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ITERATIVE SOLUTION OF RECTANGULAR SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

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Introduction

The contents of the main part of my lecture held at the Banach Center in Warsaw during the semester on Computational Mathematics – a general description of iterative methods for solving rectangular linear systems – was already published in [2]. Therefore in part 1 there are only given the most important results without proofs. Part 2 contains an extension of the semi-iterative (Čebyšev) method to rectangular linear systems.

1. Stationary one-step iterative methods

To solve the linear system

$$Ax = b, \quad A\colon \boldsymbol{R}^N \to \Re(A) \subset \boldsymbol{R}^M \tag{1}$$

iteratively we use the equation

$$x = x + Qr, \quad Q\colon \boldsymbol{R}^M \to \Re(Q) \subset \boldsymbol{R}^N, \tag{2}$$

where Q and r denote a given $M \times N$-matrix and the residual vector

$$r = b - Ax, \tag{3}$$

respectively. Obviously, (1) is consistent to (2) but generally not reciprocally consistent (cf. [10], p. 65 ff.). A solution of (2) which is not solution of (1) we will call a (with respect to Q) *generalized solution of* (1). Equation (2) yields the iterative process

$$x^k = Tx^{k-1} + Qb, \quad r^k = Sr^{k-1} \tag{4}$$

for the approximations and the residuals, respectively. The iteration matrices T and S are defined by

$$T = I_N - QA, \quad S = I_M - AQ. \tag{5}$$

Using the changing rules

$$AT^n = S^n A, \quad T^n Q = QS^n \tag{6}$$

we obtain the explicit representations

$$x^k = T^k x^0 + B_k b, \quad r^k = S^k r^0 \tag{7}$$

with

$$B_k = \sum_{n=0}^{k-1} T^n Q = \sum_{n=0}^{k-1} QS^n \tag{8}$$

and

$$T^k = I_N - B_k A, \quad S^k = I_M - AB_k. \tag{9}$$

Under the assumptions

(A1) it exists $B_\infty = \lim_{k\to\infty} B_k$,

(A2) $\mathfrak{N}(AQ) = \mathfrak{N}(Q)$, $\mathfrak{R}(QA) = \mathfrak{R}(Q)$,

(A3) $\mathfrak{N}(QA) = \mathfrak{N}(A)$, $\mathfrak{R}(AQ) = \mathfrak{R}(A)$,

(A4) $b \in \mathfrak{R}(A)$ (solvability condition),

(A5) $\operatorname{rank} AQ = M$,

(A6) $\operatorname{rank} QA = N$

we can prove the following propositions (cf. [2], pp. 25–31):

THEOREM 1. *From* (A1) *it follows: There exist the limit matrices* T^∞ *and* S^∞,

$$T^\infty = I_N - B_\infty A, \quad S^\infty = I_M - AB_\infty, \tag{10}$$

which are projectors with the properties

$$(T^\infty)^2 = T^\infty = T^\infty T = TT^\infty, \quad (S^\infty)^2 = S^\infty = S^\infty S = SS^\infty \tag{11}$$

and the following nullspace and range relations

$$\mathfrak{R}(T^\infty) = \mathfrak{N}(QA), \quad \mathfrak{R}(S^\infty) = \mathfrak{N}(AQ), \tag{12}$$

$$\mathfrak{N}(T^\infty) = \mathfrak{R}(QA), \quad \mathfrak{N}(S^\infty) = \mathfrak{R}(AQ). \tag{13}$$

There exist the limit vectors x^∞ *and* r^∞ *with the representations*

$$x^\infty = (I_N - B_\infty A)x^0 + B_\infty b, \quad r^\infty = (I_M - AB_\infty)r^0. \tag{14}$$

The matrix B_∞ is an outer (generalized) inverse of the matrix A and x^∞ is a generalized (with respect to Q) solution of (1):

$$B_\infty A B_\infty = B_\infty, \quad Q(b - Ax^\infty) = 0. \tag{15}$$

Proof. Here we only prove the last assertion since (15) was proved in [2] under stronger conditions. From the identity $TB_k + Q = B_k$ we get $TB_\infty + Q = B_\infty$ and with (5) and (10) further $QAB_\infty = Q$ and finally

$$QS^\infty = T^\infty Q = 0. \tag{16}$$

So we have

$$\sum_{n=0}^{\infty} T^n Q S^\infty = B_\infty (I_M - AB_\infty) = 0$$

and

$$0 = QS^\infty r^0 = Qr^\infty = Q(b - Ax^\infty). \ \blacksquare$$

THEOREM 2. *From* (A1) *and* (A3) *it follows: B_∞ is an inner (generalized) inverse of A,*

$$AB_\infty A = A. \tag{17}$$

If in addition (A4) *holds, then x^∞ is a solution of* (1):

$$Ax^\infty = b. \tag{18}$$

THEOREM 3. *From* (A1) *and* (A5) *(maximal row rank of A) it follows: The matrix B_∞ is an $A^{1,2,3}$-inverse of A* (for this terminology cf. for instance the lecture of G. Zielke in this issue), *S^∞ and AB_∞ are equal to the null and the unit matrix, respectively, and x^∞ is solution of* (1).

THEOREM 4. *From* (A1) *and* (A6) *(maximal column rank of A) it follows: The matrix B_∞ is an $A^{1,2,4}$-inverse of A, T^∞ and $B_\infty A$ are equal to the null and the unit matrix, respectively, and $x^\infty = B_\infty b$ is the — in this case unique — solution (or generalized solution) of* (1).

At last we define the average and the asymptotic rate of convergence according to [10], p. 84, by

$$R_k(T) = -\frac{1}{k}\log \|(T - T^\infty)^k\|, \quad R_\infty(T) = -\log \varrho(T - T^\infty), \tag{19}$$

where $\|\cdot\|$ and $\varrho(\cdot)$ denote a certain norm and the spectral radius, respectively.

Until now one question has not been answered: how to choose the matrix Q, defining the iterative process? If the matrix A is non-singular, the best Q is the inverse A^{-1}, since in this case the iteration terminates after one step. For this reason the Jacobi and the Gauss–Seidel methods use the inverse of diag A and of the left lower triangular matrix of A, respectively, as approximations of A^{-1}. Varga [9] proposed a regular splitting of A into a regular summand (which should be easy to invert) and a nonpositive summand. Plemmons [6] extended this idea to rectangular systems by splitting A into the difference $B-C$ and using B^+ or other g-inverses ($B^{1,2}$, $B^{1,3}$, $B^{1,4}$) as Q. Other possibilities are given by cyclic projection methods (cf. [1]) especially by approximation by columns or rows of A (or linear combinations of columns or rows) (cf. [2]–[5], [7], [8]).

Unfortunately, all the above-mentioned iterative methods converge very slowly, so it is necessary to look for convergence-accelerating methods.

2. Accelerating of convergence

Similarly to the semi-iterative (Čebyšev) method for linear systems with a nonsingular matrix A (cf. [9], p. 134, [10], p. 345), we will construct a sequence of linear combinations of the approximations x^k

$$y^n = \sum_{k=0}^{n} a_{n,k} x^k \tag{20}$$

and ask for coefficients such that $\{y^n\}$ converges faster than $\{x^k\}$.

If condition (A1) is fulfilled then the sequence $\{x^k\}$ converges to the limit vector $x^\infty = T^\infty x^0 + B_\infty b$. It is a solution of (2)

$$x^\infty = Tx^\infty + Qb \tag{21}$$

since $TT^\infty = T^\infty$ and $TB_\infty + Q = B_\infty$. Choosing x^∞ as a starting vector, we obtain

$$x^1 = x^2 = \ldots = x^n = x^\infty.$$

Therefore the new and — as we hope — "better" sequence $\{y^n\}$ should yield x^∞ in this case too. So we get the conditions

$$\sum_{k=0}^{n} a_{n,k} = 1, \qquad n = 0, 1, 2, \ldots \tag{22}$$

The difference of equations (4) and (21) yields

$$x^k - x^\infty = (T - T^\infty)^k (x^0 - x^\infty) \tag{23}$$

since $T^k = T^\infty + (T - T^\infty)^k$ and $T^\infty(x^0 - x^\infty) = 0$. With the help of (20) and (22) we get

$$y^n - x^\infty = p_n(\tilde{T})(x^0 - x^\infty) \tag{24}$$

where we used the abbreviations

$$\tilde{T} = T - T^{\infty} \tag{25}$$

and

$$p_n(\tilde{T}) = \sum_{k=0}^{n} a_{n,k}\tilde{T}^k. \tag{26}$$

With the Euclidean norm $\|\cdot\|$ it follows

$$\|y^n - x^{\infty}\| \leqslant \|p_n(\tilde{T})\| \|x^0 - x^{\infty}\|. \tag{27}$$

To make $\|y^n - x^{\infty}\|$ as small as possible we have to minimize the spectral norm of $p_n(\tilde{T})$. Let us assume that the coefficients $a_{n,k}$ are real and that $\tilde{T}$ is a hermitean matrix. Then $p_n(\tilde{T})$ is also hermitean, and it holds

$$\|p_n(\tilde{T})\| = \max_{1\leqslant i\leqslant N} |p_n(\tilde{\tau}_i)| \leqslant \max_{a\leqslant t\leqslant b} |p_n(t)|, \tag{28}$$

where $\tilde{\tau}_i$ denote the eigenvalues of $\tilde{T}$ and a and b are lower and upper bounds of the spectrum of $\tilde{T}$, respectively. If the sequence $\{T^n\}$ is converging, then T has (with the possible exception of the eigenvalue $\tau_j = 1$) only eigenvalues with $|\tau_i| < 1$. From $T^k y_i = \tau_i^k y_i$ and the fact that the projector T^{∞} has the eigenvalues 1 and 0 only, we have

$$\tilde{\tau}_i = \begin{cases} 0 & \text{if} \quad \tau_i = 1, \\ \tau_i & \text{if} \quad |\tau_i| < 1. \end{cases} \tag{29}$$

So there exist bounds a and b with

$$-1 < a \leqslant |\tilde{\tau}_i| \leqslant b < 1 \tag{30}$$

and we can use the theory of the semi-iterative methods: Instead of $\|p_n(\tilde{T})\|$ we minimize $\max_{a\leqslant t\leqslant b} |p_n(t)|$ in the class of polynomials with $p_n(1) = \sum_{k=0}^{n} a_{n,k} = 1$. The solutions are suitable normed Čebyšev polynomials

$$p_n(t) = T_n(t/b)/T_n(1/b). \tag{31}$$

For simplicity reasons we regard here only the special case of a symmetric interval (a, b)

$$\varrho(\tilde{T}) \leqslant b = -a. \tag{32}$$

With the well-known recurrence relation for the Čebyšev polynomials $T_n(s)$

$$T_{n+1}(s) = 2sT_n(s) - T_{n-1}(s), \qquad n \geqslant 1, \tag{33}$$

one obtains, using (31) and (24),

$$T_{n+1}(1/b)(y^{n+1} - x^{\infty}) = \frac{2}{b}\tilde{T}T_n(1/b)(y^n - x^{\infty}) - T_{n-1}(1/b)(y^{n-1} - x^{\infty})$$

and with the usual abbreviations

$$(34)\qquad \omega_{n+1} = \frac{2}{b} T_n(1/b)/T_{n+1}(1/b) = 1+T_{n-1}(1/b)/T_{n+1}(1/b)$$

finally

$$(35)\qquad y^{n+1} = \omega_{n+1}(Ty^n+Qb-y^{n-1})+y^{n-1}, \quad n \geqslant 1,$$

since $T^\infty(y^n-x^\infty) = T^\infty p_n(\tilde{T})(x^0-x^\infty) = 0$. The starting vectors are

$$(36)\qquad y^0 = x^0, \quad y^1 = Ty^0+Qb$$

since $y^1-x^\infty = p_1(\tilde{T})(x^0-x^\infty)$ and $p_1(t) = t$. The parameters ω_{n+1} can be computed directly from

$$(37)\qquad \omega_{n+1} = \omega\frac{1+(\omega-1)^n}{1+(\omega-1)^{n+1}}, \quad n \geqslant 1, \quad \omega = 2/(1+\sqrt{1-b^2})$$

or by the recurrence relation (cf. [9], [10])

$$(38)\qquad \omega_{n+1} = 1/(1-b^2\omega_n/4), \quad n \geqslant 2, \quad \omega_2 = 2/(2-b^2).$$

To estimate the rate of convergence we now must use $p_n(\tilde{T})$ instead of $\tilde{T}^n$ (cf. (19)). If the bound b is sharp, $b = \varrho(\tilde{T})$, we obtain

$$\|p_n(\tilde{T})\| = 1/T_n(1/b),$$

so the average rate of convergence is

$$(39)\qquad R_n(T\text{-}SI) = -\tfrac{1}{2}\log(\omega-1)-(1/n)\log\bigl(2/(1+(\omega-1)^n)\bigr)$$

and the asymptotic rate of convergence reads

$$(40)\qquad R_\infty(T\text{-}SI) = -\tfrac{1}{2}\log(\omega-1).$$

From (37) it follows (for $b\to 0$)

$$\omega-1 = b^2/4+O(b^4),$$

therefore the rate of convergence of the SI-method is asymptotically

$$(41)\qquad R_\infty(T\text{-}SI) = -\log\bigl(b/2+O(b^3)\bigr)$$

instead of

$$R_\infty(T) = -\log b$$

(cf. (19)) in the case of the initial process (4).

We summarize the results in the following

THEOREM 5. *Let the iterative process* (2) *converge to the limit vector* x^∞, *let the iteration matrix* T *be hermitean, and let* b *be a bound for the spectral radius* $\varrho(\tilde{T}) = \varrho(T-T^\infty)$; *then the sequence* $\{y^n\}$, *defined by the semi-iterative method* (35)–(37) *converges to the same limit vector* x^∞ *and has the rate of convergence* (40).

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