{j=n_0+k-1}^{\infty}\binom{j-n_0+1}{k}b_j+\\
&+(-1)^{k+1}\sum_{j=n_0+k-1}^{\infty}\binom{j-n_0+1}{k}F(j, y_j,\ldots,y_{j+m-1})+\\
&+(-1)^{k+1}\sum_{j=n+1}^{\infty}\binom{k+j-(n+1)}{k}b_j+\\
&+(-1)^{k+2}\sum_{j=n+1}^{\infty}\binom{k+j-(n+1)}{k}F(j, y_j,\ldots,y_{j+m-1}), \qquad n \geqslant n_0+k-1.
\end{aligned}$$

If we put

$$\begin{aligned}
L_{k+1} = {} & \sum_{i=1}^{k-1}\Big(\sum_{j=1}^{i} a_j^{i,k}L_j\Big)b_0^{k-i}-L_k(n_0+k-1)+\Delta^{m-(k+1)}y_{n_0+k-1}+\\
&+(-1)^k\sum_{j=n_0+k-1}^{\infty}\binom{j-n_0+1}{k}b_j+\\
&+(-1)^{k+1}\sum_{j=n_0+k-1}^{\infty}\binom{j-n_0+1}{k}F(j, y_j,\ldots,y_{j+m-1}),
\end{aligned}$$

and replace $n+1$ by n, we obtain

$$\begin{aligned}
\Delta^{m-(k+1)}y_n = {} & \sum_{i=1}^{k}\Big(\sum_{j=1}^{i} a_j^{i,k+1}L_j\Big)n^{k+1-i}+L_{k+1}+(-1)^{k+1}\sum_{j=n}^{\infty}\binom{k+j-n}{k}b_j+\\
&+(-1)^{k+2}\sum_{j=n}^{\infty}\binom{k+j-n}{k}F(j, y_j,\ldots,y_{j+m-1}), \qquad n \geqslant n_0+k.
\end{aligned}$$

Hence, formula (25) is valid for $k+1$, and the inductive proof of this formula is finished. Since the second and the third term of (25) tend to zero, we obtain (23).

As an example illustrating the presented theorems we consider the difference equation of the form

$$\Delta^m y_n + a_n y_n^p = b_n, \qquad n \in N.$$

The theorems cannot be applied in the cases of $p < 0$ (there is no function B

for which (i) holds), and $p > 1$. In the second case we have $\int\limits_{\varepsilon}^{\infty} ds/A(s)$ $= \int\limits_{\varepsilon}^{\infty} ds/s^p < \infty$ and therefore condition (ii) does not hold.

The same remark concerns the equation $\Delta^m y_n + \sum\limits_{j=0}^{m-1} a_n^j y_n^{p_j} = b_n$, under the same condition on at least one of the p_j.

Let now $p_j \in (0, 1\rangle$, $j = 0, 1, \ldots, m-1$, and let there exist non-negative constants a^k and a sequence $\{c_n\}$ such that $|a_n^k| \leqslant a^k c_n$ for $n \in N$. Take $Q_n = n$ for all $n \in N$. If $\lim\limits_{n\to\infty} b_n = L \neq 0$ (condition (4)), c_n are such that $\sum\limits_{j=n_0+m}^{\infty} \frac{1}{j} c_j \sum\limits_{k=0}^{m-1} a^k (j+k)^{p_k m} < \infty$ (condition (5) in the form given in remark), then $\lim\limits_{n\to\infty} \frac{\Delta^{m-1} y_n}{n} = L$ (from (6) in Theorem 1) and $\lim\limits_{n\to\infty} \frac{\Delta^{m-k} y_n}{n^k} = Lk!$, $k = 1, \ldots, m$ (from (11) in the corollary). Let now b_n be such that $\sum\limits_{j=n_0}^{\infty} |b_j| < \infty$ (condition (14) in Theorem 2) and c_n such that $\sum\limits_{j=n_0}^{\infty} c_j \sum\limits_{k=0}^{m-1} a^k (j+k)^{p_k(m-1)} < \infty$ (condition (15)); then $\lim\limits_{n\to\infty} \frac{\Delta^{m-l} y_n}{\prod\limits_{k=0}^{l-2} (n-k)} = \frac{L}{(l-1)!}$, $m \geqslant l \geqslant 1$, $L = \text{const}$ (Theorem 2).

Let b_n be such that $\sum\limits_{j=n_0}^{\infty} j^{m-1} |b_j| < \infty$ (condition (21)) and c_n such that $\sum\limits_{j=n_0}^{\infty} j^{m-1} c_j \sum\limits_{k=0}^{m-1} a^k (j+k)^{p_k(m-1)} < \infty$ (condition (22)); then $\Delta^{m-k} y_n$ $= \sum\limits_{i=1}^{k-1} (\sum\limits_{j=1}^{i} a_j^{i,k} L_j) n^{k-i} + L_k + o(1)$, $L_i = \text{const}$, $i = 1, \ldots, k-1$; $k = 1, \ldots, m$; $a_j^{i,k}$ are defined in Theorem 3 (this follows from Theorem 3).

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