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ON ORTHOGONAL DECOMPOSITION OF TWO-DIMENSIONAL VECTOR FIELDS

Construction of difference approximations (of arbitrary order) of the operators grad and div is given for the two-dimensional case. Moreover, a theorem is proved, providing a difference analogue of Weyl's theorem on orthogonal decomposition of two-dimensional periodic vector fields into two subspaces of divergence-free vectors and vectors having the form of a gradient.

1. Introduction. Let $u: R^n \to R^n$ be a sufficiently regular vector field. According to Weyl's theorem [8], if certain additional conditions are fulfilled, this field can be uniquely represented as a sum of two mutually orthogonal components $u = u_d + u_g$, the first one being a divergence-free field, i.e. $\operatorname{div} u_d = 0$, and the second taking a gradient form $u_g = \operatorname{grad}\varphi$, where $\varphi: R^n \to R$.

In this paper we are concerned with two-dimensional periodic vector fields $u: R^2 \to R^2$ with period one in both variables. Introducing the square grid $R_h \times R_h$ ($0 < h$ denotes the mesh width with respect to each of the variables) we define the discrete periodic fields $u_h: R_h^2 \to R^2$ for which we prove the difference version of Weyl's theorem.

Weyl's theorem plays a fundamental role in solving the Navier-Stokes equations [4], similarly as the difference version of this theorem formulated below does in the construction of difference schemes for these equations (cf., e.g., [1]).

To this aim, for a given natural m we construct in Section 2 (on the basis of interpolation methods) the (fundamental in further constructions) difference operators DG_x^m, DD_x^m, DG_y^m, DD_y^m approximating with the order m the differential operators $\partial/\partial x$ and $\partial/\partial y$ on the grid R_h^2 (see Theorem 2.1). Using the operators DG_x^m, DD_x^m, DG_y^m, DD_y^m we introduce (see Definition 2.7) the operators Grad_m and Div_m being difference analogues of the differential operators grad and div.

In Section 3 we examine the properties of the constructed operators Grad_m and Div_m. We derive also conditions for the choice of the parameter

h ($h = 1/N$) of the grid R_h^2 in order to satisfy the implication

$$\operatorname{Grad}_m f_h = 0 \Rightarrow f_h = \text{const}$$

for an arbitrary discrete periodic function $f_h: R_h^2 \to R$ (see the proof of Theorem 3.1).

In Section 4 we deal with divergence-free discrete vector fields, i.e. with fields such that $\operatorname{Div}_m u_h = 0$. We construct divergence-free fields in a certain special form, we investigate their properties, and we prove that they form the basis of the subspace D_h of all divergence-free fields of the space U_h of discrete periodic fields.

We determine also the basis of the subspace G_h of all periodic vector fields of the form $u_h = \operatorname{Grad}_m \varphi_h$, where $\varphi_h: R_h^2 \to R$.

We prove that both subspaces G_h and D_h are mutually orthogonal and that the space U_h can be uniquely expressed as a direct sum of the spaces G_h and D_h: $U_h = G_h + D_h$.

The present paper is an extension and generalization of the results obtained by Krzywicki [3] who proved a similar theorem for flat periodic fields, restricting himself, however, to first order approximation.

Because of complexity of calculations needed in the proofs we divided the proofs into appropriately labelled parts which, we hope, will make the paper easier to read.

In Section 2, where basic expressions are defined and well-known facts are used, we decide to distinguish neither particular theorems nor conclusions, apart from Theorem 2.1 which is fundamental for this paper and to which we refer frequently in further parts of the paper.

The sets of real and integer numbers are denoted by R and Z, respectively.

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2. Construction of difference operators Grad_m **and** Div_m. Let us begin with some facts connected with the interpolation of a real function of a single variable $f: R \to R$ and with the approximation of its derivative.

Definition 2.1. The set $R_h = \{x_i \in R: x_i = ih,\ i \in Z\}$, where $h = 1/N$ for a fixed natural number N, is called a *grid of equidistant nodes* x_i.

Put $f_i = f(x_i)$ for $x_i \in R_h$.

Let us construct, for the function f the Lagrange interpolating polynomial based on the values of the function f at the nodes $\{x_0, x_1, \ldots, x_n\}$, where n is a fixed natural number. If $f \in C^{n+1}(R)$, then for $x \in R$ the equality

$$f(x) = \sum_{i=0}^{n} \frac{(x-x_0)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} f_i + r(x) \tag{2.1}$$

holds, where the remainder $r(x)$ is given by

$$r(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{i=0}^{n}(x-x_i),$$

ξ_x being an intermediate point between $\min\{x, x_0\}$ and $\max\{x, x_n\}$.

Introducing a new variable t such that $x = x_0+th$ we can rewrite (2.1) in the form

$$f(x_0+th) = \sum_{i=0}^{n}\frac{(-1)^{n+i}}{i!(n-i)!}\,\frac{t(t-1)\dots(t-n)}{t-i}f_i+r(x_0+th).$$

Assume now that $f\in C^{n+2}(R)$ and differentiate the equality with respect to t ($x = x_0+th$) at $t = k$ ($0\leqslant k\leqslant n$). Then we obtain the equality

$$(2.2)\qquad f'(x_k) = \frac{1}{hn!}\sum_{i=0}^{n}C_n^i\left\{\frac{d}{dt}\left[\frac{t(t-1)\dots(t-n)}{t-i}\right]\right\}\bigg|_{t=k}f_i+r_1(k),$$

where $C_n^i = \binom{n}{i}$ denote the Newton binomial coefficients, and

$$(2.3)\qquad r_1(k) = \frac{h^nf^{(n+1)}(\xi_k)}{(n+1)!}\left\{\frac{d}{dt}[t(t-1)\dots(t-n)]\right\}\bigg|_{t=k}$$

(see [6], p. 86).

Renumbering the nodes $x_0, x_1, \dots, x_n$ from $-k$ to $n-k$ we can write (2.2) in the form

$$(2.4)\quad f'(x_0) = \frac{(-1)^{n+k}}{hn!}\sum_{i=-k}^{n-k}(-1)^iC_n^{k+i}\times$$
$$\times\left\{\frac{d}{dt}\left[\frac{(t+k)(t+k-1)\dots(t+k-n)}{t-i}\right]\right\}\bigg|_{t=0}f_i+r_1(k).$$

Denote the difference expression appearing on the right-hand side of equation (2.4) by $(D^{n,k}f)_0$. More generally, assume that

$$(2.5)\qquad (D^{n,k}f)_p = \frac{(-1)^{n+k}}{hn!}\sum_{i=-k}^{n-k}(-1)^iC_n^{k+i}\times$$
$$\times\left\{\frac{d}{dt}\left[\frac{(t+k)(t+k-1)\dots(t+k-n)}{t-i}\right]\right\}\bigg|_{t=0}f_{p+i}$$

for $p\in Z$ and $0\leqslant k\leqslant n$.

By (2.3)-(2.5), for $f \in C^{n+2}(R)$ and any arbitrary $x_p \in R_h$ we have the equality

$$(2.6) \qquad f'(x_p) = (D^{n,k}f)_p + O(h^n).$$

Hence we get the expression for calculating the value of the derivative, taken at the point x_p, of the Lagrange interpolating polynomial constructed at the nodes $x_{p-k}, x_{p-k+1}, \ldots, x_{p-k+n} \in R_h$.

In derivation of (2.3) for the remainder $r_1(k)$ it is necessary to assume that $f \in C^{n+2}(R)$. This assumption could be weakened to the form $f \in C^{n+1}(R)$ if we were interested only in the determination of the order of magnitude of the remainder (with respect to h). To show this, note that (2.5) is of the form

$$(2.7) \qquad (D^{n,k}f)_p = \frac{1}{h}\sum_{i=-k}^{n-k} a_i f_{p+i},$$

where the coefficients a_i do not depend on h, whereas (2.6) means that after introducing into the right-hand side of the above formula, instead of the function f, a finite sum of its Taylor series at the point x_p, and after grouping all terms containing derivatives of the function f of the same order at the point x_p (denote them by $f_p^{(j)}$ for $j = 0, 1, \ldots$) we can write (2.7) in the form

$$(D^{n,k}f)_p = \frac{1}{h}\sum_{j\geqslant 0} b_j f_p^{(j)},$$

where the coefficients b_j satisfy the following equalities:

$$b_0 = b_2 = b_3 = \ldots = b_n = 0, \qquad b_1 = h.$$

Now, if only the $(n+1)$-st derivative of the function f is continuous, then using the equality

$$f_{p+i} = \sum_{j=0}^{n} \frac{(ih)^j}{j!} f_p^{(j)} + \frac{(ih)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_i)$$

for $i = -k, -k+1, \ldots, -k+n$ we get, taking into account previous considerations,

$$(D^{n,k}f)_p = \frac{1}{h}\sum_{j=0}^{n} b_j f_p^{(j)} + r_2(k) = f'(x_p) + r_2(k),$$

where

$$r_2(k) = \frac{1}{h}\sum_{i=-k}^{n-k} a_i \frac{(ih)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_i).$$

Hence we can write

$$r_2(k) = O(h^n).$$

Finally, we may conclude that if $f \in C^{n+1}(R)$, then for $x_p \in R_h$ the euality

(2.8) $$f'(x_p) = (D^{n,k}f)_p + O(h^n)$$

holds, whereas if $f \in C^{n+2}(R)$, then the error appearing in this equality could be written in the explicit form using formula (2.3).

Let us write now the complete formulae for calculating the values of the expressions $(D^{2n,n}f)_p$, $(D^{2n+1,n}f)_p$, and $(D^{2n+1,n+1}f)_p$ which are the only ones we shall use later on.

By the equality

$$\left\{\frac{d}{dt}\left[\frac{(t+n)\dots(t-n)}{t-i}\right]\right\}\Bigg|_{t=0} = \begin{cases} 0 & \text{for } i = 0, \\ \dfrac{(-1)^n(n!)^2}{-i} & \text{for } -n \leqslant i \leqslant n, i \neq 0, \end{cases}$$

we get from (2.5) the relation

(2.9) $$(D^{2n,n}f)_p = \frac{(n!)^2}{h}\sum_{\substack{i=-n \\ i\neq 0}}^{n}\frac{(-1)^{i+1}}{(n+i)!(n-i)!i}f_{p+i}.$$

Analogously, from the equality

$$\left\{\frac{d}{dt}\left[\frac{(t+n)\dots\big(t-(n+1)\big)}{t-i}\right]\right\}\Bigg|_{t=0} = \begin{cases} (-1)^n(n!)^2 & \text{for } i = 0, \\ \dfrac{(-1)^n n!(n+1)!}{i} & \text{for } -n \leqslant i \leqslant n+1, i \neq 0, \end{cases}$$

we get

(2.10) $$(D^{2n+1,n}f)_p = \frac{(n!)^2}{h}\sum_{i=-n}^{n+1}\frac{(-1)^{i+1}a(i)}{(n+i)!(n+1-i)!}f_{p+i},$$

where

$$a(i) = \begin{cases} 1 & \text{for } i = 0, \\ (n+1)/i & \text{for } -n \leqslant i \leqslant n+1,\ i \neq 0, \end{cases}$$

and since

$$\left\{\frac{d}{dt}\left[\frac{(t+n+1)\dots(t-n)}{t-i}\right]\right\}\Bigg|_{t=0}$$
$$= \begin{cases} (-1)^n (n!)^2 & \text{for } i = 0, \\ \dfrac{(-1)^n (n+1)!n!}{-i} & \text{for } -(n+1) \leqslant i \leqslant n,\ i \neq 0, \end{cases}$$

we can write

$$(2.11) \qquad (D^{2n+1,n+1}f)_p = \frac{(n!)^2}{h} \sum_{i=-(n+1)}^{n} \frac{(-1)^{i+1}\beta(i)}{(n+1+i)!(n-i)!} f_{p+i},$$

where

$$\beta(i) = \begin{cases} -1 & \text{for } i = 0, \\ (n+1)/i & \text{for } -(n+1) \leqslant i \leqslant n,\ i \neq 0. \end{cases}$$

As an example we give the explicit form of the expressions defined by (2.5) for several values of n and k:

$$(D^{1,0}f)_p = \frac{1}{h}(f_{p+1}-f_p), \qquad (D^{1,1}f)_p = \frac{1}{h}(f_p - f_{p-1}),$$

$$(D^{2,1}f)_p = \frac{1}{2h}(f_{p+1}-f_{p-1}).$$

We shall need also an expression which, using the values of the function $\{f_{p-k}, \dots, f_{p-1}, f_{p+1}, \dots, f_{p+n-k}\}$, permits us to calculate an approximate value of $f(x_p)$ with the accuracy $O(h^n)$.

This problem is solved again by an interpolating polynomial constructed at the nodes $\{x_{p-k}, \dots, x_{p-1}, x_{p+1}, \dots, x_{p+n-k}\}$:

$$\sum_{\substack{i=-k \\ i\neq 0}}^{n-k} f_{p+i} \prod_{\substack{j=-k \\ j\neq i, j\neq 0}}^{n-k} \frac{x - x_{p+j}}{x_{p+i} - x_{p+j}}.$$

Denoting its value at the point $x = x_p$ by $(S^{n,k}f)_p$ we get the relation

$$(2.12) \qquad (S^{n,k}f)_p = (C_n^k)^{-1} \sum_{\substack{i=-k \\ i\neq 0}}^{n-k} (-1)^{i+1} C_n^{k+i} f_{p+i}$$

for $p \in Z$, and we note that if $f \in C^n(R)$, then for arbitrary $x_p \in R_h$

$$(2.13) \qquad f(x_p) = (S^{n,k}f)_p + O(h^n),$$

where $0 \leqslant k \leqslant n$.

As an example we give an explicit form of $(S^{n,k}f)_p$ for a few values of n and k:

$$(S^{1,0}f)_p = f_{p+1}, \quad (S^{1,1}f)_p = f_{p-1},$$
$$(S^{2,1}f)_p = \tfrac{1}{2}(f_{p-1}+f_{p+1}).$$

From now on we shall be concerned with periodic functions and vector fields defined on the plane R^2.

Let us denote by F the set of functions $f\colon R^2 \to R$ satisfying the conditions

$$f(x+1, y) = f(x, y+1) = f(x, y) \quad \text{for all } (x, y) \in R^2,$$

and by U the set of vector fields $u\colon R^2 \to R^2$ of the form

$$u = [u^{(1)}, u^{(2)}],$$

where $u^{(1)}, u^{(2)} \in F$.

Definition 2.2. By a *discrete grid function* or *a discrete scalar field* we mean the mapping $f_h\colon R_h^2 \to R$, where we put $f_{i,j} = f_h(x_i, y_j)$ for arbitrary $(x_i, y_j) \in R_h^2$.

Due to the one-to-one correspondence between the discrete grid function f_h and the numerical sequence $\{f_{i,j}\}_{i,j\in Z}$ of its values, we write and use both symbols interchangeably. (The name *discrete scalar field* has been introduced because the name *vector field* is commonly used for mappings in R^k for $k \geqslant 2$.)

The mapping $u_h\colon R_h^2 \to R^2$ of the form $u_h = [u_h^{(1)}, u_h^{(2)}]$, where $u_h^{(1)}, u_h^{(2)}$ are discrete scalar fields, is called a *discrete vector field.*

Hence we can also assume that $u_h = \{[u_{i,j}^{(1)}, u_{i,j}^{(2)}]\}_{i,j\in Z}$, where $[u_{i,j}^{(1)}, u_{i,j}^{(2)}]$ is the value (in R^2) of the mapping u_h at the node $(x_i, y_j) \in R_h^2$.

The discretization mapping r_h is defined by $r_h f = f|R_h^2$, where $f\colon R^2 \to R$, $r_h f\colon R_h^2 \to R$.

Definition 2.3. Define (for $h = 1/N$) the sets

$$\omega_h = \{(p, q)\colon 0 \leqslant p, q \leqslant N-1, p, q \in Z\},$$
$$\Omega_h = \{(x_p, y_q) \in R_h^2\colon (p, q) \in \omega_h\}.$$

Let F_h denote the *linear space of periodic grid functions* with the period equal to one with respect to each variable, i.e.

$$f_h \in F_h \quad \text{iff} \quad f_{i+N,j} = f_{i,j+N} = f_{i,j} \text{ for all } i, j \in Z.$$

In the space F_h we define the scalar product

$$(f_h, g_h)_h = h^2 \sum_{(i,j)\in\omega_h} f_{i,j} g_{i,j}$$

and the norm

$$|f_h|_h = \sqrt{(f_h, f_h)_h}\,.$$

Let $U_h = \{u_h : u_h = [u_h^{(1)}, u_h^{(2)}],\ u_h^{(1)}, u_h^{(2)} \in F_h\}$ denote the *linear space of discrete periodic vector fields* with the scalar product

$$\big((u_h, v_h)\big)_h = (u_h^{(1)}, v_h^{(1)})_h + (u_h^{(2)}, v_h^{(2)})_h$$

and the norm

$$\|u_h\|_h = \sqrt{\big((u_h, u_h)\big)_h}\,.$$

The elements of F_h will be called simply *periodic with period N*.

Note that for the full description of a function $f_h \in F_h$ or a vector field $u_h \in U_h$ it is sufficient, due to their periodicity, to know their values on Ω_h only.

Let us introduce now some auxiliary operators defined on the space F_h and approximating partial derivatives. They play an essential role in the next parts of this paper.

Definition 2.4. Let $D_x^{n,k}$, $D_y^{n,k}$, $S_x^{n,k}$, $S_y^{n,k}$ denote, for any arbitrary fixed $n > 0$ and $k = 0, 1, \ldots, n$, the operators mapping the space F_h into itself and defined by the following relations:

$$(2.14)\qquad (D_x^{n,k} f_h)_{p,q} = \frac{(-1)^{n+k}}{hn!} \sum_{i=-k}^{n-k} (-1)^i C_n^{k+i} \times \left\{\frac{d}{dt}\left[\frac{(t+k)(t+k-1)\ldots(t+k-n)}{t-i}\right]\right\}\Bigg|_{t=0} f_{p+i,q}$$

for all $p, q \in Z$,

$$(2.15)\qquad (D_y^{n,k} f_h)_{p,q} = \frac{(-1)^{n+k}}{hn!} \sum_{j=-k}^{n-k} (-1)^j C_n^{k+j} \times \left\{\frac{d}{dt}\left[\frac{(t+k)(t+k-1)\ldots(t+k-n)}{t-j}\right]\right\}\Bigg|_{t=0} f_{p,q+j}$$

for all $p, q \in Z$,

$$(2.16)\qquad (S_x^{n,k} f_h)_{p,q} = (C_n^k)^{-1} \sum_{\substack{i=-k\\ i\neq 0}}^{n-k} (-1)^{i+1} C_n^{k+i} f_{p+i,q} \quad \text{for all } p, q \in Z,$$

$$(2.17)\qquad (S_y^{n,k} f_h)_{p,q} = (C_n^k)^{-1} \sum_{\substack{j=-k\\ j\neq 0}}^{n-k} (-1)^{j+1} C_n^{k+j} f_{p,q+j} \quad \text{for all } p, q \in Z.$$

Let us define also

$$(2.18)\qquad \begin{aligned} DG_x^m &= S_y^{m,k}\circ D_x^{m,k}, & DG_y^m &= S_x^{m,k}\circ D_y^{m,k},\\ DD_x^m &= S_y^{m,l}\circ D_x^{m,l}, & DD_y^m &= S_x^{m,l}\circ D_y^{m,l}, \end{aligned}$$

where $m > 0$, $k = \text{entier}(m/2)$, $l = \text{entier}\big((m+1/2)\big)$, and the symbol $\circ$ means the operation of the composition of operators.

The above operators, as will be seen later, are difference approximations of the derivatives appearing in the operators grad and div.

It is easy to note that if the function $f^{j_0}: R\to R$ is defined by

$$f^{j_0}(x) = f(x, y_{j_0})$$

(for a fixed $j_0 \in Z$), then

$$(2.19)\qquad (D_x^{m,k}\circ r_h f)_{i,j_0} = (D^{m,k} f^{j_0})_i$$

for all integers i (see (2.5)). Using (2.5) and (2.12) we can write similar relations for the expressions (2.15)-(2.17). This allows us to use formulae (2.8) and (2.12) also in the case of grid functions.

Denote by $C^{k,l}(R^2)$ the class of functions from R^2 in R, differentiable continuously k times with respect to the first variable and l times with respect to the second one.

THEOREM 2.1. *Let $m > 0$. If $f \in C^{m+1,m}(R^2)$, then for all integers p, q we have*

(2.20)

$$(DG_x^m\circ r_h f)_{p,q} = \frac{\partial f}{\partial x}\bigg|_{(x_p,y_q)} + O(h^m), \qquad (DD_x^m\circ r_h f)_{p,q} = \frac{\partial f}{\partial x}\bigg|_{(x_p,y_q)} + O(h^m).$$

Similarly, if $f \in C^{m,m+1}(R^2)$, then

(2.21)

$$(DG_y^m\circ r_h f)_{p,q} = \frac{\partial f}{\partial y}\bigg|_{(x_p,y_q)} + O(h^m), \qquad (DD_y^m\circ r_h f)_{p,q} = \frac{\partial f}{\partial y}\bigg|_{(x_p,y_q)} + O(h^m).$$

(We say that the difference operator DG_x^m *approximates with the order m the differential operator $\partial/\partial x$ on the grid R_h^2* (see [7], p. 19), and similarly for the operators DD_x^m, DG_y^m, DD_y^m.)

We now write the full formulae defining $(DG_x^m f_h)_{p,q}$ and other expressions of that form corresponding to the operators given by formulae (2.18). We do that separately for even and odd values of the parameter m.

The case of even $m = 2n$ $(n > 0)$. Taking

$$(2.22)\qquad I^{(1)} = \{(i,j)\colon -n \leqslant i, j \leqslant n,\ ij \neq 0,\ i, j \in Z\}$$

and using (2.14)-(2.18) and (2.9) we get the relations

$$(DG_x^{2n}f_h)_{p,q} = \sum_{(i,j)\in I^{(1)}} a_{i,j} f_{p+i,q+j}, \tag{2.23}$$

where

$$a_{i,j} = \frac{(-1)^{i+j}(n!)^4}{h(n-i)!(n+i)!(n-j)!(n+j)!i}, \tag{2.24}$$

and

$$(DG_y^{2n}f_h)_{p,q} = \sum_{(i,j)\in I^{(1)}} \bar{a}_{i,j} f_{p+i,q+j}, \tag{2.25}$$

where

$$\bar{a}_{i,j} = \frac{(-1)^{i+j}(n!)^4}{h(n-i)!(n+i)!(n-j)!(n+j)!j}. \tag{2.26}$$

It is easy to see that formulae (2.25) and (2.26) can be also obtained by interchanging the indices i, j in the coefficients $a_{i,j}$ of $f_{p+i,q+j}$ (see (2.23) and (2.24)).

In order to obtain expressions analogous to (2.23)-(2.26) for the operators DD_x^{2n} and DD_y^{2n} it suffices to use the identities

$$(DD_x^{2n}f_h)_{p,q} = (DG_x^{2n}f_h)_{p,q}, \quad (DD_y^{2n}f_h)_{p,q} = (DG_y^{2n}f_h)_{p,q} \tag{2.27}$$

following from (2.18).

The case of odd $m = 2n+1$. Let us put

$$I^{(2)} = \{(i,j): -n \leqslant i,j \leqslant n+1,\ j \neq 0,\ i,j \in Z\}, \tag{2.28}$$

$$I^{(3)} = \{(i,j): -n \leqslant i,j \leqslant n+1,\ i \neq 0,\ i,j \in Z\}. \tag{2.29}$$

By (2.14)-(2.18) and (2.11) we get

$$(DG_x^{2n+1}f_h)_{p,q} = \sum_{(i,j)\in I^{(2)}} b_{i,j} f_{p+i,q+j}, \tag{2.30}$$

where

$$b_{i,j} = \frac{(-1)^{i+j}(n!)^3(n+1)!\,a(i)}{h(n+i)!(n+1-i)!(n+j)!(n+1-j)!}, \tag{2.31}$$

and

$$(DG_y^{2n+1}f_h)_{p,q} = \sum_{(i,j)\in I^{(3)}} \bar{b}_{i,j} f_{p+i,q+j}, \tag{2.32}$$

where

$$\bar{b}_{i,j} = \frac{(-1)^{i+j}(n!)^3(n+1)!\,a(j)}{h(n+i)!(n+1-i)!(n+j)!(n+1-j)!} \tag{2.33}$$

and

$$\alpha(i) = \begin{cases} 1 & \text{for } i = 0, \\ (n+1)/i & \text{for } -n \leqslant i \leqslant n+1, i \neq 0. \end{cases}$$

For the case of operators DD_x^{2n+1} and DD_y^{2n+1}, we define the sets of indices

(2.34) $$I^{(4)} = \{(i,j)\colon -(n+1) \leqslant i,j \leqslant n, j \neq 0, \; i,j \in Z\},$$

(2.35) $$I^{(5)} = \{(i,j)\colon -(n+1) \leqslant i,j \leqslant n, i \neq 0, \; i,j \in Z\}$$

and we additionally use (2.11). Then we get

(2.36) $$(DD_x^{2n+1} f_h)_{p,q} = \sum_{(i,j)\in I^{(4)}} c_{i,j} f_{p+i,q+j},$$

where

(2.37) $$c_{i,j} = \frac{(-1)^{i+j}(n!)^3(n+1)!\beta(i)}{(n+1+i)!(n-i)!(n+1+j)!(n-j)!},$$

and

(2.38) $$(DD_y^{2n+1} f_h)_{p,q} = \sum_{(i,j)\in I^{(5)}} \bar{c}_{i,j} f_{p+i,q+j},$$

where

(2.39) $$\bar{c}_{i,j} = \frac{(-1)^{i+j}(n!)^3(n+1)!\beta(j)}{(n+1+i)!(n-i)!(n+1+j)!(n-j)!}$$

and

$$\beta(i) = \begin{cases} -1 & \text{for } i = 0, \\ (n+1)/i & \text{for } -(n+1) \leqslant i \leqslant n, \; i \neq 0. \end{cases}$$

Similarly as for even m, formulae (2.29), (2.32), (2.33) can be obtained also from (2.28), (2.30), (2.31), and formulae (2.35), (2.38), (2.39) from (2.34), (2.36), (2.37) by interchanging the indices i, j.

As we see, the operators DG_x^m, DG_y^m, DD_x^m, DD_y^m are completely defined by the coefficients $a_{i,j}$, $\bar{a}_{i,j}$, $b_{i,j}$, $\bar{b}_{i,j}$, $c_{i,j}$, $\bar{c}_{i,j}$ in formulae (2.23), (2.25), (2.30), (2.32), (2.36), (2.38). It will be useful to introduce auxiliary scalar fields with values at the nodes equal to the above-mentioned coefficients. This will enable us to calculate the values of $DG_x^m f_h$, $DG_y^m f_h$, $DD_x^m f_h$, $DD_y^m f_h$ at the nodes of the grid R_h^2 by taking scalar products of the appropriate fields and the function f_h.

Definition 2.5. For arbitrary integers l, k the *translation operators* P_x^k and P_y^l are defined by the formulae

$$(P_x^k f_h)_{p,q} = f_{p-k,q} \quad \text{and} \quad (P_y^l f_h)_{p,q} = f_{p,q-l}$$

for all integers p, q and $f_h \in F_h$.

From now on the translation operators acting on vector fields are also denoted by the same symbols.

Definition 2.6. A *discrete scalar field* $dg_x^{m,0,0}$ for even $m = 2n$ $(n > 0)$ is defined on R_h^2 in the following way:

We first specify the values taken by the field at the nodes (x_i, y_j) for $(i, j) \in I^{(1)}$ by the formula

$$(dg_x^{2n,0,0})_{i,j} = a_{i,j},$$

where $a_{i,j}$ are given by (2.24). Then we translate these values on the whole grid R_h^2 in a periodic way (with the period equal to N). At the remaining grid points let us put zero.

In what follows we assume that $m < N$, which is necessary for the correctness of the above construction.

Let us now define the family of scalar fields $\{dg_x^{m,p,q}\}_{p,q \in Z}$ by putting

$$dg_x^{m,p,q} = P_x^p \circ P_y^q dg_x^{m,0,0}$$

for all integers p, q.

Similarly we define the families of scalar fields

$$\{dg_x^{2n+1,p,q}\}_{p,q \in Z}, \quad \{dg_y^{m,p,q}\}_{p,q \in Z}, \quad \{dd_x^{m,p,q}\}_{p,q \in Z}, \quad \{dd_y^{m,p,q}\}_{p,q \in Z}$$

starting with the fields

$$\begin{aligned}
(dd_x^{2n,0,0})_{k,l} &= (dg_x^{2n,0,0})_{k,l} = a_{k,l}, \\
(dd_y^{2n,0,0})_{k,l} &= (dg_y^{2n,0,0})_{k,l} = \bar{a}_{k,l} \quad \text{for } (k, l) \in I^{(1)}, \\
(dg_x^{2n+1,0,0})_{k,l} &= b_{k,l} \quad \text{for } (k, l) \in I^{(2)}, \\
(dg_y^{2n+1,0,0})_{k,l} &= \bar{b}_{k,l} \quad \text{for } (k, l) \in I^{(3)}, \\
(dd_x^{2n+1,0,0})_{k,l} &= c_{k,l} \quad \text{for } (k, l) \in I^{(4)}, \\
(dd_y^{2n+1,0,0})_{k,l} &= \bar{c}_{k,l} \quad \text{for } (k, l) \in I^{(5)},
\end{aligned}$$

where the coefficients $a_{i,j}$, $\bar{a}_{i,j}$, $b_{i,j}$, $\bar{b}_{i,j}$, $c_{i,j}$, $\bar{c}_{i,j}$ are defined by (2.24) (2.26), (2.31), (2.33), (2.37), (2.39) and the sets $I^{(a)}$ for $a = 1, 2, \ldots, 5$ by (2.22), (2.28), (2.29), (2.34), (2.35), respectively.

Using the above scalar fields we can write $(DG_x^m f_h)_{p,q}$ as a scalar product:

$$(DG_x^m f_h)_{p,q} = \frac{1}{h^2} (dg_x^{m,p,q}, f_h)_h \tag{2.40}$$

and similarly

$$(DG_y^m f_h)_{p,q} = \frac{1}{h^2} (dg_y^{m,p,q}, f_h)_h, \quad (DD_x^m f_h)_{p,q} = \frac{1}{h^2} (dd_x^{m,p,q}, f_h)_h,$$
$$(DD_y^m f_h)_{p,q} = \frac{1}{h^2} (dd_y^{m,p,q}, f_h)_h \tag{2.41}$$

for all integers p, q.

Since the introduced fields are periodic, in every family there are N^2 different fields; in particular, they are the fields corresponding to the elements $(p, q) \in \omega_h$.

As an illustration we present now the values of the scalar fields $dg_x^{m,p,q}$, $dg_y^{m,p,q}$, $dd_x^{m,p,q}$, $dd_y^{m,p,q}$ for $m = 1, 2$, $p = 0$, and $q = 0$.

It is sufficient to determine the values of the scalar fields $dg_x^{m,0,0}$, $dg_y^{m,0,0}$, $dd_x^{m,0,0}$, $dd_y^{m,0,0}$ at the grid points (x_i, y_j) such that (i, j) belong to the appropriate sets $I^{(\alpha)}$, and then to follow the procedure outlined in Definition 2.6 in order to determine their values at arbitrary nodes of the grid R_h^2. In the cases $m = 1$ or $m = 2$ the scalar fields, which are of interest to us, take only three different values. For the case $m = 1$ they are equal to 0, $1/h$, $-1/h$, respectively, and in the case $m = 2$ they are 0, $1/4h$, $-1/4h$. The fields can be presented visually if we denote by $\circ$ the grid points (x_i, y_j) where the fields are positive, and by $\bullet$ the nodes where they are negative. Such figures provide a clear picture of the fields in question (see Fig. 2.1).

The above considerations were necessary to define the basic operators which are the difference equivalents of the grad and div operators.

Definition 2.7. By the symbol Grad_m we denote the operator

$$\mathrm{Grad}_m\colon F_h \to U_h$$

defined by the formula

$$\mathrm{Grad}_m f_h = [DG_x^m f_h, DG_y^m f_h] \qquad (f_h \in F_h),$$

and by the symbol Div_m the operator

$$\mathrm{Div}_m\colon U_h \to F_h$$

defined by

$$\mathrm{Div}_m u_h = DD_x^m u_h^{(1)} + DD_y^m u_h^{(2)}, \qquad \text{where } u_h = [u_h^{(1)}, u_h^{(2)}] \in U_h.$$

It follows from Theorem 2.1 that the above-defined operators Grad_m and Div_m approximate with order m the classical operators grad and div.

3. Properties of the operators Grad_m **and** Div_m. Now we formulate and prove some theorems concerning the operators defined in the last section. They will be used in the construction and investigation of the properties of discrete and divergence-free vector fields in the next section.

The condition $N > m$ imposed on N still holds.

THEOREM 3.1. *For an arbitrary natural number m and arbitrary $h_0 > 0$ there exists h $(0 < h < h_0)$ such that if* $\mathrm{Grad}_m f_h = 0$, $f_h \in F_h$, *then* $f_h = \mathrm{const}$.

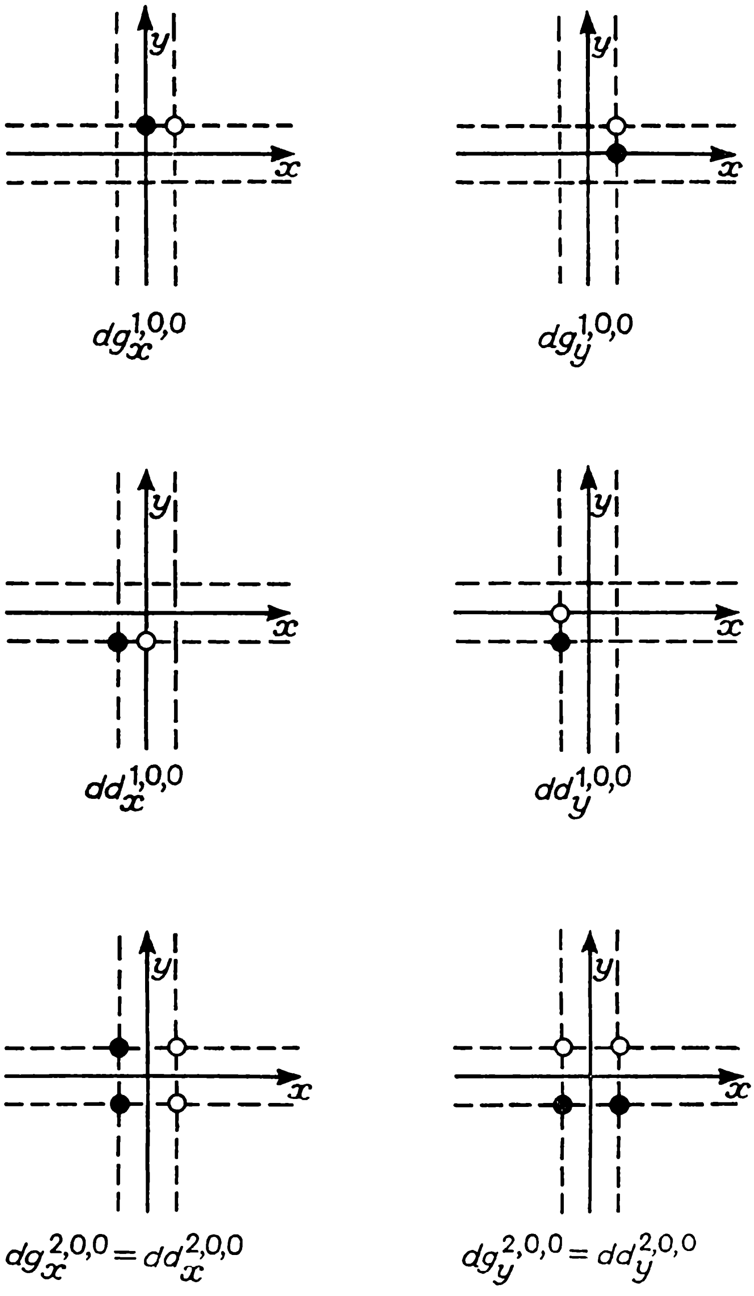

Fig. 2.1

Proof. By assumption the function f_h satisfies the equations

$$(DG_x^m f_h)_{i,j} = 0 \quad \text{for all } (i,j) \in \omega_h, \tag{3.1}$$

$$(DG_y^m f_h)_{i,j} = 0 \quad \text{for all } (i,j) \in \omega_h. \tag{3.2}$$

By Definition 2.4, equations (3.1) can be written in the form

$$(S_y^{m,k} g_h)_{i,j} = 0 \quad \text{for all } (i,j) \in \omega_h, \tag{3.3}$$

where $g_h = D_x^{m,k} f_h$, $k = \text{entier}(m/2)$.

The theorem will be proved if we show that

(a) Equations (3.3) have only null solutions (which follows from the non-vanishing of any eigenvalue of the matrix of system (3.3)).

(b) The equation

$$D_x^{m,k} f_h = 0 \tag{3.4}$$

has the only solution $f_h = \text{const}$ with respect to the variable x. Similarly, we show that from (3.2) it follows that f_h is independent of y. Hence

$$f_h = \text{const}.$$

Step I. (3.3) $\Rightarrow g_h = 0$. Let us write equations (3.3) for arbitrary fixed $i_0 \in \{0, 1, \ldots, N-1\}$ and for $j = k, k+1, \ldots, N-1, 0, 1, \ldots, k-1$ (for arbitrary k $(0 \leqslant k \leqslant N)$). By (2.17) we get a set of linear homogeneous equations with respect to the unknowns $g_{i_0,0}, g_{i_0,1}, \ldots, g_{i_0,N-1}$ having the cyclic matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \ldots & a_{N-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}. \tag{3.5}$$

The matrix elements $a_0, a_1, \ldots, a_{N-1}$ can be expressed by the coefficients appearing in formula (2.17).

It can easily be checked (see [9], p. 231) that the cyclic matrix (3.5) has eigenvalues of the form

$$\lambda_k = a_0 + a_1\varepsilon_k + a_2\varepsilon_k^2 + \ldots + a_{N-1}\varepsilon_k^{N-1} \tag{3.6}$$

for $k = 0, 1, \ldots, N-1$, where $\varepsilon_k = \exp\{i\cdot 2k\pi/N\}$ $(i = \sqrt{-1})$, and the eigenvector x_k corresponding to the eigenvalue λ_k is of the form

$$x_k = \begin{vmatrix} 1 \\ \varepsilon_k \\ \varepsilon_k^2 \\ \vdots \\ \varepsilon_k^{N-1} \end{vmatrix}.$$

The eigenvectors $x_0, x_1, \ldots, x_{N-1}$ form a linearly independent set of vectors in R^N.

We prove now that the eigenvalues λ_k (for $k = 0, 1, \ldots, N-1$) of the matrix (3.5) are nonzero. To do this we consider separately the case of even and odd m.

The case of odd $m = 2n+1$. The quantities a_j for $0 \leqslant j \leqslant N-1$ are given by (see (2.17))

(3.7)
$$a_j = \begin{cases} (C_{2n+1}^n)^{-1}(C_{2n+1}^j)(-1)^{n+j+1} & \text{for } j = 0, 1, \ldots, 2n+1, j \neq n, \\ 0 & \text{for } j = n, 2n+2, 2n+3, \ldots, N-1. \end{cases}$$

Assume that there exists $k \in \{0, 1, \ldots, N-1\}$ such that $\lambda_k = 0$. Using (3.6) and (3.7) we can write this condition in the following form ($a_n = 0$, see (3.7)):

$$\sum_{j=0}^{2n+1} C_{2n+1}^j (-1)^j \exp\left\{i\frac{2k\pi}{N}j\right\} - (-1)^n C_{2n+1}^n \exp\left\{i\frac{2k\pi}{N}n\right\} = 0,$$

i.e.

$$\left(1 - \exp\left\{i\frac{2k\pi}{N}\right\}\right)^{2n+1} = (-1)^n C_{2n+1}^n \exp\left\{i\frac{2k\pi}{N}n\right\}.$$

Multiplying both sides of the last equality by $\exp\{-ik\pi(2n+1)/N\}$ we get

$$\left(\exp\left\{-i\frac{k\pi}{N}\right\} - \exp\left\{i\frac{k\pi}{N}\right\}\right)^{2n+1} = (-1)^n C_{2n+1}^n \exp\left\{-i\frac{k\pi}{N}\right\},$$

and since its left-hand side is equal to

$$\left(-2i\sin\frac{k\pi}{N}\right)^{2n+1} = (-1)^{n+1}2^{2n+1}i\sin^{2n+1}\frac{k\pi}{N},$$

we get

$$-2^{2n+1}i\sin^{2n+1}\frac{k\pi}{N} = C_{2n+1}^n\left(\cos\frac{k\pi}{N} - i\sin\frac{k\pi}{N}\right). \tag{3.8}$$

The left-hand side of (3.8) is purely imaginary, and k should satisfy the inequality $0 \leqslant k \leqslant N-1$; hence (3.8) could be satisfied only if $k = N/2$ or $k = 3N/2$. In this case, for (3.8) we have $2^{2n+1} = C_{2n+1}^n$.

To complete the proof that the eigenvalues λ_k of the matrix (3.5) are nonzero for all $k \in \{0, 1, \ldots, N-1\}$ it is sufficient to use the inequality $2^{2n+1} > C_{2n+1}^n$ holding for any $n \geqslant 0$. This can be easily proved by induction.

The case of even $m = 2n$ ($n > 0$). The elements $\{a_0, a_1, \ldots, a_{N-1}\}$ of the matrix (3.5) can be expressed in this case as

$$a_j = \begin{cases} (C_{2n}^n)^{-1}(-1)^{n+1+j}C_{2n}^j & \text{for } j = 0, 1, \ldots, 2n, j \neq n, \\ 0 & \text{for } j = n, 2n+1, 2n+2, \ldots, N-1. \end{cases}$$

The condition for any the eigenvalues λ_k $(0 \leqslant k \leqslant N-1)$ to be equal to zero takes the form of the equality

$$\sum_{j=0}^{2n} (-1)^j C_{2n}^j \exp\left\{i\frac{2k\pi}{N}j\right\} = (-1)^n C_{2n}^n \exp\left\{i\frac{2k\pi}{N}n\right\}$$

(since $a_n = 0$).

After writing the left-hand side of this equality in the form

$$\left(1-\exp\left\{i\frac{2k\pi}{N}\right\}\right)^{2n}$$

and multiplying both sides by $\exp\{-ik\pi\cdot 2n/N\}$ we get

$$\left(\exp\left\{-i\frac{k\pi}{N}\right\}-\exp\left\{i\frac{k\pi}{N}\right\}\right)^{2n} = (-1)^n\frac{(2n)!}{(n!)^2},$$

i.e.

$$\sin^{2n}\frac{k\pi}{N} = w_n, \tag{3.9}$$

where

$$w_n = \frac{(2n)!}{(n!)^2\cdot 2^{2n}}.$$

It is easy to check that $0 < w_n < 1$.

Since n is fixed, there exist infinitely many natural numbers N such that for all $k \in \{0, 1, \ldots, N-1\}$ equality (3.9) is not satisfied.

Therefore, we have proved that for N chosen as above, for the case of $m = 2n$, all eigenvalues λ_k of the matrix (3.5) are nonzero. For example, for $n = 1$ one should choose N from natural numbers prime with respect to 4.

We have shown, therefore, that for m even as well as for m odd, for properly chosen $N > 1/h_0$ and $h = 1/N$, the matrix of the homogeneous system (3.3) is nonsingular, and hence the function g_h, which is the solution of this system, vanishes at every node of the grid.

Step II. (3.4) $\Rightarrow f_h = \text{const}$ with respect to the variable x. In the previous step we have proved that the function f_h satisfying equations (3.1) has to fulfill equation (3.4), i.e.

$$(D_x^{m,k}f_h)_{i,j} = 0 \tag{3.10}$$

for all integers i, j and $k = \text{entier}(m/2)$.

Fixing $j_0 \in \{0, 1, \ldots, N-1\}$, we can treat the infinite sequence of numbers $\{f_{i,j_0}\}_{i\in Z}$, which are the values of the function f_h satisfying the system of equations (3.10), as a solution of a homogeneous recurrence linear equation with constant coefficients, which for the case $m = 2n+1$ ($n \geqslant 0$) takes the form

$$\sum_{l=-n}^{n+1} a_l f_{i+l,j_0} = 0, \tag{3.11}$$

where

$$a_l = \begin{cases} -\dfrac{1}{n!(n+1)!} & \text{for } l = 0, \\ \dfrac{(-1)^{l+1}(n+1)}{(n+l)!(n+1-l)!l} & \text{for } -n \leqslant l \leqslant n+1,\ l \neq 0 \end{cases}$$

(see (2.14) and (2.10)). For the case of $m = 2n$ this recurrence equation is of the form

$$\sum_{l=-n}^{n} \beta_l f_{i+l,j_0} = 0, \tag{3.12}$$

where

$$\beta_l = \begin{cases} 0 & \text{for } l = 0, \\ \dfrac{(-1)^{l+1}}{(n+l)!(n-l)!l} & \text{for } -n \leqslant l \leqslant n,\ l \neq 0 \end{cases}$$

(see (2.14) and (2.9)).

Since the function f_h is periodic, so is the sequence $\{f_{i,j_0}\}_{i\in Z}$.

As is known (see [2]), every periodic solution of a homogeneous linear recurrence equation with constant coefficients is a finite linear combination of sequences, each one of which being formed of consecutive powers of one of the roots, of modulus one, of the characteristic polynomial of the recurrence equation.

Hence, in order to show that the function f_h is constant with respect to the variable x it is sufficient to prove that the only root, of modulus one, of the characteristic polynomial is equal to 1. We show that this is true for infinitely many N.

Let us notice first that $z = 1$ is the characteristic root of every one of the considered recurrence equations (see (3.11) and (3.12)) for $m > 0$. Since m is positive, for every constant function f_h we have $D_x^{m,k} f_h = 0$, i.e. the sum of the coefficients of the characteristic polynomial is zero.

Similarly as in the first part of the proof, we consider separately the cases of even and odd m.

The case of odd $m = 2n+1$. Denote by z some of the characteristic roots, of modulus one, of the recurrence equation (3.11). Then a sequence of the form $\{z^l\}_{l\in Z}$ is one of the possible periodic solutions of equation (3.11), which means that the following equality must hold:

$$C^0_{2n+1}\frac{(-1)^{-n+1}(n+1)}{-n}z^{l-n}+C^1_{2n+1}\frac{(-1)^{-n+2}(n+1)}{-(n-1)}z^{l-(n-1)}+\ldots+$$
$$+C^{n-1}_{2n+1}\frac{(-1)^{-1+1}(n+1)}{-1}z^{l-1}+(-1)C^n_{2n+1}z^l+C^{n+1}_{2n+1}\frac{(-1)^2(n+1)}{1}z^{l+1}+\ldots+$$
$$+C^{2n}_{2n+1}\frac{(-1)^{n+1}(n+1)}{n}z^{l+n}+C^{2n+1}_{2n+1}\frac{(-1)^{n+2}(n+1)}{n+1}z^{l+n+1}=0.$$

Multiplying this equality by $(-1)^n z^{-l}/(n+1)$ we get

$$\frac{(-1)^n}{1}(C^n_{2n+1}z-C^{n-1}_{2n+1}z^{-1})+\frac{(-1)^{n+1}}{2}(C^{n+2}_{2n+1}z^2-C^{n-2}_{2n+1}z^{-2})+\ldots+$$
$$+\frac{(-1)^{2n-1}}{n}(C^{2n}_{2n+1}z^n-C^0_{2n+1}z^{-n})+\frac{1}{n+1}\left(z^{n+1}+(-1)^{n-1}C_{2n+1}\right)=0,$$

which can be transformed into

(3.13)
$$\frac{(-1)^n}{1}C^n_{2n+2}\left(n(z-z^{-1})+2z\right)+\frac{(-1)^{n-1}}{2}C^{n-1}_{2n+2}\left((n-1)(z^2-z^{-2})+4z^2\right)+\ldots+$$
$$+\frac{-1}{n}C^1_{2n+2}\left((z^n-z^{-n})+2nz^n\right)+(-1)^{n+1}C^{n+1}_{2n+2}+2z^{n+1}=0.$$

Then substituting $z=e^{i\varphi}$, using the identity

$$e^{ik\varphi}-e^{-ik\varphi}=2i\sin k\varphi,$$

and equating the real part of the left-hand side of (3.13) to zero, we get

$$2\left[\frac{(-1)^{n+1}}{2}C^{n+1}_{2n+2}+(-1)^nC^n_{2n+2}\cos\varphi+(-1)^{n-1}C^{n-1}_{2n+2}\cos2\varphi+\ldots+\right.$$
$$\left.+(-1)C^1_{2n+2}\cos n\varphi+C^0_{2n+2}\cos(n+1)\varphi\right]=0.$$

We prove now that for $n \geqslant 0$ the following identity holds:

(3.14) $2^{n-1}(1-\cos\varphi)^n = \frac{1}{2}C_{2n}^n - C_{2n}^{n-1}\cos\varphi + \ldots + (-1)^n C_{2n}^0 \cos n\varphi$.

Using this identity we can write the previous equality in the form

$$2^{n+1}(1-\cos\varphi)^{n+1} = 0.$$

It is obvious that only $\varphi = 0$ fulfills this equality. Hence the only root, of modulus one, of the characteristic polynomial of the considered recurrence equation, is the number 1.

Proof of identity (3.14). Using the formulae

(3.15) $\cos\varphi = 1-2\sin^2\varphi/2$ and $T_k(\cos\varphi) = \cos k\varphi$,

where T_k is the k-th Chebyshev polynomial, we can write (3.14) as

(3.16) $$2^{2n-1}\sin^{2n}\frac{\varphi}{2} = \sum_{i=0}^{n}{}^{\circ} C_{2n}^i(-1)^{n-i}T_{n-i}(\cos\varphi),$$

where the symbol $\sum^{\circ}$ means that the summand containing the polynomial T_0 is to be multiplied by $1/2$.

Substituting $x = \sin^2\varphi/2$, we get

$$2^{2n-1}x^n = \sum_{i=0}^{n}{}^{\circ} C_{2n}^i(-1)^{n-i}T_{n-i}(1-2x) = \sum_{i=0}^{n}{}^{\circ} C_{2n}^i T_{n-i}^*(x),$$

where

$$T_k^*(x) = T_k(2x-1) = (-1)^k T_k(1-2x).$$

Finally, (3.14) takes now the form of the identity

$$2^{2n-1}x^n = \sum_{i=0}^{n}{}^{\circ} C_{2n}^i T_{n-i}^*(x),$$

which was proved in [5], p. 29.

The case of even $m = 2n (n > 0)$. Similarly as in the case of odd m, by (3.12), for z being a characteristic root, of modulus one (i.e. $z = e^{i\varphi}$), of the recurrence equation, we get the following equality:

(3.17) $$\sum_{\substack{l=-n \\ l\neq 0}}^{n} \frac{(-1)^{l+1}}{l} C_{2n}^{n+l} e^{i\varphi l} = 0.$$

Since $C_{2n}^{n+l} = C_{2n}^{n-l}$, we obtain

$$\sum_{l=1}^{n} \frac{(-1)^{l+1}}{l} C_{2n}^{n-l}(e^{i\varphi l} - e^{-i\varphi l}) = 0,$$

i.e.

(3.18) $$H(\varphi) = 0,$$

where

$$H(\varphi) \equiv C_{2n}^{0}\frac{\sin n\varphi}{n} - C_{2n}^{1}\frac{\sin(n-1)\varphi}{n-1} + \ldots + (-1)^{n-1}C_{2n}^{n-1}\sin\varphi.$$

Note that equation (3.18) has at least two solutions, $\varphi = 0$ and $\varphi = \pi$, belonging to the interval $[0, 2\pi)$.

We show now that there are no other solutions to that equation, or, more precisely, we prove that $H(\varphi)$ does not vanish in the interval $(0, \pi)$, which suffices because the function H is odd.

Let us consider the derivative $H'(\varphi)$:

$$H'(\varphi) = C_{2n}^{0}\cos n\varphi - C_{2n}^{1}\cos(n-1)\varphi + \ldots + (-1)^{n-1}C_{2n}^{n-1}\cos\varphi.$$

Using the identity

$$(-1)^n H'(\varphi) + \tfrac{1}{2}C_{2n}^{n} = 2^{n-1}(1-\cos\varphi)^n,$$

following from (3.14), we can write immediately

$$H'(\varphi) = (-1)^n 2^{n-1}[(1-\cos\varphi)^n - 2^{-n}C_{2n}^{n}],$$

and then

$$H(\varphi) = (-1)^n 2^{n-1}\int_0^{\varphi}[(1-\cos\varphi)^n - 2^{-n}C_{2n}^{n}]d\varphi \tag{3.19}$$

because $H(0) = 0$.

Using the relation

$$\int(1-\cos\varphi)^n d\varphi = -\frac{1}{n}(1-\cos\varphi)^{n-1}\sin\varphi + \left(1 + \frac{n-1}{n}\right)\int(1-\cos\varphi)^{n-1}d\varphi$$

we write the first part of the integral in (3.19) in the form

$$\begin{aligned}\int_0^{\varphi}(1-\cos\varphi)^n d\varphi = &-\frac{1}{n}(1-\cos\varphi)^{n-1}\sin\varphi + \\ &+\frac{2n-1}{n}\left[-\frac{1}{n-1}(1-\cos\varphi)^{n-2}\sin\varphi + \right.\\ &+\frac{2(n-1)-1}{n-1}\left[-\frac{1}{n-2}(1-\cos\varphi)^{n-3}\sin\varphi + \ldots + \right.\\ &\left.\left.+\frac{5}{3}\left[-\frac{1}{2}(1-\cos\varphi)\sin\varphi + \frac{3}{2}[\varphi - \sin\varphi]\right]\ldots\right]\right]\\ = &\,\varphi\frac{3\cdot 5\cdot\ldots\cdot(2n-1)}{2\cdot 3\cdot\ldots\cdot n} - \sin\varphi\left\{\frac{1}{n}(1-\cos\varphi)^{n-1} + \right.\end{aligned}$$

$$+\frac{2n-1}{n}\left[\frac{1}{n-1}(1-\cos\varphi)^{n-2}+\right.$$

$$\left.\left.+\frac{2(n-1)-1}{n-1}\left[\frac{1}{n-2}(1-\cos\varphi)^{n-3}+\ldots+\frac{5}{3}\left[\frac{1}{2}(1-\cos\varphi)+\frac{3}{2}\right]\ldots\right]\right]\right\}.$$

The second part of the integral in (3.19) is given by

$$-2^{-n}C_{2n}^{n}\int_{0}^{\varphi}d\varphi = -\varphi\frac{(2n)!}{(n!)^2 2^n} = -\varphi\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{n!}.$$

Therefore, after summing up both parts we get $H(\varphi)$ in the form from which it follows immediately that $H(\varphi)$ does not vanish in the interval $(0, \pi)$.

Finally, the only roots, of modulus one, of the characteristic polynomial of the recurrence equation (3.17) are the numbers 1 and -1. Consequently, the only periodic sequences satisfying (3.17) are the sequences of the form $\{c_1+c_2(-1)^l\}_{l\in Z}$, where c_1, c_2 are arbitrary constants.

If we now require these sequences to be periodic and, moreover, if we assume that N is odd, then these sequences are constant.

Of course, the intersection of the set of odd numbers and of the set determined in Step I contains infinitely many elements.

Thus we have proved that there exists an arbitrary large N such that for arbitrary fixed $j_0 \in \{0, 1, \ldots, N-1\}$ the sequence $\{f_{i,j_0}\}_{i\in Z}$ satisfying (3.11) or (3.12) is constant. This means that the grid functions f_h for $h = 1/N$, satisfying the system of equations (3.1), are constant with respect to the variable x, i.e. $f_{i,j} = f_{i+1,j}$ for arbitrary fixed $j \in \{0, 1, \ldots, N-1\}$ and all integers i.

Now, considering the system of equations (3.2), by similar reasoning we come to the conclusion that for the same N the functions f_h satisfying (3.2) are constant with respect to the variable y, i.e., $f_{i,j} = f_{i,j+1}$ for arbitrary fixed $i \in \{0, 1, \ldots, N-1\}$ and all integers j. Therefore, since equations (3.1) and (3.2) are satisfied simultaneously, we obtain $f_{i,j} = \text{const}$ for all integers i, j, which completes the proof.

Theorem 3.1 can also be formulated in the following way:

For an arbitrary fixed $m > 0$ there exists a strictly increasing sequence $\{N_i\}_{i=1}^{\infty}$ of natural numbers such that for $h = h_i = 1/N_i$ $(1 \leqslant i < \infty)$ the condition $\mathrm{Grad}_m f_h = 0$, $f_h \in F_h$, *implies* $f_h = \text{const}$.

From now on we assume that, for a given $m > 0$, N has been chosen according to the thesis of the theorem.

Consider now the operator

$$\Delta_h = \mathrm{Div}_m \circ \mathrm{Grad}_m, \tag{3.20}$$

which is a difference approximation of the Laplace operator, and let us investigate with what error the grid function $\Delta_h \circ r_h f$ approximates Δf (Δ denotes the Laplace operator) for a sufficiently regular function $f \in F$.

We use the notation Δ_h remembering, however, that this operator depends on m.

Using the definitions of the operators Div_m and Grad_m we can immediately write Δ_h in the form

$$\Delta_h = DD_x^m \circ DG_x^m + DD_y^m \circ DG_y^m,$$

where $DD_x^m \circ DG_x^m$ approximates the second order derivative of the function with respect to the variable x, and $DD_y^m \circ DG_y^m$ approximates the second order derivative with respect to the variable y. We can, therefore, expect different regularity requirements for the function $f \in F$ to make optimal estimations from below for the orders r and s of the error in the equalities

$$(DD_x^m \circ DG_x^m \circ r_h f)_{p,q} = \frac{\partial^2 f}{\partial x^2}\bigg|_{(x_p, y_q)} + O(h^r),$$

$$(DD_y^m \circ DG_y^m \circ r_h f)_{p,q} = \frac{\partial^2 f}{\partial y^2}\bigg|_{(x_p, y_q)} + O(h^s).$$

As the reasoning for the operators $DD_x^m \circ DG_x^m$ and $DD_y^m \circ DG_y^m$ is analogous, we investigate only the first one.

Using (2.18) we write

$$DD_x^m \circ DG_x^m = S_y^{m,l} \circ D_x^{m,l} \circ S_y^{m,k} \circ D_x^{m,k}, \tag{3.21}$$

where

$$k = \mathrm{entier}(m/2), \qquad l = \mathrm{entier}\big((m+1)/2\big). \tag{3.22}$$

Note that the operators appearing on the right-hand side of equation (3.21) commute (this property follows from (2.14)-(2.17)). Hence we write (3.21) in the form

$$DD_x^m \circ DG_x^m = S \circ D, \tag{3.23}$$

where

$$S = S_y^{m,l} \circ S_y^{m,k}, \qquad D = D_x^{m,l} \circ D_x^{m,k}. \tag{3.24}$$

Observe that the operators S and D have the following property: if we calculate the values of $(Sf_h)_{p,q}$ and $(Df_h)_{p,q}$, then we have to use only the values of the function f_h at grid nodes with one of the coordinates fixed. Therefore, it is more convenient to return to functions of one variable and consider $S^{m,k}$ and $D^{m,k}$ (see (2.12) and (2.5)) as operators acting on the functions of one real variable $f\colon R \to R$. To this aim it is sufficient to replace, in (2.12) and (2.5), the discrete variable x_p by a variable x taking values on the whole real axis. Therefore, we deal with the operators S and D of the form

$$S = S^{m,l} \circ S^{m,k} \quad \text{and} \quad D = D^{m,l} \circ D^{m,k},$$

where k and l are defined by (3.22).

As at the beginning of Section 2, let f_i denote the value of the function $f\colon R \to R$ at the respective node of the grid R_h, i.e., $f_i = f(x_i)$ for $x_i \in R_h$. Assume additionally that

$$f \in C^{m+2}(R). \tag{3.25}$$

Then the equality

$$f_{p+k} = \sum_{l=0}^{m+1} \frac{(kh)^l}{l!} f_p^{(l)} + \frac{(kh)^{m+2}}{(m+2)!} f^{(m+2)}(\xi_k) \tag{3.26}$$

holds, where p, k are arbitrary integers, $f_p^{(l)}$ denotes the l-th order derivative of the function f at the node x_p, and ξ_k is an intermediate point between x_p and x_{p+k}.

Now we investigate the operator D. The cases of even and odd m will be considered separately.

The case of even $m = 2n$ $(n > 0)$. Using directly (2.9) we write

$$(D^{2n,n} \circ D^{2n,n} f)_p = \frac{(n!)^4}{h^2 [(2n)!]^2} \sum_{\substack{i=-n \\ i \neq 0}}^{n} \sum_{\substack{j=-n \\ j \neq 0}}^{n} \frac{(-1)^{i+j}}{ij} C_{2n}^{n+i} C_{2n}^{n+j} f_{p+i+j}. \tag{3.27}$$

Making use of (2.6), (2.9), and the fact that $f \in C^{2n+2}(R)$ (see (3.25)) we get

$$\begin{aligned}
(D^{2n,n} \circ D^{2n,n} f)_p &= \left(D^{2n,n} \left[f' + \frac{h^{2n} c}{(2n+1)!} f^{(2n+1)}(\xi_x) \right] \right)_p \\
&= (D^{2n,n} f')_p + \left(D^{2n,n} \left[\frac{h^{2n} c}{(2n+1)!} f^{(2n+1)}(\xi_x) \right] \right)_p \\
&= \big(f''(x_p) + O(h^{2n}) \big) + \frac{h^{2n-1} (n!)^2 c}{(2n+1)!} \sum_{\substack{i=-n \\ i \neq 0}}^{n} \frac{(-1)^{i+1}}{(n+i)!(n-i)!\,i} f^{(2n+2)}(\xi_{x_{p+i}}),
\end{aligned}$$

where $c = (-1)^n (n!)^2$. Consequently, we obtain

$$(D^{2n,n} \circ D^{2n,n} f)_p = f''(x_p) + O(h^r), \tag{3.28}$$

where $r \geqslant 2n-1$.

We prove now that $r \geqslant 2n$.

Notice that in (3.27) the coefficients at f_{p+i+j} for $i = k$, $j = l$ and $i = -k$, $j = -l$ (where k and l satisfy the conditions $-n \leqslant k, l \leqslant n$, $k \neq 0, l \neq 0$) are equal. Hence the coefficients at the values of the function f at the nodes of the grid R_h located symmetrically with respect to the grid point x_p are also equal.

Let us replace now the values f_{p+i+j} of the function f appearing in (3.27) by the expressions (3.26). It follows from previous considerations that the expressions containing the values $f_p^{(l)}$ of odd order derivatives of the function f vanish and, therefore, the order r of the error appearing in (3.28) must be an even number. Since $r \geqslant 2n-1$, we have $r \geqslant 2n$.

The estimation from below of the order r of the error ($r \geqslant 2n$) cannot be improved by increasing the regularity of the function. Indeed, it can be illustrated by the following example for $n = 1$:

$$(D^{2,1}f)_p = \frac{1}{2h}(f_{p+1}-f_{p-1}),$$

$$(D^{2,1}\circ D^{2,1}f)_p = \frac{1}{4h^2}(f_{p-2}-2f_p+f_{p+2}).$$

Replacing now the function f by its Taylor series at the point x_p we get

$$\frac{1}{4h^2}(f_{p-2}-2f_p+f_{p+2}) = \frac{1}{4h^2}\left(4h^2f_p''+\frac{16}{12}h^4f_p^{(4)}+\ldots\right) = f_p''+O(h^2),$$

i.e., $r = 2$.

The case of odd $m = 2n+1$. We use again formulae (2.7) and (2.8) for two cases $(D^{2n+1,n}f)_p$ and $(D^{2n+1,n+1}f)_p$. Similarly as for even m, we write

$$(D^{2n+1,n+1}\circ D^{2n+1,n}f)_p = f''(x_p)+O(h^r), \tag{3.29}$$

where $r \geqslant 2n$ for $f \in C^{2n+3}(R)$ (see (3.25)).

Notice that if $f \in C^{2n+4}(R)$, we cannot generally increase the order r of the error. We can only state that the error takes the form

$$h^{2n}\sum_{j=-(n+1)}^{n} c_j f^{(2n+2)}(\xi_j)+O(h^{2n+1}),$$

where ξ_j are intermediate points between nodes x_{p-n} and x_{p+n+1}, and the constants c_j do not depend on the parameter h. We prove now that if, however, $f \in C^{2n+4}(R)$, then it turns out that the order r of the error from formula (3.29) is at least equal to $2n+2$, i.e., $r \geqslant 2n+2$.

By (2.10) and (2.11) we have

$$(3.30)\qquad (D^{2n+1,n+1}\circ D^{2n+1,n}f)_p$$
$$= \frac{(n!)^4}{h^2}\sum_{j=-(n+1)}^{n}\sum_{i=-n}^{n+1}\frac{(-1)^{i+j}a(i)\beta(j)}{(n+i)!(n+1-i)!(n+1+j)!(n-j)!}f_{p+i+j},$$

where

$$a(i) = \begin{cases}1 & \text{for } i = 0,\\ (n+1)/i & \text{for } -n\leqslant i\leqslant n+1,\ i\neq 0,\end{cases}$$

$$\beta(j) = \begin{cases}-1 & \text{for } j = 0,\\ (n+1)/j & \text{for } -(n+1)\leqslant j\leqslant n,\ j\neq 0.\end{cases}$$

Notice that the coefficients at f_{p+i+j} for $i = k$, $j = -l$ and for $i = l$, $j = -k$ (where $-n\leqslant k, l\leqslant n+1$) are equal. Therefore, if we introduce into the right-hand side of equation (3.30) instead of the values f_{p+i+j} of the function f the expressions (3.26) written for $m = 2n+2$, then the coefficients at the odd order derivatives $f_p^{(2k+1)}$ vanish for $k = 0, 1, \ldots, n+1$.

Therefore, the proof of the inequality $r\geqslant 2n+2$ reduces to showing that the coefficient at $f_p^{(2n+2)}$ is also zero.

This coefficient is equal to

$$\frac{(n!)^4h^{2n}}{(2n+2)!}\sum_{j=-(n+1)}^{n}\sum_{\substack{i=-n\\ i+j\neq 0}}^{n+1}\frac{(-1)^{i+j}a(i)\beta(j)(i+j)^{2n+2}}{(n+i)!(n+1-i)!(n+1+j)!(n-j)!}.$$

The double sum appearing in the above formula can be expressed as a sum of three components A, B, C, where

$$A = (n+1)^2\sum_{\substack{j=-(n+1)\\ j\neq 0}}^{n}\sum_{\substack{i=-n\\ i\neq 0}}^{n+1}\frac{(-1)^{i+j}(i+j)^{2n+2}}{(n+i)!(n+1-i)!(n+1+j)!(n-j)!\,ij},$$

$$B = -\frac{n+1}{n!(n+1)!}\sum_{\substack{i=-n\\ i\neq 0}}^{n+1}\frac{(-1)^i i^{2n+1}}{(n+1-i)!(n-i)!},$$

$$C = \frac{n+1}{n!(n+1)!}\sum_{\substack{j=-(n+1)\\ j\neq 0}}^{n}\frac{(-1)^j j^{2n+1}}{(n+1+j)!(n-j)!}.$$

Replacing in the last sum the summation index j by $-i$ we see immediately that the second and third elements are equal ($B = C$).

Let us investigate now the first component A changing the summation index j into $-j$. We get

$$A = (n+1)^2 \sum_{\substack{j=-n\\ j\neq 0}}^{n+1} \frac{(-1)^{j+1}}{(n+1-j)!(n+j)!j} \sum_{\substack{i=-n\\ i\neq 0}}^{n+1} \frac{(-1)^i}{(n+1-i)!(n+i)!i} (i-j)^{2n+2};$$

and using the formula

$$(i-j)^{2n+2} = (2n+2)! \sum_{k=0}^{2n+2} \frac{(-1)^k i^k j^{2n+2-k}}{k!(2n+2-k)!}$$

$$= (2n+2)! \sum_{k=-(n+1)}^{n+1} \frac{(-1)^{n+k+1} i^{n+k+1} j^{n-k+1}}{(n+k+1)!(n-k+1)!}$$

we obtain

$$A = (2n+2)!(n+1)^2 \sum_{k=-(n+1)}^{n+1} \frac{(-1)^{n+k}}{(n+k+1)!(n-k+1)!} \gamma(-k)\gamma(k),$$

where

$$\gamma(k) = \sum_{\substack{i=-n\\ i\neq 0}}^{n+1} \frac{(-1)^i i^{n+k}}{(n+1-i)!(n+i)!} \tag{3.31}$$

for values of k under consideration. Hence

$$A = (2n+2)!(n+1)^2 \times \tag{3.32}$$

$$\times \left\{ \frac{(-1)^n}{[(n+1)!]^2} \gamma^2(0) + 2 \sum_{k=1}^{n+1} \frac{(-1)^{n+k}}{(n+k+1)!(n-k+1)!} \gamma(-k)\gamma(k) \right\}.$$

We can obtain some information about $\gamma(k)$ from the relation

$$(D^{2n+1,n}f)_p = f'_p + O(h^{2n+1})$$

using additionally formulae (3.26), i.e., the expansion of the function f.

Namely, writing the above relation in the form

$$(D^{2n+1,n}f)_p = \sum_{j=0}^{2n+1} h^{j-1}\sigma(j) f_p^{(j)} + O(h^{2n+1}),$$

where, by (3.26) and (2.10),

(3.33)

$$\sigma(j) = \begin{cases} -\dfrac{n!(n+1)!}{j!} \displaystyle\sum_{\substack{i=-n \\ i\neq 0}}^{n+1} \frac{(-1)^i i^{j-1}}{(n+1-i)!(n+i)!} & \text{for } 1 \leqslant j \leqslant 2n+1, \\ -n!(n+1)! \displaystyle\sum_{\substack{i=-n \\ i\neq 0}}^{n+1} \frac{(-1)^i}{(n+1-i)!(n+i)i!} - \frac{1}{n+1} & \text{for } j = 0, \end{cases}$$

we conclude that the quantities $\sigma(j)$ satisfy the equalities

$$\sigma(j) = \begin{cases} 1 & \text{for } j = 1, \\ 0 & \text{for } j = 0, 2, 3, \ldots, 2n+1. \end{cases} \tag{3.34}$$

Using (3.31) and (3.32) we can express $\gamma(k)$ by means of $\sigma(j)$. From the above relations and from (3.34) we deduce that $\gamma(k) = 0$ for $k = -n+1, -n+2, \ldots, n$. Hence in expression (3.32) only those factors remain which correspond to the values $k = -(n+1), n+1$. We get

$$A = -2(n+1)^2\gamma(-n-1)\gamma(n+1),$$

and since

$$\gamma(-n-1) = -\frac{1}{[(n+1)!]^2},$$

we obtain

$$A = \frac{2}{(n!)^2} \sum_{\substack{i=-n \\ i\neq 0}}^{n+1} \frac{(-1)^i i^{2n+1}}{(n+1-i)!(n+i)!}.$$

Comparing the formula obtained with those defining the components B and C we see that $A+B+C = 0$, which is just the coefficient at $f_p^{(2n+2)}$. Therefore, we eventually obtain the inequality $r \geqslant 2n+2$.

In general, it is impossible to increase the order r of the error ($r \geqslant 2n+2$) by increasing the regularity of the function f. This is illustrated by the following example for $n = 0$:

$$\begin{aligned}(D^{1,1} \circ D^{1,0} f)_p &= \frac{1}{h^2}(f_{p-1} - 2f_p + f_{p+1}) \\ &= \frac{1}{h^2}\left[-2f_p + 2\left(f_p + \frac{h^2}{2!}f_p'' + \frac{h^4}{4!}f_p^{(4)} + \ldots\right)\right] = f_p'' + \frac{h^2}{12}f_p^{(4)} + \ldots;\end{aligned}$$

hence $r = 2$.

The error we make if we replace the value of the function f at the grid point x_p by the expression

$$(Sf)_p = (S^{m,l} \circ S^{m,k} f)_p$$

(see (3.24)) under the assumption that $f \in C^m(R)$ is determined by the formula

$$(Sf)_p = f(x_p) + O(h^m),$$

following from relation (2.13) and from the fact that the coefficients in the expression $(S^{m,l}f)_p$ (see (2.12)) do not depend on the parameter h.

The above considerations for a function of one variable can be generalized in a straightforward way for functions of two variables in the space F (provided we treat functions of two variables with fixed value of one of the variables as a one-parameter family of one-variable functions) and for operator compositions in question (see (3.33) and (3.24)) $D_x^{m,i}$, $D_y^{m,i}$, $S_x^{m,i}$, $S_y^{m,i}$ for $i = k, l$ (see (3.22)). Hence we obtain

THEOREM 3.2. *For arbitrary $m > 0$ and a function $f \in F$ the following implications hold:*

if $f \in C^{m+2,m}(R^2)$, then

$$(DD_x^m \circ DG_x^m \circ r_h f)_{p,q} = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_p, y_q)} + O(h^m);$$

if $f \in C^{m,m+2}(R^2)$, then

$$(DD_y^m \circ DG_y^m \circ r_h f)_{p,q} = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_p, y_q)} + O(h^m);$$

if $f \in C^{m+2,m+2}(R^2)$, then

$$(\Delta_h \circ r_h f)_{p,q} = \Delta f|_{(x_p, y_q)} + O(h^m).$$

We prove now one more property of the operators Grad_m and Div_m, which will be used later.

THEOREM 3.3. *For an arbitrary function $f_h \in F_h$ and an arbitrary vector field $u_h \in U_h$ the following equality holds:*

$$(\mathrm{Div}_m u_h, f_h)_h = -((u_h, \mathrm{Grad}_m f_h))_h \tag{3.35}$$

(scalar products $(\cdot,\cdot)_h$ and $((\cdot,\cdot))_h$ have been determined in Definition 2.3).

Proof. Since an arbitrary vector field $u_h = [u_h^{(1)}, u_h^{(2)}]$ can be expressed as a sum of the fields of the form $[u_h^{(1)}, 0]$ and $[0, u_h^{(2)}]$, it is sufficient to prove the thesis for each of the fields separately. We do this for the field $[u_h^{(1)}, 0]$. Analogous calculations for $[0, u_h^{(2)}]$ are omitted. Let us consider separately two cases: of even and odd m.

The case of even $m = 2n$ $(n > 0)$. The complete form of the left-hand side L of equality (3.35) is

$$L = h^2 \sum_{p,q=0}^{N-1} (DD_x^{2n} u_h^{(1)})_{p,q} f_{p,q} = \sum_{p,q=0}^{N-1} \sum_{\substack{i=-n\\ i\neq 0}}^{n} \sum_{\substack{j=-n\\ j\neq 0}}^{n} A_{i,j} u_{p+i,q+j}^{(1)} f_{p,q},$$

where

$$A_{i,j} = h(n!)^4 \frac{(-1)^{i+j}}{(n-i)!(n+i)!(n-j)!(n+j)!i}$$

(by (2.22)-(2.24) and (2.27)).

Since the functions $u_h^{(1)}$ and f_h are periodic, we can transform the equality obtained to the form

$$L = \sum_{\substack{i=-n\\ i\neq 0}}^{n} \sum_{\substack{j=-n\\ j\neq 0}}^{n} A_{i,j} \sum_{p,q=0}^{N-1} u_{p,q}^{(1)} f_{p-i,q-j}.$$

Replacing the summation indices i, j by $-i$, $-j$, respectively, and using the equality $A_{i,j} = -A_{-i,-j}$, we get

$$L = -\sum_{p,q=0}^{N-1} \sum_{\substack{i=-n\\ i\neq 0}}^{n} \sum_{\substack{j=-n\\ j\neq 0}}^{n} A_{i,j} u_{p,q}^{(1)} f_{p+i,q+j},$$

which, by (2.22)-(2.24) and (2.27), takes the form

$$L = -h^2 \sum_{p,q=0}^{N-1} u_{p,q}^{(1)} (DG_x^{2n} f_h)_{p,q}.$$

This is just the right-hand side of equality (3.35), which completes the proof for even m.

The case of odd $m = 2n+1$. Following the procedure of the previous case we obtain

$$L = h^2 \sum_{p,q=0}^{N-1} (DD_x^{2n+1} u_h^{(1)})_{p,q} f_{p,q}$$

$$= \sum_{p,q=0}^{N-1} \sum_{i=-(n+1)}^{n} \sum_{\substack{j=-(n+1)\\ j\neq 0}}^{n} \beta(i) B_{i,j} u_{p+i,q+j}^{(1)} f_{p,q},$$

where, by (2.34), (2.36), and (2.37),

$$B_{i,j} = h(n!)^3(n+1)!\,\frac{(-1)^{i+j}}{(n+1+i)!(n-i)!(n+1+j)!(n-j)!},$$

$$\beta(i) = \begin{cases} -1 & \text{for } i = 0, \\ (n+1)/i & \text{for } -(n+1) \leqslant i \leqslant n,\ i \neq 0. \end{cases}$$

Since the functions $u_h^{(1)}$ and f_h are periodic, we obtain

$$L = \sum_{p,q=0}^{N-1} \sum_{i=-(n+1)}^{n} \sum_{\substack{j=-(n+1)\\ j\neq 0}}^{n} \beta(i) B_{i,j} u_{p,q}^{(1)} f_{p-i,q-j}.$$

After changing i into $-i$, j into $-j$, and applying the equality

$$B_{-i,-j} = C_{i,j},$$

where

$$C_{i,j} = h(n!)^3(n+1)!\,\frac{(-1)^{i+j}}{(n+1-i)!(n+i)!(n+1-j)!(n+j)!}$$

we get

$$L = -\sum_{p,q=0}^{N-1} \sum_{i=-n}^{n+1} \sum_{\substack{j=-n\\ j\neq 0}}^{n+1} \gamma(i) C_{i,j} u_{p,q}^{(1)} f_{p+i,q+j},$$

where

$$\gamma(i) = \begin{cases} 1 & \text{for } i = 0, \\ (n+1)/i & \text{for } -n \leqslant i \leqslant n+1,\ i \neq 0. \end{cases}$$

Using (2.28), (2.30), and (2.31) we obtain

$$L = -h^2 \sum_{p,q=0}^{N-1} u_{p,q}^{(1)} (DG_x^{2n+1} f_h)_{p,q},$$

which completes the proof.

COROLLARY 3.1. *If $\Delta_h f_h = 0$ for $f_h \in F_h$, then $f_h =$ const.*

Proof. From the formula $\Delta_h = \mathrm{Div}_m \circ \mathrm{Grad}_m$ and from Theorem 3.3 it follows that

$$0 = (\Delta_h f_h, f_h)_h = (\mathrm{Div}_m \circ \mathrm{Grad}_m f_h, f_h)_h = -\big((\mathrm{Grad}_m f_h, \mathrm{Grad}_m f_h)\big)_h$$
$$= \|\mathrm{Grad}_m f_h\|_h^2.$$

Hence we get $\mathrm{Grad}_m f_h = 0$, and further, applying Theorem 3.1, we conclude that $f_h = \mathrm{const}$.

4. Construction and properties of discrete divergence-free vector fields. Orthogonal decomposition of the space U_h. As before, m is a fixed natural number and N is chosen in such a way that if $\mathrm{Grad}_m f_h = 0$, then $f_h = \mathrm{const}$ for arbitrary $f_h \in F_h$. The condition $N > m$ imposed on N still holds.

Definition 4.1. By *discrete divergence-free vector fields* we mean such elements u_h of the space U_h which satisfy the equality

$$\mathrm{Div}_m u_h = 0. \tag{4.1}$$

Notice that every constant vector field $u_h = [u_h^{(1)}, u_h^{(2)}]$, i.e., such that $u_h^{(1)} = c_1 = \mathrm{const}$ and $u_h^{(2)} = c_2 = \mathrm{const}$, satisfies condition (4.1). From Theorem 3.3 we obtain

COROLLARY 4.1. *Each discrete divergence-free vector field $u_h \in U_h$ is orthogonal to each vector field of the form $\mathrm{Grad}_m f_h, f_h \in F_h$.*

Proof. The corollary follows from the equalities

$$\big((u_h, \mathrm{Grad}_m f_h)\big)_h = -(\mathrm{Div}_m u_h, f_h)_h = 0.$$

In this section we deal with the construction of all discrete divergence-free vector fields.

We associate with the operator Div_m the family $\{dd_h^{m,p,q}\}_{p,q\in Z}$ of periodic vector fields $dd_h^{m,p,q} \in U_h$ defined by

$$dd_h^{m,p,q} = [dd_x^{m,p,q}, dd_y^{m,p,q}] \tag{4.2}$$

for all integers p and q, where $dd_x^{m,p,q}, dd_y^{m,p,q} \in F_h$ were determined in Definition 2.6.

We can write now a relation valid for an arbitrary vector field $u_h \in U_h$:

$$(\mathrm{Div}_m u_h)_{p,q} = \frac{1}{h^2}\big((dd_h^{m,p,q}, u_h)\big)_h. \tag{4.3}$$

As we can see, by the periodicity of the introduced fields, there exist precisely N^2 different fields $dd_h^{m,p,q}$; they are in one-to-one correspondence with the elements (p, q) of the set ω_h. The fields $dd_h^{m,p,q}$ are used in defining discrete divergence-free vector fields.

Definition 4.2. We define the operators

$$\mathrm{Refl}\colon U_h \to U_h \quad \text{and} \quad \mathrm{Rev}\colon U_h \to U_h$$

as follows: for $u_h, v_h, w_h \in U_h$

$$v_h = \mathrm{Refl}\, u_h \quad \text{and} \quad w_h = \mathrm{Rev}\, u_h$$

if

$$v^{(1)}_{i,j} = u^{(1)}_{-i,-j}, \qquad v^{(2)}_{i,j} = u^{(2)}_{-i,-j},$$
$$w^{(1)}_{i,j} = u^{(2)}_{i,j}, \qquad w^{(2)}_{i,j} = -u^{(1)}_{i,j}$$

($u_h = [u_h^{(1)}, u_h^{(2)}]$ and similarly for v_h and w_h).

THEOREM 4.1. *If $u_h \in U_h$, then for arbitrary integers s, t the field $v_h^{s,t} \in U_h$ defined by*

$$v_h^{s,t} = \mathrm{Rev} \circ \mathrm{Refl} \circ P_x^s \circ P_y^t u_h \tag{4.4}$$

is orthogonal to the field u_h, i.e. $((u_h, v_h^{s,t}))_h = 0$ (the translation operators P_x^s, P_y^t have been determined in Definition 2.5).

Proof. For $i, j \in Z$ denote by $[u^{(1)}_{i,j}, u^{(2)}_{i,j}]$ the value of the vector field u_h at the node $(x_i, y_j) \in R_h^2$. By (4.4) and the definition of the operators appearing there, the vector

$$[u^{(2)}_{-(i+s),-(j+t)}, \; -u^{(1)}_{-(i+s),-(j+t)}]$$

is the value of the field $v_h^{s,t}$ at the grid point $(x_i, y_j) \in R_h^2$. Since the fields u_h and $v_h^{s,t}$ are periodic, the equality

$$\frac{2}{h^2}((u_h, v_h^{s,t}))_h = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} (u^{(1)}_{i,j} u^{(2)}_{-(i+s),-(j+t)} - u^{(2)}_{i,j} u^{(1)}_{-(i+s),-(j+t)}) + \tag{4.5}$$
$$+ \sum_{k=-N-s+1}^{-s} \; \sum_{l=-N-t+1}^{-t} (u^{(1)}_{k,l} u^{(2)}_{-(k+s),-(l+t)} - u^{(2)}_{k,l} u^{(1)}_{-(k+s),-(l+t)})$$

holds. Changing in this formula the summation indices k, l into i, j according to the rule $k = -s-i$, $l = -t-j$, we conclude that the right-hand side of (4.5) is zero, which means that $((u_h, v_h^{s,t}))_h = 0$.

Using Theorem 4.1 we are able to construct a finite family of discrete divergence-free vector fields. We use them together with constant vector fields for forming a basis in the space of vector fields $u_h \in U_h$ such that $\mathrm{Div}_m u_h = 0$.

COROLLARY 4.2. *Vector fields $z_h^{m,p,q}$ defined by*

$$z_h^{m,p,q} = \mathrm{Rev} \circ \mathrm{Refl}\, dd_h^{m,N-p,N-q} \tag{4.6}$$

for all integers p and q are discrete divergence-free vector fields of the space U_h, i.e., $\mathrm{Div}_m z_h^{m,p,q} = 0$.

(The choice of indices of the form $N-p$ and $N-q$ appearing in (4.6) leads to the simplest and easiest formula (4.7) which gives the field $z_h^{m,p,q}$ in an explicit form).

Note that the family $\{z_h^{m,p,q}\}_{p,q\in Z}$ contains exactly N^2 different vector fields. This follows from the fact that there is a one-to-one correspondence between these fields and the vector fields $dd_h^{m,s,t}$.

It is easy to check, for the complete form of the field $z_h^{m,p,q}$ (by (4.2), (4.6) and Definitions 2.6 and 4.2), that discrete divergence-free fields $z_h^{m,p,q}$ can be expressed as

$$z_h^{m,p,q} = [dg_y^{m,p,q}, \ -dg_x^{m,p,q}], \tag{4.7}$$

where the scalar fields $dg_x^{m,p,q}$ and $dg_y^{m,p,q}$ have been determined in Definition 2.6.

COROLLARY 4.3. *Discrete, constant vector fields are orthogonal to the above-constructed fields $z_h^{m,p,q}$ for all $(p, q) \in \omega_h$.*

Proof. It is sufficient to use the form (4.7) of the vector fields $z_h^{m,p,q}$ and the fact that the operators DG_x^m and DG_y^m fulfill equations (2.40), (2.41) and (2.20), (2.21).

Let us assign to the operator Δ_h the family of scalar fields $\delta_h^{m,p,q} \in F_h$ for $(p, q) \in \omega_h$ such that the equalities

$$(\Delta_h f_h)_{p,q} = \frac{1}{h^2}(\delta_h^{m,p,q}, f_h)_h \tag{4.8}$$

hold for all $(p, q) \in \omega_h$ and arbitrary $f_h \in F_h$.

Remark 4.1. The scalar fields $\delta_h^{m,p,q}$, similarly as the fields assigned to the difference operators $DG_x^m, DD_x^m, \ldots$ determined earlier are given by

$$\delta_h^{m,p,q} = P_x^p \circ P_y^q \delta_h^{m,0,0},$$

where P_x^p and P_y^q are the translation operators introduced in Definition 2.5. Therefore, it is sufficient to determine the values of the field $\delta_h^{m,0,0}$ only at the nodes of the grid R_h^2. We follow here the procedure from Section 2, i.e., we define first the set I of nodes of the grid R_h^2, and we specify values of the field $\delta_h^{m,0,0}$ at each of the grid points belonging to this set. Next, we transfer the values of the field from nodes belonging to the set I onto the whole grid R_h^2 (with a period equal to N), taking zero as the value of the field at the remaining grid points.

We consider separately the cases of m even and odd.

The case of even $m = 2n$ $(n > 0)$. By (3.20) and (2.22)-(2.27) we have

$$(\Delta_h f_h)_{0,0} = M \sum_{\substack{p=-n \\ p\neq 0}}^{n} \sum_{\substack{q=-n \\ q\neq 0}}^{n} \Big\{ \frac{w(p, q)}{p} \sum_{\substack{i=-n \\ i\neq 0}}^{n} \sum_{\substack{j=-n \\ j\neq 0}}^{n} \frac{w(i, j)}{i} f_{p+i,q+j} +$$

$$+\frac{w(p,q)}{q}\sum_{\substack{i=-n\\i\neq 0}}^{n}\sum_{\substack{j=-n\\j\neq 0}}^{n}\frac{w(i,j)}{j}f_{p+i,q+j}\Bigg\}$$

$$= M\sum_{\substack{p,q,i,j=-n\\pqij\neq 0}}^{n} w(p,q)w(i,j)\left[\frac{1}{pi}+\frac{1}{qj}\right]f_{p+i,q+j},$$

where $M = (n!)^8/h^2$ and

$$w(i,j) = \frac{(-1)^{i+j}}{(n-i)!(n+i)!(n-j)!(n+j)!}. \tag{4.9}$$

As we see, to calculate $(\Delta_h f_h)_{0,0}$ we use the values $f_{k,l}$ of the function f_h for $-m \leqslant k, l \leqslant m$.

Remark 4.2. In order to define scalar fields $\delta_h^{m,p,q} \in F_h$ having property (4.8) and not to go into complicated calculations, we impose on N the condition

$$N > 2m, \tag{4.10}$$

where m is a fixed natural number. An additional complication for $N \leqslant 2m$ is connected with the fact that the intersection of supports of factors of the scalar product to appear later (equality (4.13)) contains more than one component.

Condition (4.10) is not a serious restriction, as the ultimate goal of the approximation is to make possible calculations for arbitrary small h ($h = 1/N$).

From now on we restrict ourselves to the case of sufficiently large N satisfying inequality (4.10).

Hence we can assume that

$$I = \{(x_k, y_l) \in R_h^2\colon -m \leqslant k, l \leqslant m\} \tag{4.11}$$

and, as can be easily checked, the field $\delta_h^{2n,0,0}$ takes at the nodes $(x_k, y_l) \in I$ the values

$$\delta_{k,l}^{2n,0,0} = M\sum_{\substack{i=\max(-n,k-n)\\i\neq 0, i\neq k}}^{\min(n,k+n)}\ \sum_{\substack{j=\max(-n,l-n)\\j\neq 0, j\neq l}}^{\min(n,l+n)} w(p,q)w(i,j)\left[\frac{1}{pi}+\frac{1}{qj}\right], \tag{4.12}$$

where $p = k-i$ and $q = l-j$.

There is also another way, which we shall use later, to determine the values $\delta_{k,l}^{2n,0,0}$ of the scalar field $\delta_h^{2n,0,0}$ for (k, l) such that $(x_k, y_l) \in I$. Namely, we prove that

$$\delta_{k,l}^{2n,0,0} = -\frac{1}{h^2}\big((z_h^{2n,0,0}, z_h^{2n,k,l})\big)_h, \tag{4.13}$$

where $z_h^{2n,i,j}$ are discrete divergence-free vector fields introduced in Corollary 4.2.

Proof of equality (4.13). Using the periodicity of the fields $z_h^{2n,k,l}$, equality (4.7), and Definition 2.6, we obtain

$$\big((z_h^{2n,0,0}, z_h^{2n,k,l})\big)_h = (n!)^8 \sum_{\substack{i=\max(-n,k-n)\\ i\neq 0, i\neq k}}^{\min(n,k+n)} \sum_{\substack{j=\max(-n,l-n)\\ j\neq 0, j\neq l}}^{\min(n,l+n)} \left\{\frac{w(i,j)}{j}\,\frac{w(i-k,j-l)}{j-l} + \right.$$
$$\left. +\frac{w(i,j)}{i}\,\frac{w(i-k,j-l)}{i-k}\right\},$$

where $w(i,j)$ are given by (4.9). Comparing the above expression with the right-hand side of formula (4.12) we see at once that for even m equality (4.13) holds.

The case of odd $m = 2n+1$. To simplify the notation we put

$$\alpha(i) = \begin{cases} 1 & \text{for } i = 0, \\ (n+1)/i & \text{for } i \neq 0, \end{cases}$$

$$\beta(i) = \begin{cases} -1 & \text{for } i = 0, \\ (n+1)/i & \text{for } i \neq 0, \end{cases}$$

$$\gamma(i) = \begin{cases} 0 & \text{for } i = 0, \\ 1 & \text{for } i \neq 0, \end{cases}$$

$$w(i,j) = \frac{(-1)^{i+j}}{(n+i)!\,(n+1-i)!\,(n+j)!\,(n+1-j)!},$$

$$v(i,j) = \frac{(-1)^{i+j}}{(n+1+i)!\,(n-i)!\,(n+1+j)!\,(n-j)!},$$

$$M = \frac{[(n+1)!]^2 (n!)^6}{h^2}.$$

Now, by (3.20) and (2.28)-(2.39), we obtain

$$(\Delta_h f_h)_{0,0} = M \sum_{p=-(n+1)}^{n} \sum_{q=-(n+1)}^{n} \sum_{i=-n}^{n+1} \sum_{j=-n}^{n+1} v(p,q)\,w(i,j) \times$$
$$\times \{\gamma(q)\beta(p)\gamma(j)\alpha(i) + \beta(q)\gamma(p)\alpha(j)\gamma(i)\} f_{p+i,q+j}.$$

After simple transformations we see that the values $\delta_{k,l}^{2n+1,0,0}$ of the field $\delta_h^{2n+1,0,0}$ at the grid points $(x_k, y_l) \in I$ (the set I is still defined by (4.11)) are given by

$$(4.14)\quad \delta_{k,l}^{2n+1,0,0} = \sum_{i=\max(-n,k-n)}^{\min(n+1,k+n+1)} \sum_{j=\max(-n,l-n)}^{\min(n+1,l+n+1)} v(k-i, l-j)w(i,j)\times$$
$$\times\{\gamma(l-j)\beta(k-i)\gamma(j)\alpha(i)+\beta(l-j)\gamma(k-i)\alpha(j)\gamma(i)\}.$$

Let us calculate now $((z_h^{2n+1,0,0}, z_h^{2n+1,k,l}))_h$ for (k, l) such that $(x_k, y_l) \in I$. We get

$$(4.15)\quad \frac{1}{h^2}((z_h^{2n+1,0,0}, z_h^{2n+1,k,l}))_h$$
$$= M \sum_{\substack{i=\max(-n,k-n)\\ i\neq 0, i\neq k}}^{\min(n+1,k+n+1)} \sum_{j=\max(-n,l-n)}^{\min(n+1,l+n+1)} w(i,j)w(i-k,j-l)\alpha(j)\alpha(j-l)+$$
$$+M \sum_{i=\max(-n,k-n)}^{\min(n+1,k+n+1)} \sum_{\substack{j=\max(-n,l-n)\\ j\neq 0, j\neq l}}^{\min(n+1,l+n+1)} w(i,j)w(i-k,j-l)\alpha(i)\alpha(i-k)$$
$$= M \sum_{i=\max(-n,k-n)}^{\min(n+1,k+n+1)} \sum_{j=\max(-n,l-n)}^{\min(n+1,l+n+1)} w(i,j)w(i-k,j-l)\times$$
$$\times\{\gamma(i)\gamma(i-k)\alpha(j)\alpha(j-l)+\alpha(i)\alpha(i-k)\gamma(j)\gamma(j-l)\}.$$

It is sufficient now to use the relations

$$\alpha(i) = -\beta(-i), \quad w(i,j) = v(-i, -j)$$

to ensure, by comparing the right-hand sides of equalities (4.14) and (4.15), that for all k, l considered the equality

$$(4.16)\quad \delta_{k,l}^{m,0,0} = -\frac{1}{h^2}((z_h^{m,0,0}, z_h^{m,k,l}))_h$$

holds for $m = 2n+1$.

Thus, we have proved that the values $\delta_{k,l}^{m,0,0}$ of the scalar field $\delta_h^{m,0,0}$ are given by (4.16) at all grid points $(x_k, y_l) \in I$, where I is determined by (4.11) and $z_h^{m,k,l}$ are discrete divergence-free vector fields defined by (4.6).

By Remark 4.1 the obtained values of the field $\delta_h^{m,0,0}$ permit to determine the values of this field on the whole grid R_h^2.

The above considerations prove the following

THEOREM 4.2. *The values* $\delta_{p+k,q+l}^{m,p,q}$ *of the periodic scalar field* $\delta_h^{m,p,q} \in F_h$ *are given by*

$$\delta_{k,l}^{m,p,q} = -\frac{1}{h^4}\big((z_h^{m,p,q}, z_h^{m,p+k,q+l})\big)_h.$$

Using this theorem we can formulate

COROLLARY 4.4. *For an arbitrary function* $f_h \in F_h$ *the relation*

$$(\Delta_h f_h)_{p,q} = -\frac{1}{h^2}\Big(\Big(z_h^{m,p,q}, \sum_{(k,l)\in\omega_h} f_{p+k,q+l}\, z_h^{m,p+k,q+l}\Big)\Big)_h$$

holds for all integers p *and* q.

Now we prove the following

THEOREM 4.3. *In the set of discrete divergence-free vector fields* $\{z_h^{m,p,q}\}_{(p,q)\in\omega_h}$ *each subset of* N^2-1 *different elements forms a system of linearly independent vector fields in the space* U_h.

Proof. We prove this theorem by showing that if the equality

$$\sum_{(p,q)\in\omega_h} c_{p,q} z_h^{m,p,q} = 0 \tag{4.17}$$

holds, then $c_{p,q} = \text{const}$ for all $(p, q) \in \omega_h$.

Let us treat the set of coefficients $\{c_{p,q}\}$ as a discrete function defined on Ω_h and denote by c_h its periodic continuation on the whole grid R_h^2. By (4.17) the equality

$$\Big(\Big(\sum_{(p,q)\in\omega_h} c_{p,q} z_h^{m,p,q}, z_h^{m,k,l}\Big)\Big)_h = 0$$

holds for every $(k, l) \in \omega_h$. Therefore, from Corollary 4.4 we infer that $(\Delta_h c_h)_{k,l} = 0$ for all $(k, l) \in \omega_h$, i.e. $\Delta_h c_h = 0$. Hence, by Corollary 3.1, we get equality $c_h = \text{const}$, which completes the proof.

COROLLARY 4.5. *In the set of vector fields* $\{dd_h^{m,p,q}\}_{(p,q)\in\omega_h}$ (see (4.2)) *every subset of* N^2-1 *different elements forms a system of linearly independent vector fields in the space* U_h.

Proof. It is sufficient to use formula (4.6) and the fact that linear dependence (or independence) of the functions $u_h \in U_h$ $(u_h: R_h^2 \to R^2)$ is invariant with respect to the transformation

$$u_h \to \mathrm{Rev} \circ \mathrm{Refl} \circ u_h,$$

where the mappings $\mathrm{Refl}: R_h^2 \to R_h^2$ and $\mathrm{Rev}: R^2 \to R^2$ are linear and non-singular.

Now we investigate the set of vector fields of the form $\mathrm{Grad}_m f_h$ for $f_h \in F_h$, forming a linear space. We determine the dimension of that space.

Let us consider the set of special, periodic grid functions $\{f_h^{k,l}\}_{(k,l)\in\omega_h} \subset F_h$ of the form

$$f_{i,j}^{k,l} = \delta_{i,k}\delta_{l,j},$$

where

$$\delta_{i,j} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j \end{cases}$$

for $(i,j) \in \omega_h$.

It follows directly from (2.22)-(2.27) (for even m), from (2.28)-(2.39) (for odd m), and from Definition 2.6 that for all $(k,l) \in \omega_h$ we have

$$\mathrm{Grad}_m f_h^{k,l} = -[dd_x^{m,k,l}, dd_y^{m,k,l}], \quad \text{i.e.,} \quad \mathrm{Grad}_m f_h^{k,l} = -dd_h^{m,k,l}.$$

Hence, by Corollary 4.5, we obtain

COROLLARY 4.6. *Every subset of N^2-1 different elements from the family $\{\mathrm{Grad}_m f_h^{k,l}\}_{(k,l)\in\omega_h}$ is a set of linearly independent vector fields in the space U_h.*

Since every field of the form $\mathrm{Grad}_m f_h$, $f_h \in F_h$, is a linear combination of the fields $\mathrm{Grad}_m f_h^{k,l}$ defined above, the following theorem is true:

THEOREM 4.4. *The dimension of the space G_h of all vector fields of the form $\mathrm{Grad}_m f_h$, $f_h \in F_h$, is equal to N^2-1.*

Using the results obtained we know that in the space U_h of all discrete periodic vector fields there exist two orthogonal spaces:

the space G_h, spanned by the fields $\mathrm{Grad}_m f_h^{k,l}$:

$$G_h = \mathrm{Lin}\{\{\mathrm{Grad}_m f_h^{k,l}\}_{(k,l)\in\omega_h}\},$$

the space D_h, spanned by the fields $z_h^{m,p,q}$ and by two constant fields:

$$D_h = \mathrm{Lin}\{\{z_h^{m,p,q}\}_{(p,q)\in\omega_h}, [1,0], [0,1]\}.$$

Moreover, we know (Theorems 4.4, 4.3, and Corollary 4.3) that

$$\dim G_h = N^2-1 \quad \text{and} \quad \dim D_h \geqslant N^2+1.$$

Since the dimension of the whole space U_h is exactly equal to $2N^2$, and the spaces D_h and G_h are mutually orthogonal (see Corollary 4.2), we get $\dim D_h = N^2+1$.

COROLLARY 4.7. *We have $U_h = D_h + G_h$.*

This gives the desired orthogonal decomposition of the space U_h of discrete periodic vector fields.

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