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Improvement on Davenport's iterative method and new results in additive number theory, I

by

K. Thanigasalam (Monaca, Penn.)

1. Introduction (added on February 19, 1985). It was proved by Davenport [4] in 1940 that (in Waring's problem) (previously best known bounds were given in [11] and [8])

$$G^*(4) \leqslant 14, \quad G(5) \leqslant 23, \quad G(6) \leqslant 36. \tag{1.1}$$

It was also about the same time that the bounds

$$G(7) \leqslant 53, \quad G(8) \leqslant 73 \tag{1.2}$$

were established. Since then, several attempts have been made to improve on these bounds. In this series of papers, these long standing estimates will be improved to the following:

$$G^*(4) \leqslant 13, \quad G(5) \leqslant 21, \quad G(6) \leqslant 32, \quad G(7) \leqslant 45, \quad G(8) \leqslant 62, \quad G(9) \leqslant 82, \quad G(10) \leqslant 102. \tag{1.3}$$

(In a recent paper [14], the author has shown that $G(9) \leqslant 88$ and $G(10) \leqslant 104$.)

In this Part I of the series, the following results (less precise than (1.3)) will be established.

THEOREM 1. $G(7) \leqslant 50$, $G(8) \leqslant 68$.

THEOREM 2. $G(9) \leqslant 87$, $G(10) \leqslant 103$.

In Part II of this series, we again prove some general results, and deduce that $G(5) \leqslant 22$. Proof of (1.3) (which requires some modifications) will be completed in another paper elsewhere.

(When the new method was discovered, the author first obtained the bounds $G(5) \leqslant 22$, $G(6) \leqslant 34$, ... Later, using a variaton of the method, the results (1.3) have been obtained. It is understood that R. C. Vaughan also has obtained these later improvements as in (1.3).)

Several results in Hua's book [12] can be improved on by the method in this paper. Most significantly, in Waring–Goldbach problem, Hua showed that

$$H(5) \leqslant 25, \quad H(6) \leqslant 37, \quad H(7) \leqslant 55, \quad H(8) \leqslant 75 \tag{1.4}$$

(the bounds for $H(6)$, $H(7)$ and $H(8)$ being explicitly mentioned only in [10], § 33, p. 91.)

Corresponding to (1.3), it can now be shown that

$$H(6) \leqslant 33, \quad H(7) \leqslant 47, \quad H(8) \leqslant 63, \quad H(9) \leqslant 83, \quad H(10) \leqslant 107.$$

However, in this paper, only the earlier improvements over (1.4) are given. These are now improved to

THEOREM 3. $H(5) \leqslant 23$, $H(6) \leqslant 35$, $H(7) \leqslant 51$, $H(8) \leqslant 69$.

(Thus, all large odd integers are representable as the sum of 23 fifth powers of primes.)

Another additive problem considered by some mathematicians is the solubility of homogeneous additive equations. (For the definitions, see [7].) The method in this paper should give the corresponding bounds for $G^*(k)$ in that problem. Of special interest is the case $k = 7$, as the desired bound

$$G^*(k) \leqslant k^2+1 \tag{1.5}$$

is now established (for $k = 7$). R. C. Vaughan [18] showed that (1.5) holds for $11 \leqslant k \leqslant 17$. With the results in [7], the cases for which (1.5) remains to be settled are $k = 8$, 9 and 10.

The main new idea used in this paper is the Fundamental Lemma in Section 2. Davenport's improvements over the earlier bounds for $G(k)$ was based on obtaining better bounds for $U_s^{(k)}(N)$, the number of integers $\leqslant N$ that are expressible as sums of s non-negative kth powers. Together with this, Davenport used (for small values of k) Weyl's inequality for dealing with the minor arcs in the Hardy–Littlewood method. In Vinogradov's method (and its improvement given by the author in [13] and [14]), in place of Weyl's inequality, bounds for a special kind of trigonometric sum is used. In this paper, we use Weyl's inequality for $5 \leqslant k \leqslant 8$, and the method in [14] for $k = 9$ and 10. For $k \geqslant 11$, it might be possible to obtain slightly better bounds for $G(k)$ than those given in [14]. However, for this, the use of the Fundamental Lemma would involve numerous computations. Most of the methods used for obtaining better bounds for $U_s^{(k)}(N)$ have been iterative in nature. A special feature in this paper is that of devicing a framework in which Hardy–Littlewood method itself becomes iterative. (After all, the final goal of the Hardy–Littlewood method itself is to obtain the ideal bound $U_s^{(k)}(N) > N-C$ for some constant C with minimal s.)

Throughout, P is a large positive number (finally taken to be $N^{1/k}$), δ_0 a small positive constant, and ε an arbitrarily small positive number. As defined in [5], for $k \geqslant 2$, the set $\{\lambda_1, \ldots, \lambda_s\}$ of positive numbers are said to form admissible exponents if the number of solutions of the equation

$$\sum_{i=1}^{s} x_i^k = \sum_{i=1}^{s} y_i^k \quad (P^{\lambda_i} < x_i < 2P^{\lambda_i},\ P^{\lambda_i} < y_i < 2P^{\lambda_i})$$

is

$$\ll P^{\lambda_1+\ldots+\lambda_s+\varepsilon}.$$

2. THE FUNDAMENTAL LEMMA. *Let* $k \geqslant 4$, $s \geqslant 2$, $0 \leqslant \delta_i \leqslant 1$, $\mu_i = (k-1+\delta_i)/k$, $\lambda_i = \mu_s \mu_{s-1} \ldots \mu_i$, $P_i = P^{\lambda_i}$ $(1 \leqslant i \leqslant s)$, *and* $U = P_1 P_2 \ldots P_s$. *Suppose further that the sets* $\{\lambda_1, \ldots, \lambda_j\}$ *form admissible exponents for* $1 \leqslant j \leqslant s$, *and define* σ_j $(1 \leqslant j \leqslant s)$ *as follows*:

$$\sigma_j' = \max_{1 \leqslant l \leqslant k-2} \{\min(\lambda_j/2^{l+1}, [(l+2)\lambda_j - (\lambda_1+\ldots+\lambda_j)]/2^{l+1})\}, \tag{2.1}$$

$$\sigma_j'' = \begin{cases} \lambda_j/2^{k-1} & \text{if} \quad k < 12, \\ \max\limits_{l} (\lambda_j \{1-(1/2)(k-1)^2((k-2)/(k-1))^l\}/2(k-1)l) & \text{if} \quad k \geqslant 12, \end{cases} \tag{2.2}$$

and

$$\sigma_j = \begin{cases} \sigma_j' & \text{if} \quad \sigma_j' > \sigma_j'', \\ \min((1/2)\{k\lambda_j - (\lambda_1+\ldots+\lambda_j)\}, \sigma_j'') & \text{otherwise}. \end{cases} \tag{2.3}$$

Further let

$$\tau_i = k\lambda_{s-i} - (k-i-1) \quad (1 \leqslant i \leqslant l-1), \tag{2.4}$$

and S *denote the number solutions of the equation*

$$x^k + \Big(\sum_{i=1}^{s} x_i^k\Big) = y^k + \Big(\sum_{i=1}^{s} y_i^k\Big) \tag{2.5}$$

with

$$\begin{cases} P < x < 2P,\ P < y < 2P, \\ P^{\lambda_i} < x_i < 2P^{\lambda_i},\ P^{\lambda_i} < y_i < 2P^{\lambda_i}. \end{cases}$$

Then, for $1 \leqslant l \leqslant \min(k-2, s-1)$, *with*

$$\theta_r = \sum_{j=0}^{r} (\sigma_{s-j}/2^j), \qquad \varphi_r = \sum_{i=1}^{r} (\tau_i/2^{i+1}), \tag{2.6}$$

$$S \ll P^{1+\varepsilon} U + S_0 + \max_{1 \leqslant r \leqslant l-1} (S_r) + T, \tag{2.7}$$

where

$$S_0 = P^{(1/2+\delta_s+\varepsilon)} (P_s)^{1/2} U P^{-\sigma_s+\delta_0}, \tag{2.8}$$

(2.9) $$S_r = (P^{\delta_s/2+\varepsilon}U)\,P^{(1/2^{r+1}+\varphi_r+\tau_r/2^{r+1})}\Big\{\prod_{i=0}^{r}(P_{s-i})^{1/2^{i+1}}\Big\}P^{-\theta_r+\delta_0},$$

and

(2.10) $$T = (P^{\delta_s/2+\varepsilon}U)(P^{\varphi_{l-1}})\Big\{\prod_{i=0}^{l-1}(P_{s-i})^{1/2^{i+1}}\Big\}(P_1\ldots P_{s-l})^{1/2^l}P^{-\theta_{l-1}+\delta_0}$$

(δ_0 *being the sufficiently small positive constant*).

Before proving this lemma (in § 5), we introduce some notations and prove some auxiliary results. Variations of this lemma giving rise to slightly better bounds for $U_s^{(k)}(N)$ are possible. Such variations are also used for $k = 5$ and 6, but for most of our purposes, the lemma itself is sufficient.

3. Notation. For $k \geqslant 2$, define

$$f^{(k)}(X, Y; \alpha) = \sum_{X<x<Y} e(\alpha x^k), \qquad S^{(k)}(a, q) = \sum_{x=1}^{q} e_q(ax^k),$$

$$J^{(k)}(X, Y; \beta) = (1/k)\sum_{X^k<y<Y^k} y^{(1/k-1)}e(\beta y),$$

$$g^{(k)}(X, Y; a, q, \alpha) = q^{-1}S^{(k)}(a, q)J^{(k)}(X, Y; \alpha-a/q).$$

Throughout the paper, $a \leqslant q$ and $(a, q) = 1$. In some estimates, multiples of δ_0 or ε (in the exponents of large numbers) are (for convenience) simply taken as δ_0 or ε (as these can be easily adjusted).

4. Auxiliary lemmas. The next lemma is a generalization of Theorem 1 in [3].

LEMMA 4.1. *Let* $k \geqslant 3$, $s \geqslant 2$, $0 \leqslant \delta \leqslant 1$, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_s$ *with* $\lambda_s = (k-1+\delta)/k$. *Write* $P_i = P^{\lambda_i}$ $(1 \leqslant i \leqslant s)$, *and let the number of solutions of the equation*

(4.1) $$\sum_{i=1}^{s} x_i^k = \sum_{i=1}^{s} y_i^k \qquad (P_i < x_i < 2P_i,\ P_i < y_i < 2P_i)$$

be M. *Then, for* $1 \leqslant l \leqslant k-2$, *the number of solutions* S *of the equation*

(4.2) $$x^k + \Big(\sum_{i=1}^{s} x_i^k\Big) = y^k + \Big(\sum_{i=1}^{s} y_i^k\Big) \qquad (\textit{with } P < x < 2P,\ P < y < 2P)$$

satisfies

(4.3) $$S \ll PM + P^{1+\delta+\varepsilon}M\{P^{-1} + P^{-\delta-l-1}(P_1P_2\ldots P_s)^2M^{-1}\}^{1/2^l}.$$

Proof. The proof is similar to that of Theorem 1 in [3] (which we quote as Lemma 4.2 below), but requires some modifications. For a detailed proof see Lemma 4 in [18]. (The lemma can also be proved by expressing S as an integral, and using part of the argument in the proof of the Fundamental Lemma in § 5 of this paper.)

LEMMA 4.2 (Davenport). *Let* $0 \leqslant \delta \leqslant 1$, $1 < u_1 < \ldots < u_U < P^{k\lambda}$ *with*

(4.4) $$\lambda = (k-1+\delta)/k.$$

Then, for $1 \leqslant l \leqslant k-2$, *the number of solutions of*

(4.5) $$x^k + u_i = y^k + u_j \qquad (\textit{with } P < x < 2P,\ P < y < 2P)$$

is

(4.6) $$\ll PU + P^{1+\delta+\varepsilon}U\{P^{-2} + P^{-\delta-l-1}U\}^{1/2^l}.$$

For later reference, we note the following (basic) difference in the proofs of the above two lemmas. In Davenport's proof of Lemma 4.2, in order to estimate the number of solutions M_h of

(4.7) $$\Delta_{t,t_1,\ldots,t_{h-1}}(x^k) + u_j = u_i,$$

the equation

(4.8) $$\Delta_{t,t_1,\ldots,t_{h-1}}(x^k) + u_j = \Delta_{t,t_1,\ldots,t_{h-1}}(y^k) + u_{j'} = u_i$$

is considered, and the number of solutions of (4.8) is estimated to be $\ll M_h + M_{h+1}$. From these, the inequality

(4.9) $$M_h \ll P^\delta P^{h-1}U + (P^\delta P^{h-1}U)^{1/2}M_{h+1}$$

is derived. However, if the value of l is decided on, this argument (which is based on the fact that the u_i's are distinct) needs to be carried out only (once) at the last iterative step. In the previous steps, it is sufficient to use

(4.10) $$M_h \ll P^\delta P^h U + (P^\delta P^{h-1}U)^{1/2}M_{h+1},$$

which is the estimate occurring (at the iterative steps) in the proof of Lemma 4.1 (not requiring the distinctness of the integers $\sum_{i=1}^{s} x_i^k$, and M replacing U).

The next lemma is Theorem 2 in [3].

LEMMA 4.3. *If* $U_s^{(k)}(N) > N^{\alpha-\varepsilon}$, *then Lemma* 4.2 *gives the bound*

(4.11) $$U_s^{(k)}(N) > N^{\beta-\varepsilon},$$

where

(4.12) $$\beta = \max_{1\leqslant l\leqslant k-2}(1/k)\Big\{1 + \frac{(2^l-1)(k-1)+(l+1)}{2^l-1+\alpha}\alpha\Big\}.$$

Also, in this estimate, δ *in Lemma* 4.2 *is given by*

(4.13) $$\delta = \{k\beta - 1 - (k-1)\alpha\}/\alpha.$$

Note. In estimating $U_s^{(k)}(N)$, the term P^{-2} in Lemma 4.2 allows δ in

$$1/2^l < \delta \leqslant 1/2^{l-1}, \tag{4.14}$$

which is important for small values of s. When s gets large, δ has to be taken $< 1/2^l$ (generally only with $l = k-2$), so that, Lemma 4.1 is equally effective. The ideas in the two lemmas have to be combined suitably before adapting the proof of the Fundamental Lemma in estimating $U_s^{(k)}(N)$. Such a combination will be indicated in Section 9.

COROLLARY 4.1. *Let* $\{\lambda_1, \ldots, \lambda_s\}$ *form admissible exponents with*

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{s-1} \leqslant 1, \quad \lambda_s = 1, \quad P_i = P^{\lambda_i}, \quad \text{so that } P_s = P; \quad U = P_1 \ldots P_s. \tag{4.15}$$

Then, for $1 \leqslant l \leqslant k-2$, *the number of solutions of the equation*

$$x^k + \Big(\sum_{i=1}^{s} x_i^k\Big) = y^k + \Big(\sum_{i=1}^{s} y_i^k\Big) \quad \text{with} \quad \begin{cases} P_i < x_i < 2P_i,\ P_i < y_i < 2P_i, \\ P < x < 2P,\ P < y < 2P \end{cases} \tag{4.16}$$

is

$$\ll P^{1+\varepsilon} U + P^{2+\varepsilon} U \{P^{-1} + P^{-l-2} U\}^{1/2^l}. \tag{4.17}$$

Proof. In Lemma 4.1, take $\delta = 1$, and use $M \ll P^{\varepsilon} U$.

With the definitions in Section 3, the next two lemmas correspond to Lemmas 5 and 8 in [2].

LEMMA 4.4. *If* $\alpha = a/q + \beta$, $|\beta| \leqslant 1/2$, *then*

$$g^{(k)}(P, 2P; a, q, \alpha) \ll q^{-1/k} \min(P, P^{1-k} |\beta|^{-1}).$$

LEMMA 4.5. *If* $\alpha = a/q + \beta$, $q \leqslant P^{1-\delta_0}$, $|\beta| \leqslant q^{-1} P^{1-k-\delta_0}$, *then*

$$f^{(k)}(P, 2P; \alpha) - g^{(k)}(P, 2P; a, q, \alpha) \ll q^{3/4+\varepsilon}.$$

LEMMA 4.6 (Weyl's inequality). *If* $\alpha = a/q + \beta$, *with* $P^{1-\delta_0} < q \leqslant P^{k-1+\delta_0}$, *and* $|\beta| \leqslant q^{-1} P^{1-k-\delta_0}$, *then*

$$f^{(k)}(P, 2P; \alpha) \ll P^{1-1/2^{k-1}+\delta_0}.$$

LEMMA 4.7 (Vinogradov). *If* $\alpha = a/q + \beta$, $P < q \leqslant P^{k-1}$ *and* $|\beta| \leqslant q^{-1} P^{1-k}$, *then*

$$f^{(k)}(P, 2P; \alpha) \ll P^{1-\sigma''(k)+\varepsilon} \quad (\text{for } k \geqslant 12),$$

where

$$\sigma''(k) = \max_l \Big(\big\{1 - (1/2)(k-1)^2 \big((k-2)/(k-1)\big)^l\big\}/2(k-1)l\Big). \tag{4.18}$$

(For this, see Theorem 5.3 in [19], which is an improved version of Theorem 9 in [12].)

LEMMA 4.8. *Let* $k \geqslant 4$. *Then, with the same premises as in Corollary* 4.1, *and the condition that* $\lambda_i = (k-1+\delta_i)\lambda_{i+1}/k$ (*with* $0 \leqslant \delta_i \leqslant 1$ *for* $1 \leqslant i \leqslant s-1$), *the number of solutions of* (4.16) *is also*

$$\ll P^{2-2\sigma(k)+\delta_0} U, \tag{4.19}$$

where (with $\sigma''(k) = 1/2^{k-1}$ *for* $k < 12$, *and defined by* (4.18) *for* $k \geqslant 12$)

$$\sigma(k) = \min\big((1/2)\{k - (\lambda_1 + \ldots + \lambda_s)\},\ \sigma''(k)\big). \tag{4.20}$$

Proof. We prove the result for $4 \leqslant k < 12$, and only a slight modification will be required for the cases $k \geqslant 12$.

Write

$$f = f(\alpha) = f^{(k)}(P, 2P; \alpha), \quad f_i = f_i(\alpha) = f^{(k)}(P_i, 2P_i; \alpha), \tag{4.21}$$

$$g = g^{(k)}(P, 2P; a, q, \alpha), \quad g_i = g^{(k)}(P_i, 2P_i; a, q, \alpha), \tag{4.22}$$

$$Q = P^{k-1+\delta_0}. \tag{4.23}$$

(Note that, here $f_s = f$, $g_s = g$ since $P_s = P$.)

Then, the number of solutions S of (4.16) is given by

$$S = \int_{Q^{-1}}^{1+Q^{-1}} |f f_1 f_2 \ldots f_s|^2 \, d\alpha. \tag{4.24}$$

Divide $Q^{-1} < \alpha < 1 + Q^{-1}$ into basic intervals $\mathfrak{M}$ and supplementary intervals m as follows:

$$\text{For } 1 \leqslant q \leqslant P^{k/2^{k-1}},\ \mathfrak{M}_{a,q} = \{\alpha\colon |\alpha - a/q| \leqslant q^{-1} Q^{-1}\},\ \mathfrak{M} = \bigcup_{a,q} \mathfrak{M}_{a,q}, \quad \text{and } m = (Q^{-1}, 1+Q^{-1}) \backslash \mathfrak{M}. \tag{4.25}$$

I. *Integral over* m. First consider $\alpha = a/q + \beta$ with $P^{k/2^{k-1}} < q \leqslant P^{1-\delta_0}$, $|\beta| \leqslant q^{-1} Q^{-1}$. Then, by Lemmas 4.4 and 4.5,

$$f \ll q^{3/4+\varepsilon} + q^{-1/k} P \ll P^{1-1/2^{k-1}} \tag{4.26}$$

since $P^{3/4} \ll P^{1-1/2^{k-1}}$ (for $k \geqslant 3$). This, together with Lemma 4.6 shows that on m, $f \ll P^{1-1/2^{k-1}+\delta_0}$.

Hence, since $\{\lambda_1, \ldots, \lambda_s\}$ form admissible exponents,

$$\int_m |f f_1 f_2 \ldots f_s|^2 d\alpha \ll P^{2(1-1/2^{k-1}+\delta_0)} \int_{Q^{-1}}^{1+Q^{-1}} |f_1 f_2 \ldots f_s|^2 \, d\alpha \ll P^{2(1-1/2^{k-1}+\delta_0)} (P_1 P_2 \ldots P_s) P^{\varepsilon}. \tag{4.27}$$

II. *Integral over* $\mathfrak{M}$. First we consider $s \geqslant k-1$. It is an easy verification (see (20) in [13]) that

$$(4.28)\qquad (1-1/k)^{k-2} > \begin{cases} 1/4 & \text{for} \quad k \geqslant 6, \\ 1/2 & \text{for} \quad k = 4,\ 5. \end{cases}$$

Since $\lambda_s = 1$, it follows that for $s-k+2 \leqslant i \leqslant s$ (using (4.28)),

$$(4.29)\qquad \lambda_i > (1-1/k)^{k-2} > k/2^{k-1} \qquad (k \geqslant 4).$$

Hence,

$$P_i = P^{\lambda_i} > P^{k/2^{k-1}} \quad \text{and} \quad q^{-1}Q^{-1} \leqslant q^{-1}P_i^{1-k-\delta_0} \qquad (s-k+2 \leqslant i \leqslant s).$$

Thus, from (4.25) and Lemma 4.5, it follows that on $\mathfrak{m}$,

$$(4.30)\qquad f_i - g_i \ll q^{3/4+\varepsilon} \qquad (s-k+2 \leqslant i \leqslant s).$$

It is an easy deduction from this and Lemma 4.4, that (on $\mathfrak{m}$)

$$(4.31)\qquad f_i \ll q^{-1/k} P_i \qquad (s-k+2 \leqslant i \leqslant s).$$

Also, on $\mathfrak{m}$,

$$q^{-1/k} P^{1-k} |\beta|^{-1} \geqslant q^{-1/k} P^{1-k} (qP^{k-1+\delta_0}) \geqslant q^{3/4+\varepsilon} \qquad (k \geqslant 4).$$

Hence, from (4.31) and Lemma 4.4 (on $\mathfrak{m}$),

$$(4.32)\qquad f \ll q^{-1/k} \min(P,\ P^{1-k}|\beta|^{-1}).$$

From (4.31) and (4.32) (using $s \geqslant k-1$),

$$(4.33)\qquad \int\limits_{\mathfrak{m}} |ff_1 f_2 \ldots f_s|^2\,d\alpha$$

$$\ll \sum_{q \leqslant P^{k/2^{k-1}}} \sum_a q^{-2} (PP_1 P_2 \ldots P_s)^2 \int\limits_0^{q^{-1}Q^{-1}} \min(P,\ P^{1-k}|\beta|^{-1})^2\,d\beta$$

$$\ll (PP_1 P_2 \ldots P_s)^2 P^{-k} \sum_{q \leqslant P^{k/2^{k-1}}} q^{-1} \ll (PP_1 P_2 \ldots P_s)^2 P^{-k+\varepsilon}.$$

If $s < k-1$, estimate (4.33) has to be replaced by

$$(4.34)\qquad \int\limits_{\mathfrak{m}} |ff_1 f_2 \ldots f_s|^2\,d\alpha \ll (PP_1 P_2 \ldots P_s)^2 P^{-k} \sum_{q \leqslant P^{k/2^{k-1}}} q^{-2(s+1)/k+1}$$

$$\ll (PP_1 P_2 \ldots P_s)^2 P^{-k} P^{(k-s-1)/2^{k-2}}$$

$$\ll (P^2 U) P^{-(k-s)} P^{(k-s-1)/2^{k-2}}$$

using $U = P_1 P_2 \ldots P_s$ and $U \ll P^s$. Obviously,

$$(4.35)\qquad P^{-(k-s)} P^{(k-s-1)/2^{k-2}} \ll P^{-2\sigma(k)} \quad \text{with} \quad \sigma(k) = 1/2^{k-1}.$$

Also,

$$(P_1 P_2 \ldots P_s) P^{-k+\varepsilon} \ll P^{-\{k-(\lambda_1+\ldots+\lambda_s)\}+\varepsilon} \ll P^{-2\sigma(k)+\varepsilon}$$

with

$$\sigma(k) = (1/2)\{k-(\lambda_1+\ldots+\lambda_s)\}.$$

Hence, result follows from (4.24), (4.27), (4.33), (4.34) and (4.20). For $k \geqslant 12$, it is necessary to make only the following changes. Take $Q = P^{k-1}$ in place of (4.23), and define the $\mathfrak{m}_{a,q}$'s with $|\alpha - a/q| \leqslant q^{-1}Q^{-1}$ for $q \leqslant P^{k\sigma''(k)}$, where $\sigma''(k)$ is defined by (4.18). (4.31) and (4.32) can be derived from $f_i - g_i \ll q^{3/4+\varepsilon}\max(1,\ P_i^k|\beta|)$ (and similar estimate for $f-g$), which is obtained by a partial summation with the result in [6]. In place of (4.35), we use

$$P^{-(k-s)} P^{2(k-s-1)\sigma''(k)} \ll P^{-2\sigma''(k)} \qquad (\text{since } \sigma''(k) < 1/2).$$

The rest of the argument is precisely the same as for $k < 12$.

Lemma 4.9. *Let the λ's be defined as in Lemma 4.8, and σ_j by* (2.3). *Then, if $\{\lambda_1, \ldots, \lambda_j\}$ form admissible exponents, and the functions f, f_i are defined by* (4.21),

$$(4.36)\qquad \int\limits_0^1 |f_j|^2 |f_1 f_2 \ldots f_j|^2\,d\alpha \ll P_j^2 P^{-2\sigma_j+\sigma_0} (P_1 P_2 \ldots P_j)$$

(*using* $P_j^{\sigma(k)} = P^{\sigma_j}$).

Proof. The proof follows from Corollary 4.1 and Lemma 4.8, where P is replaced by P_j and U by $(P_1 P_2 \ldots P_j)$.

Lemma 4.10. *Let the integers $t_1, \ldots, t_s$ be such that $1 \leqslant t_i \leqslant L_i$ $(1 \leqslant i \leqslant s)$. Then, the number of (different) sets $\{t_1, t_2, \ldots, t_s\}$ satisfying $1 \leqslant t_1 t_2 \ldots t_s \leqslant L$ is*

$$\ll L^{1+\varepsilon}.$$

Proof. This is easily proved by induction on s. Assume the result for s. Since the number of representations of m in the form $m = t_1 t_2 \ldots t_s$ is $\ll L^\varepsilon$, the number of (different) sets $\{t_1, t_2, \ldots, t_s, t_{s+1}\}$ with $1 \leqslant t_1 t_2 \ldots \ldots t_s t_{s+1} \leqslant L$ is

$$\ll L^\varepsilon \sum_{t_{s+1}=1}^{L} \Big\{ \sum_{1 \leqslant t_1 t_2 \ldots t_s \leqslant (L/t_{s+1})} 1 \Big\} \ll L^\varepsilon \sum_{t_{s+1}=1}^{L} (L/t_{s+1}) \cdot L^\varepsilon \ll L^{1+3\varepsilon}.$$

The result now follows since it is true for $s = 1$.

5. Proof of the Fundamental Lemma. With the $\delta_i, \mu_i, \lambda_i, P_i$ $(1 \leqslant i \leqslant s)$ as defined in the lemma, and (as in (4.21)) $f_i = f_i(\alpha) = \sum_{P_i < x < 2P_i} e(\alpha x^k)$, write

$$(5.1)\qquad I_j = \int\limits_0^1 |f_j|^2 |f_1 f_2 \ldots f_j|^2\,d\alpha, \qquad R_j = \int\limits_0^1 |f_1 f_2 \ldots f_j|^2\,d\alpha \qquad (1 \leqslant j \leqslant s).$$

The functions $\Delta_{t,t_1,\ldots,t_r}(x^k)$ have the same meaning as in [3]. Since $\{\lambda_1, \ldots, \lambda_s\}$ form admissible exponents, the number of solutions of (2.5) with $x = y$ is $\ll P^{1+\varepsilon} U$ (using $U = P_1 P_2 \ldots P_s$).

Let the number of solutions with $y = x+t$, $t > 0$ be M_1. Then,

$$S \ll P^{1+\varepsilon} U + 2M_1. \tag{5.2}$$

M_1 is the number of solutions of

$$\Delta_t(x^k) + \Big(\sum_{i=1}^{s} y_i^k\Big) = \Big(\sum_{i=1}^{s} x_i^k\Big) \quad (t > 0). \tag{5.3}$$

Let $\mathscr{A}$ be the set of t's that actually count in the solutions of (5.3). Since (cf. (6) in [3]) $0 < t < P^{\delta_s}$, we have

$$\operatorname{Card} \mathscr{A} < P^{\delta_s}. \tag{5.4}$$

Write

$$F = F(\alpha) = \sum_{t \in \mathscr{A}} \sum_{P < x < 2P} e\big(\Delta_t(x^k)\alpha\big), \tag{5.5}$$

so that

$$M_1 = \int_0^1 F(\alpha) |f_1 f_2 \ldots f_s|^2 \, d\alpha.$$

Hence, by Schwarz's inequality,

$$\begin{aligned} M_1 &\leqslant \int_0^1 |F(\alpha)| \, |f_1 f_2 \ldots f_s|^2 \, d\alpha \\ &\leqslant \Big\{\int_0^1 |F(\alpha)|^2 |f_1 \ldots f_{s-1}|^2 \, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_s|^2 |f_1 \ldots f_s|^2 \, d\alpha\Big\}^{1/2}. \end{aligned} \tag{5.6}$$

Now, by Cauchy's inequality,

$$|F(\alpha)|^2 \leqslant (\operatorname{Card} \mathscr{A}) \Big\{\sum_{t \in \mathscr{A}} \Big|\sum_{P < x < 2P} e\big(\Delta_t(x^k)\alpha\big)\Big|^2\Big\},$$

and

$$\begin{aligned} \Big|\sum_{P<x<2P} e\big(\Delta_t(x^k)\alpha\big)\Big|^2 &= P + \sum_{P<x<2P} \sum_{\substack{P<y<2P \\ x \neq y}} e\big((\Delta_t(y^k) - \Delta_t(x^k))\alpha\big) \\ &= P + F^*(\alpha) \quad \text{(say)}. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_0^1 |F(\alpha)|^2 |f_1 \ldots f_{s-1}|^2 \, d\alpha \\ &\leqslant (\operatorname{Card} \mathscr{A})^2 P \int_0^1 |f_1 \ldots f_{s-1}|^2 \, d\alpha + (\operatorname{Card} \mathscr{A}) \int_0^1 \Big\{\sum_{t \in \mathscr{A}} F^*(\alpha)\Big\} |f_1 \ldots f_{s-1}|^2 \, d\alpha. \end{aligned} \tag{5.7}$$

Also, $\int_0^1 \{\sum_{t \in \mathscr{A}} F^*(\alpha)\} |f_1 \ldots f_{s-1}|^2 \, d\alpha$ is equal to the number of solutions of

$$\Delta_t(y^k) - \Delta_t(x^k) = \Big(\sum_{i=1}^{s-1} y_i^k\Big) - \Big(\sum_{i=1}^{s-1} x_i^k\Big) \quad \text{with} \quad x \neq y. \tag{5.8}$$

The number of solutions of (5.8) with $x > y$ is equal to that with $x < y$; so that the number of solutions of (5.8) is $\leqslant 2M_2$, where M_2 is the number of solutions of (putting $y = x + t_1$)

$$\Delta_{t,t_1}(x^k) + \Big(\sum_{i=1}^{s-1} x_i^k\Big) = \Big(\sum_{i=1}^{s-1} y_i^k\Big) \quad (0 < t_1 \leqslant P). \tag{5.9}$$

Thus, from (5.1), (5.6) and (5.7),

$$M_1 \ll (\operatorname{Card} \mathscr{A})(P R_{s-1} I_s)^{1/2} + \{(\operatorname{Card} \mathscr{A}) I_s M_2\}^{1/2}. \tag{5.10}$$

Let $\mathscr{A}_1$ be the set of $\{t, t_1\}$ that count in the solutions of (5.9). Now

$$\Big(\sum_{i=1}^{s-1} y_i^k\Big) - \Big(\sum_{i=1}^{s-1} x_i^k\Big) \ll P_{s-1}^k \ll P^{k\lambda_{s-1}},$$

and since $x > P$, $\Delta_{t,t_1}(x^k) > t t_1 P^{k-2}$. Hence,

$$1 \leqslant t t_1 \ll P^{k\lambda_{s-1} - (k-2)} = P^{\tau_1} \quad \text{(by (2.4))}; \tag{5.11}$$

so that, from Lemma 4.10,

$$\operatorname{Card} \mathscr{A}_1 \ll P^{\tau_1 + \varepsilon}. \tag{5.12}$$

With

$$F_1 = F_1(\alpha) = \sum_{\{t,t_1\} \in \mathscr{A}_1} \sum_{P < x < 2P} e\big(\Delta_{t,t_1}(x^k)\alpha\big),$$

we see that M_2 (the number of solutions of (5.9)) is given by

$$\begin{aligned} M_2 &= \int_0^1 F_1(\alpha) |f_1 \ldots f_{s-1}|^2 \, d\alpha \\ &\leqslant \Big\{\int_0^1 |F_1(\alpha)|^2 |f_1 \ldots f_{s-2}|^2 \, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_{s-1}|^2 |f_1 \ldots f_{s-1}|^2 \, d\alpha\Big\}^{1/2}. \end{aligned}$$

Estimating $|F_1(\alpha)|^2$ in the same way as $|F(\alpha)|^2$, and arguing as above, we have

$$M_2 \ll (\operatorname{Card} \mathscr{A}_1)(P R_{s-2} I_{s-1})^{1/2} + \{(\operatorname{Card} \mathscr{A}_1) I_{s-1} M_3\}^{1/2}, \tag{5.13}$$

where M_3 is the number of solutions of

$$\Delta_{t,t_1,t_2}(x^k) + \Big(\sum_{i=1}^{s-2} x_i^k\Big) = \Big(\sum_{i=1}^{s-2} y_i^k\Big) \quad (0 < t_2 \leqslant P). \tag{5.14}$$

Repeating the above process, we define the sets $\mathscr{A}_r = [\{t, t_1, \ldots, t_r\}]$, and the functions $F_r(\alpha)$ $(1 \leqslant r \leqslant l-1)$ as follows:

M_{r+1} denotes the number of solutions of

$$\Delta_{t,t_1,\ldots,t_r}(x^k)+\Big(\sum_{i=1}^{s-r} x_i^k\Big) = \Big(\sum_{i=1}^{s-r} y_i^k\Big), \tag{5.15}$$

and the set of $\{t, t_1, \ldots, t_r\}$ that count in these solutions is $\mathscr{A}_r$. Also,

$$F_r(\alpha) = \sum_{\{t,t_1,\ldots,t_r\}\in\mathscr{A}_r} \sum_{P<x<2P} e\big((\Delta_{t,t_1,\ldots,t_r}(x^k))\alpha\big).$$

As in the proof of (5.12),

$$\mathrm{Card}\,\mathscr{A}_r \ll P^{\tau_r+\varepsilon} \quad (1 \leqslant r \leqslant l-1). \tag{5.16}$$

At the rth stage, we use Schwarz's inequality in the form

$$M_{r+1} = \int_0^1 F_r(\alpha)|f_1 \ldots f_{s-r}|^2\, d\alpha$$
$$\leqslant \Big\{\int_0^1 |F_r(\alpha)|^2 |f_1 \ldots f_{s-r-1}|^2\, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_{s-r}|^2 |f_1 \ldots f_{s-r}|^2\, d\alpha\Big\}^{1/2},$$

and for the transition from M_{r+1} to M_{r+2}, estimate $|F_r(\alpha)|^2$ as before. As in the proof of (5.13), we then have (for $1 \leqslant r \leqslant l-1$)

$$M_{r+1} \ll (\mathrm{Card}\,\mathscr{A}_r)(PR_{s-r-1}I_{s-r})^{1/2}+\{(\mathrm{Card}\,\mathscr{A}_r)I_{s-r}M_{r+2}\}^{1/2}. \tag{5.17}$$

From (5.2), (5.10) and (5.17) for $1 \leqslant r \leqslant l-1$, we have (inductively),

$$S \ll P^{1+\varepsilon}U+(\mathrm{Card}\,\mathscr{A})(PR_{s-1}I_s)^{1/2}+\Big\{\sum_{r=1}^{l-1} S_r'\Big\}+T', \tag{5.18}$$

with

(5.19)

$$S_r' = \{(\mathrm{Card}\,\mathscr{A})I_s\}^{1/2}\Big\{\prod_{i=1}^{r}[(\mathrm{Card}\,\mathscr{A}_i)I_{s-i}]^{1/2^{i+1}}\Big\}\{(\mathrm{Card}\,\mathscr{A}_r)PR_{s-r-1}\}^{1/2^{r+1}},$$

and

$$T' = \{(\mathrm{Card}\,\mathscr{A})I_s\}^{1/2}\Big\{\prod_{i=1}^{l-1}[(\mathrm{Card}\,\mathscr{A}_i)I_{s-i}]^{1/2^{i+1}}\Big\}(M_{l+1})^{1/2^l}, \tag{5.20}$$

where M_{l+1} is the number of solutions of

$$\Delta_{t,t_1,\ldots,t_l}(x^k) = \Big(\sum_{i=1}^{s-l} y_i^k\Big)-\Big(\sum_{i=1}^{s-l} x_i^k\Big) \quad (tt_1 \ldots t_l > 0). \tag{5.21}$$

For given $x_1, \ldots, x_{s-l}, y_1, \ldots, y_{s-l}$, there are $\ll P^\varepsilon$ choices for $t, t_1, \ldots, t_l$ as divisors of the right-hand side terms in (5.21), and then, x is determined uniquely. Accordingly,

$$M_{l+1} \ll (P_1P_2 \ldots P_{s-l})^2 P^\varepsilon. \tag{5.22}$$

Furthermore, since $\{\lambda_1, \ldots, \lambda_j\}$ form admissible exponents (for $j \geqslant 1$), we have (cf. (5.1))

$$R_j \ll (P_1P_2 \ldots P_j)P^\varepsilon \quad (j \geqslant 1), \tag{5.23}$$

and by (4.36),

$$I_j \ll P_j^2 P^{-2\sigma_j+\delta_0}(P_1P_2 \ldots P_j) = P_j^3 P^{-2\sigma_j+\delta_0}(P_1P_2 \ldots P_{j-1}) \quad (j \geqslant 2). \tag{5.24}$$

Hence, writing (with θ_r defined by (2.6))

$$\varrho_i = \Big(\sum_{j=1}^{i} 1/2^j\Big)+3/2^{i+1} = 1+1/2^{i+1}, \quad \xi_r = \sum_{j=1}^{r+1} 1/2^j = 1-1/2^{r+1},$$

we have

(5.25)
$$I_s^{1/2}\Big\{\prod_{i=1}^{r}(I_{s-i})^{1/2^{i+1}}\Big\}$$
$$\ll P^{-\theta_r+\delta_0}(P_s)^{3/2}\Big\{\prod_{i=1}^{r}P_{s-i}^{\varrho_i}\Big\}(P_1P_2 \ldots P_{s-r-1})^{\xi_r}$$
$$\ll P^{-\theta_r+\delta_0}P_s^{1/2}\Big\{\prod_{i=1}^{r}(P_{s-i})^{1/2^{i+1}}\Big\}\times$$
$$\times(P_sP_{s-1} \ldots P_{s-r})(P_1P_2 \ldots P_{s-r-1})^{1-1/2^{r+1}};$$

so that (from (5.23))

$$I_s^{1/2}\Big\{\prod_{i=1}^{r}(I_{s-i})^{1/2^{i+1}}\Big\}(R_{s-r-1})^{1/2^{r+1}} \ll P^{-\theta_r+\delta_0}\Big\{\prod_{i=0}^{r}(P_{s-i})^{1/2^{i+1}}\Big\}U, \tag{5.26}$$

using $P_1P_2 \ldots P_s = U$. Also, from (5.23) and (5.24),

$$(R_{s-1}I_s)^{1/2} \ll (P_1P_2 \ldots P_{s-1})^{1/2}(P^{-2\sigma_s+\delta_0})^{1/2}(P_1P_2 \ldots P_{s-1}P_s^3)^{1/2} \ll P^{-\sigma_s+\delta_0}P_s^{1/2}U; \tag{5.27}$$

from (5.4) and (5.16),

$$(\mathrm{Card}\,\mathscr{A})^{1/2}\Big\{\prod_{i=1}^{r}(\mathrm{Card}\,\mathscr{A}_i)^{1/2^{i+1}}\Big\} \ll (P^{\delta_s/2})(P^{\varphi_r+\varepsilon}) \quad (\text{cf. } (2.6)). \tag{5.28}$$

With S_0, S_r defined by (2.8) and (2.9), we see that from (5.19), (5.26) and (5.28),

$$S_r' \ll S_r \quad (1 \leqslant l \leqslant r-1); \tag{5.29}$$

from (5.27) and (5.4),

$$(\operatorname{Card} \mathscr{A})(PR_{s-1} I_s)^{1/2} \ll S_0. \tag{5.30}$$

Now, with $r = l-1$, (5.25) gives

$$I_s^{1/2} \Big\{\prod_{i=1}^{l-1} (I_{s-i})^{1/2^{i+1}}\Big\} \ll P^{-\theta_{l-1}+\delta_0} \Big\{\prod_{i=0}^{l-1} (P_{s-i})^{1/2^{i+1}}\Big\} \times \tag{5.31}$$
$$\times (P_s P_{s-1} \ldots P_{s-l+1})(P_1 P_2 \ldots P_{s-l})^{1-1/2^l}.$$

This, in conjunction with (2.10), (5.20), (5.22), and (5.28) (with $r = l-1$) yields

$$T' \ll T \quad (\text{using } U = P_1 P_2 \ldots P_s). \tag{5.32}$$

(2.7) now follows from (5.18), (5.29), (5.30), and (5.32), proving the assertion of the lemma.

6. Results pertaining to computations. The use of the Fundamental Lemma would generally involve numerous computations. In this section, we derive some results using which most of the computations can be avoided. The following notes are also useful.

Note 1. For obtaining better bounds for $G(k)$ (by using Weyl's inequality), the improvements on the bounds for $U_s^{(k)}(N)$ should be comparable to $N^{1/k(2^{k-1})}$.

Note 2. In the usage of the iterative method, the improvement N^α obtained on the bound for $U_s^{(k)}(N)$ will be reduced to approximately $N^{\alpha(1-1/k)^r}$ in estimating $U_{s+r}^{(k)}(N)$ (since the δ's are small). Hence, slight improvements on the bounds for small values of s will not generally contribute towards a better estimate for $G(k)$.

Note 3. For $k \geqslant 5$, the Fundamental Lemma gives better bounds than Davenport's method except for the first few values of s. However, these improvements are substantial only when s gets large. Thus, in order to avoid several computations, we use only Davenport's method up to certain values of s (depending on k).

Note 4. In using (2.7), we choose δ_s so that each term in the estimate is $\ll P^{1+\varepsilon} U$. The computations of the best value δ of δ_s allowable by the method would be complicated. The factor $P^{-\theta_{l-1}}$ (in (2.9) and (2.10)) provides the improvements over Davenport's method. By estimating θ_{l-1} (and the δ's given by Davenport's method), the values of δ are chosen approximately.

Note 5. The method of Theorem 4 in [3] can also be incorporated by making slight changes in the method (and would give slight improvements in some cases). However, for estimating $H(k)$, we have to avoid this. The purpose of the next lemma is to simplify computations.

LEMMA 6.1. *Let* $\{\delta_i\}$ *satisfy*

$$0 \leqslant \delta_i \leqslant 1, \tag{6.1}$$

and

$$1 - \lambda_{s-r+1} \{(1+1/k) - (2+1/k)\delta_{s-r}\} \geqslant 2\sigma_{s-r}. \tag{6.2}$$

Then (with S_r *defined by* (2.8) *and* (2.9))

$$S_{r-1} \ll S_r \quad (1 \leqslant r \leqslant l-1). \tag{6.3}$$

Proof. From (2.4) (letting $\tau_0 = \delta_s$),

$$\tau_r - \tau_{r-1} = 1 - k(\lambda_{s-r+1} - \lambda_{s-r}) = 1 - k\lambda_{s-r+1}(1-\mu_{s-r})$$
$$= 1 - \lambda_{s-r+1}(1-\delta_{s-r}).$$

Also,

$$1 - \lambda_{s-r} = 1 - \lambda_{s-r+1}(1 - 1/k + \delta_{s-r}/k).$$

Hence,

$$2(\tau_r - \tau_{r-1}) - (1-\lambda_{s-r}) = 1 - \lambda_{s-r+1} \{(1+1/k) - (2+1/k)\delta_{s-r}\}. \tag{6.4}$$

Thus, if

$$\varkappa = 2(\tau_r - \tau_{r-1}) - (1-\lambda_{s-r}) - 2\sigma_{s-r}, \tag{6.5}$$

it follows from (6.2) and (6.4) that

$$\varkappa \geqslant 0. \tag{6.6}$$

Now from (2.9) and (2.8), we have (for $1 \leqslant r \leqslant l-1$)

$$S_{r-1}/S_r = \{P^{(1+\tau_{r-1})/2^r} P^{\sigma_{s-r}/2^r}\} / \{P^{(1/2^{r+1}+\tau_r/2^r)} (P_{s-r})^{1/2^{r+1}}\}$$
$$= \{P/P_{s-r}\}^{1/2^{r+1}} \{P^{\sigma_{s-r}}/P^{(\tau_r - \tau_{r-1})}\}^{1/2^r}.$$

Hence, since $P/P_{s-r} = P^{1-\lambda_{s-r}}$, we have from (6.5) and (6.6),

$$S_{r-1}/S_r = P^{-\varkappa/2^{r+1}} \ll 1,$$

proving (6.3).

Remark. The following simplifications may be noted in using the above result. Even without the factor $P^{-\sigma_s}$ (cf. (2.8)), it is easily verified that $S_0 \ll P^{1+\varepsilon} U$ if $\delta_s \leqslant 1/(2k+1)$. Also, if $r \geqslant 2$, and the δ's are small, the expression on the left-hand side in (6.2) will be bounded from below by approximately $\{1-(1-1/k)^r(1+1/k)\}$, which (with the general choice

$\sigma_j = \lambda_j/2^{k-1}$) exceeds $2\sigma_{s-r}$ (by a sufficient margin). If (6.2) holds, there will be no need to estimate S_r for $0 \leqslant r \leqslant l-2$. However, when variations of the method are used, the changed S_i's have to be estimated separately.

The next lemma follows directly from (2.9) and (2.10).

LEMMA 6.2. $S_{l-1}/T = \{P^{(1+\tau_{l-1})}/(P_1 P_2 \ldots P_{s-l})\}^{1/2^l}$.

Once we estimate T, S_{l-1} is estimated easily by using Lemma 6.2. We also note the following, which is useful in estimating T: (With the estimates $U_i^{(k)}(N) > N^{\alpha_i - \varepsilon}$ in the iterative methods.)

(6.7) $$P_1 P_2 \ldots P_i = P^{k\lambda_i \alpha_i} \quad (\text{with } P_i = P^{\lambda_i});$$

so that, from (2.10),

(6.8) $$T \ll (P^{\delta_s/2+\varepsilon} U)(P^{\varphi_{l-1}}) \Big\{\prod_{i=0}^{l-2} (P_{s-i})^{1/2^{l+1}}\Big\} (P^{-\theta_{l-1}+\delta_0}) \{P^{k\lambda_{s-l+1}\alpha_{s-l+1}}\}^{1/2^l}.$$

7. A lemma on admissible exponents. The following lemma will be used in the proof of Lemma 8.2 in the next section.

LEMMA 7.1. *If the set $\Lambda = \{\lambda_1, \ldots, \lambda_s\}$ form admissible exponents, then every subset of Λ also form admissible exponents.*

Proof. This is easily proved by induction, by showing that every subset of Λ with $(s-1)$ terms form admissible exponents. Let

$$\Lambda' = \left\{\lambda_i \,\middle|\, \begin{matrix} 1 \leqslant i \leqslant s \\ i \neq r \end{matrix}\right\} \quad \text{for any given } r \ (1 \leqslant r \leqslant s),$$

and let the number of solutions of the equation

(7.1) $$\sum_{\substack{i=1 \\ i\neq r}}^{s} x_i^k = \sum_{\substack{i=1 \\ i\neq r}}^{s} y_i^k$$

be M. Then, the number of solutions of

(7.2) $$\sum_{i=1}^{s} x_i^k = \sum_{i=1}^{s} y_i^k$$

with $x_r = y_r$ is $\ll P^{\lambda_r} M$. But, by hypothesis, the number of solutions of (7.2) is $\ll P^{\lambda_1+\ldots+\lambda_s+\varepsilon}$, so that,

$$M \ll P^{(\lambda_1+\ldots+\lambda_{r-1})+(\lambda_{r+1}+\ldots+\lambda_s)+\varepsilon},$$

showing that Λ' form admissible exponents.

8. Further preliminary lemmas. In this section, we introduce some variations of the Fundamental Lemma. These results (while applicable for all values of k) will be used for $k = 5$ and 6.

We recall that

(8.1) $$f_i = \sum_{P_i < x < 2P_i} e(\alpha x^k)$$

with

(8.2) $$P_i = P^{\lambda_i}.$$

LEMMA 8.1. *Under the same definitions and hypotheses as in the Fundamental Lemma, the number of solutions S of* (2.5) *also satisfies* (2.7) *with S_{l-1} replaced by S''_{l-1}, and T by T'', where*

(8.3) $$S''_{l-1} = S_{l-1}\{P^{\sigma_{s-l+1}/2^{l-1}}\}/(P_{s-l+1})^{1/2^l},$$

and

(8.4) $$T'' = T\{P^{\sigma_{s-l+1}/2^{l-1}}\}$$

(*the other terms in* (2.7) *remaining the same*).

Proof. We have to make the following changes in the proof of the Fundamental Lemma. In that proof, at the last stage (with $r = l-1$), the integral

$$\int_0^1 F_{l-1}(\alpha)|f_1 \ldots f_{s-l+1}|^2 \, d\alpha$$

was estimated with

(8.5) $$\Big\{\int_0^1 |F_{l-1}(\alpha)|^2 |f_1 \ldots f_{s-l}|^2 \, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_{s-l+1}|^2 |f_1 \ldots f_{s-l+1}|^2 \, d\alpha\Big\}^{1/2}$$

to get (cf. (5.17) with $r = l-1$)

(8.6) $$M_l \ll (\operatorname{Card} \mathscr{A}_{l-1})(PR_{s-l} I_{s-l+1})^{1/2} + \{(\operatorname{Card} \mathscr{A}_{l-1}) I_{s-l+1} M_{l+1}\}^{1/2}.$$

Here, in place of (8.5), we estimate with

(8.7) $$\Big\{\int_0^1 |F_{l-1}(\alpha)|^2 |f_1 \ldots f_{s-l+1}|^2 \, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_1 \ldots f_{s-l+1}|^2 \, d\alpha\Big\}^{1/2}$$

to get (by the same arguments)

(8.8) $$M_l \ll (\operatorname{Card} \mathscr{A}_{l-1})(PR_{s-l+1}^2)^{1/2} + \{(\operatorname{Card} \mathscr{A}_{l-1}) R_{s-l+1} M_{l+1}\}^{1/2},$$

where M_{l+1} is the number of solutions of (in place of (5.21))

(8.9) $$\Delta_{l,t_1,\ldots,t_l}(x^k) = \sum_{i=1}^{s-l+1} (y_i^k - x_i^k).$$

Estimate (5.22) is now replaced by

(8.10) $$M_{l+1} \ll (P_1 P_2 \ldots P_{s-l} P_{s-l+1})^2 P^{\varepsilon}.$$

Earlier we used estimates for I_{s-l+1} and R_{s-l} (cf. (5.23), (5.24)). Now, we use $R_{s-l+1} \ll (P_1 \dots P_{s-l+1}) P^{\varepsilon}$.

Comparing these changes together with those from (8.6) to (8.8), and from (5.22) to (8.10), the relations (8.3) and (8.4) follow easily.

LEMMA 8.2. *With the same premises as in the Fundamental Lemma, let $\mathscr{B}$ be a subset of $\{1, 2, \dots, s-l+1\}$, and let*

(8.11) $$I' = \int_0^1 |f_1 \dots f_{s-l+1}|^2 \{\prod_{i\in\mathscr{B}} |f_i|\}^2 d\alpha \ll (P_1 \dots P_{s-l+1}) \{\prod_{i\in\mathscr{B}} P_i\}^2 P^{-\xi+\varepsilon}.$$

Then, the number of solutions S of (2.5) also satisfies (2.7) with S_{l-1} replaced by S'''_{l-1} and T by T''', where (with S''_{l-1}, T'' as in Lemma 8.1)

(8.12) $$S'''_{l-1} = \{\prod_{i\in\mathscr{B}} P_i\}^{1/2^l} P^{-\xi/2^l+\varepsilon} S''_{l-1},$$

and

(8.13) $$T''' = P^{-\xi/2^l+\varepsilon} T''.$$

Proof. Since (by hypothesis) $\{\lambda_1, \dots, \lambda_{s-l+1}\}$ form admissible exponents, by Lemma 7.1, the subset $\left\{\lambda_i \,\middle|\, \begin{matrix} 1 \leqslant i \leqslant s-l+1 \\ i \notin \mathscr{B} \end{matrix}\right\}$ also form admissible exponents. Hence,

(8.14) $$R' = \int_0^1 \{\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s-l+1} |f_i|^2\} d\alpha \ll \{\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s-l+1} P_i\} P^{\varepsilon}.$$

Rest of the argument proceeds as in Lemma 8.1. In place of (8.5), we estimate with

(8.15) $$\{\int_0^1 |F_{l-1}(\alpha)|^2 (\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s-l+1} |f_i|^2) d\alpha\}^{1/2} \{\int_0^1 |f_1 \dots f_{s-l+1}|^2 (\prod_{i\in\mathscr{B}} |f_i|^2) d\alpha\}^{1/2}.$$

M_{l+1} now denotes the number of solutions of

(8.16) $$\Delta_{l,t_1,\dots,t_l}(x^k) = \sum_{\substack{i=1\\ i\notin\mathscr{B}}}^{s-l+1} (y_i^k - x_i^k),$$

and (5.22) is replaced by

(8.17) $$M'_{l+1} \ll \{\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s-l+1} P_i\}^2 P^{\varepsilon}.$$

In (8.6), I_{s-l+1} and R_{s-l} are respectively replaced by I' and R' (given by (8.11) and (8.14)). Comparison of (8.3) with (8.12) and (8.4) with (8.13) gives the desired results.

In applying the Fundamental Lemma, the λ_i's (and not the δ_i's or μ_i's) will be different at each step (the λ_i at one step being proportional to the λ_i at another step). At the iterative step of estimating $U_{s+1}^{(k)}(N)$ from $U_s^{(k)}(N)$, we write

(8.18) $$\lambda_i = \lambda_i^{(s)}.$$

9. Lemmas for adapting Davenport's results. Results given in this section are for the purpose of using Davenport's estimates for $U_s^{(k)}(N)$ (modifying the requirement of admissibility of exponents) in the Fundamental Lemma.

From now on (unless otherwise specified), δ_r and l_r will denote the values of δ and l taken in Lemma 4.2 (and also in the Fundamental Lemma) in estimating $U_{r+1}^{(k)}(N)$ from $U_r^{(k)}(N)$. Also, α_r will refer to that in the estimate $U_r^{(k)}(N) > N^{\alpha_r - \varepsilon}$. For any given s, let $\mathscr{U}$ denote (uniformly for all k's) the set of distinct integers u_i of the form

(9.1) $$u_i = \sum_{i=1}^{s} x_i^k,$$

where Card $\mathscr{U}$ is estimated by Davenport's method (using Lemmas 4.2 and 4.3) with $P = N^{1/k}$, $P_i = P^{\lambda_i}$, $x_i \in (P_i, 2P_i)$ to satisfy

(9.2) $$P^{\lambda_1+\dots+\lambda_s-\varepsilon} \ll \mathrm{Card}\,\mathscr{U} \ll P^{\lambda_1+\dots+\lambda_s},$$

so that

(9.3) $$\alpha_s = (\lambda_1 + \dots + \lambda_s)/k.$$

Write

(9.4) $$U(\alpha) = \sum_{u_i\in\mathscr{U}} e(\alpha u_i) \quad \text{and} \quad U(0) = U = \mathrm{Card}\,\mathscr{U}.$$

LEMMA 9.1. *Let $s \geqslant 2$, $t \geqslant 1$, and let in the estimate (obtained with Lemma 4.2) $\alpha_{s+t} = (\lambda_1 + \dots + \lambda_{s+t})/k$,*

(9.5) $$\delta_i \leqslant 1/2^{l_i} \quad \text{for} \quad i \geqslant s+1$$

(for $1 \leqslant i \leqslant s$, δ_i may be taken in $1/2^{l_i} < \delta_i \leqslant 1/2^{(l_i-1)}$). Also, let the set $\mathscr{U}$ be defined as above (consisting of distinct u_i's of the form (9.1)).

Then, the number of solutions of the equation

(9.6) $$x_{s+t}^k + \dots + x_{s+1}^k + u_i = y_{s+t}^k + \dots + y_{s+1}^k + u_j$$

with $x_i, y_i \in (P^{\lambda_i}, 2P^{\lambda_i})$ for $i \geqslant s+1$, and $u_i, u_j \in \mathscr{U}$ is

$$\ll P^{\lambda_{s+t}+\dots+\lambda_{s+1}+\varepsilon} U.$$

Proof. When $t = 1$ (using the distinctness of the u_i's), the proof is the same as in Lemma 4.2, where δ_s may be taken in $1/2^{l_s} < \delta_s \leqslant 1/2^{(l_s-1)}$. When

$t \geqslant 2$, we use the proof of Lemma 4.1 (since δ_i now satisfies (9.5)). (Generally, δ_i satisfies (9.5) only with $l = k-2$.)

Corollary 9.1. *With the f_i's defined as before,*

$$\int_0^1 |f_{s+1} \ldots f_{s+r}|^2 |U(\alpha)|^2 \, d\alpha \ll (P_{s+1} \ldots P_{s+r}) U P^{\varepsilon}, \tag{9.7}$$

where

$$(P_{s+1} \ldots P_{s+r}) U \ll P_{s+r}^{k\alpha_{s+r}}. \tag{9.8}$$

Proof. The proof follows from the lemma on using (9.2), (9.3) and (9.4).

In place of integrals of the form I_j (cf. (5.1)) occurring in the proof of the Fundamental Lemma, we also have to consider integrals of the form

$$\int_0^1 |f_{s+r}|^2 |f_{s+1} \ldots f_{s+r}|^2 |U(\alpha)|^2 \, d\alpha. \tag{9.9}$$

The next lemma will show how the adjustments in the proofs (and the choice for the σ's) are made in these cases.

Lemma 9.2. *Let $1 \leqslant r \leqslant k-2$, and (with $\alpha_{s+r} = (\lambda_1 + \ldots + \lambda_{s+r})/k$)*

$$\alpha_{s+r} < 1-(k-r)/k(2^{k-2}). \tag{9.10}$$

Then,

$$\int_0^1 |f_{s+r}|^2 |f_{s+1} \ldots f_{s+r}|^2 |U(\alpha)|^2 \, d\alpha \ll P_{s+r}^{2-2\sigma+\delta_0} (P_{s+1} \ldots P_{s+r}) U, \tag{9.11}$$

where

$$\sigma = \sigma(k) = 1/2^{k-1}. \tag{9.12}$$

Proof. Following the proof of Lemma 4.8, the integral over the minor arcs m (using (9.7) and Weyl's inequality for $|f_{s+r}|^2$) is bounded by the estimate in (9.11). The integral over the major arcs $\mathfrak{M}$ is estimated with (using trivial estimate for $U(\alpha)$)

$$\Big\{ \sum_{q \leqslant P_{s+r}^{(k/2^{k-1})}} \sum_a q^{-2(r+1)/k} \Big\} (P_{s+r}^{-k}) (P_{s+1} \ldots P_{s+r})^2 P_{s+r}^2 U^2. \tag{9.13}$$

The double sum above is $\ll P_{s+r}^{2(k-r-1)/2^{k-1}}$, so that (using (9.8) and (9.12)), the integral over $\mathfrak{M}$ is

$$\ll P_{s+r}^{-2\sigma+(k-r)/2^{k-2}} P_{s+r}^{-k} (P_{s+1} \ldots P_{s+r}) U P_{s+r}^{k\alpha_{s+r}} P_{s+r}^2.$$

Result now follows from (9.10).

Lemma 9.3. *Let $r \geqslant k-1$, and*

$$\alpha_{s+r} < 1-1/k(2^{k-2}). \tag{9.14}$$

Then, the estimate (9.11) *holds with σ as in* (9.12).

Proof. The proof is the same as in Lemma 9.2, except that the double sum in (9.13) is estimated to be $\ll P_{s+r}^{\varepsilon}$.

10. Further auxiliary results for estimating $U_s^{(k)}(N)$ (for $k \geqslant 7$). For $k \geqslant 7$, we further simplify the computations (required in the proof of $S \ll P^{1+\varepsilon} U$) with the results in this section (together with those in Section 6). With some additional conditions (which are not required in the proof of the Fundamental Lemma itself), we obtain an explicit inequality for the iterative choice of the δ's.

We first choose $\delta_1, \ldots, \delta_{s_0}$ by using Davenport's method (for suitable s_0 depending on k), and then use the Fundamental Lemma. The verifications of $S_i \ll P^{1+\varepsilon} U$ (for the S_i's occurring in the estimate of S) are simplified with the use of Lemma 10.2.

Lemma 10.1. *For $1 \leqslant l \leqslant k-2$,*

$$(S_{l-1}/T)^{2^l} \ll P^{E(s)-(k-l-1)},$$

where

$$E(s) = \lambda_{s-l+1} \{k-(k-1+\delta_{s-l}) \alpha_{s-l}\}.$$

Proof. From (2.4),

$$1+\tau_{l-1} = k\lambda_{s-l+1} - (k-l-1) \quad \text{and} \quad P_1 \ldots P_{s-l} = P^{k\lambda_{s-l}\alpha_{s-l}}.$$

Also, by Lemma 6.2,

$$(S_{l-1}/T)^{2^l} = P^{(1+\tau_{l-1})}/(P_1 \ldots P_{s-l}).$$

Hence,

$$(S_{l-1}/T)^{2^l} = P^{k(\lambda_{s-l+1} - \lambda_{s-l}\alpha_{s-l}) - (k-l-1)}.$$

Result now follows, since $\lambda_{s-l} = \lambda_{s-l+1} \mu_{s-l} = \lambda_{s-l+1} (k-1+\delta_{s-l})/k$. From now onwards, we take $l = k-2$ (as, only this is required for our purposes).

Lemma 10.2. *Let the Fundamental Lemma be used with $s \geqslant s_0+1$, and suppose that the inequality*

$$\lambda_{s-k+3} \{k-(k-1+\delta_{s-k+2}) \alpha_{s-k+2}\} < 1 \tag{10.1}$$

holds for $s \geqslant s_0+1$. Then, subject to (6.2) *(for $1 \leqslant r \leqslant k-3$),*

$$S \ll T \tag{10.2}$$

at all the iterative steps with $s \geqslant s_0+1$.

Proof. From (10.1), and Lemma 10.1 (with $l = k-2$), it follows that

$$(S_{k-3}/T)^{2^{k-2}} \ll 1 \quad (s \geqslant s_0+1). \tag{10.3}$$

Now, by Lemma 6.1, $S_i \ll S_{k-3}$ $(0 \leqslant i \leqslant k-4)$; so that, from (10.3), and (2.7), $S \ll T$, as asserted.

Remark. In applications of the method, the inequality (10.1) needs to be verified only at the first iterative step (with $s = s_0+1$) since with small values for the δ's, $E(s)$ decreases (with the increasing α's) as s increases.

LEMMA 10.3. *Let* $\sigma_j = \lambda_j/2^{k-1}$ *for* $s-k+3 \leqslant j \leqslant s$ (cf. (2.2) and (2.3)), *and*

$$A_s = \sum_{i=0}^{k-3} \lambda_{s-i}^{(s)}/2^{i+1}, \quad B = \sum_{i=0}^{k-3} (k-i-1)/2^{i+1}, \tag{10.4}$$

(*where as previously indicated,* $\lambda_{s-i}^{(s)}$ *denotes* λ_{s-i} *at the s-th iterative step*). *Then, with* $l = k-2$,

$$T \ll P^{\{A_s(k+1-1/2^{k-2})-(k-2)+\delta_0\}} \{P^{k\lambda_{s-k+2}^{(s)}+2^{\alpha_{s-k+2}}}\}^{1/2^{k-2}} U. \tag{10.5}$$

Proof. Since $\delta_s = k\lambda_s-(k-1)$, and $\tau_i = k\lambda_{s-i}-(k-i-1)$ for $1 \leqslant i \leqslant k-3$,

$$\delta_s/2 + \Big(\sum_{i=1}^{k-3} \tau_i/2^{i+1}\Big) = kA_s - B; \tag{10.6}$$

$$\prod_{i=0}^{k-3} (P_{s-i})^{1/2^{i+1}} = P^{A_s} \quad (\text{using } P_{s-i} = P^{\lambda_{s-i}}); \tag{10.7}$$

$$\sum_{i=0}^{k-3} (\sigma_{s-i}/2^i) = (1/2^{k-2}) \Big\{\sum_{i=0}^{k-3} (\lambda_{s-i}/2^{i+1})\Big\} = A_s/2^{k-2}. \tag{10.8}$$

Also,

$$(P_1 \ldots P_{s-k+2})^{1/2^{k-2}} = \{P^{k\lambda_{s-k+2}^{(s)}+2\alpha_{s-k+2}}\}^{1/2^{k-2}}. \tag{10.9}$$

Hence, (10.5) follows from (2.10) (with $l = k-2$) as it is easily seen that $B = k-2$.

The next lemma is designed to make iterative use of the computations.

LEMMA 10.4. *With* A_s *defined by* (10.4),

$$A_{s+1} = (\lambda_{s+1}^{(s+1)}/2)(1+A_s-\lambda_{s-k+3}^{(s)}/2^{k-2}). \tag{10.10}$$

Proof. The proof follows easily from (10.4) since with $\lambda_{s+1}^{(s+1)} = (k-1+\delta_{s+1})/k$,

$$\lambda_i^{(s+1)} = \lambda_{s+1}^{(s+1)}\lambda_i^{(s)} \quad (1 \leqslant i \leqslant s). \tag{10.11}$$

Under the hypotheses of Lemmas 10.2 and 10.3, we can choose the δ's with the following:

LEMMA 10.5. *Let* $\delta_1, \ldots, \delta_s$ *be such that* $\{\lambda_1^{(s)}, \ldots, \lambda_s^{(s)}, 1\}$ *form admissible exponents. Then* (*with* A_s *defined by* (10.4)), $\{\lambda_1^{(s+1)}, \ldots, \lambda_{s+1}^{(s+1)}, 1\}$ *also form admissible exponents provided* δ_{s+1} *satisfies*

$$\{(k-1+\delta_{s+1})/k\}\,[(1/2)\{(1+A_s-\lambda_{s-k+3}^{(s)}/2^{k-2})(k+1-1/2^{k-2})\} + \{k\lambda_{s-k+3}^{(s)}\alpha_{s-k+3}/2^{k-2}\}] < k-1. \tag{10.12}$$

Proof. $\lambda_{s+1}^{(s+1)} = (k-1+\delta_{s+1})/k$ and $\lambda_{s-k+3}^{(s+1)}\alpha_{s-k+3} = \lambda_{s+1}^{(s+1)}\lambda_{s-k+3}^{(s)}\alpha_{s-k+3}$. Hence, from (10.10), (10.12) and (10.5) (with $s+1$ replacing s), we have $T \ll PU$, as required. (Note that δ_{s+1} is the only unknown in (10.12).)

For $7 \leqslant k \leqslant 10$, we use (10.12) for $s \geqslant s_0+1$ where s_0 is chosen suitably to minimize computations. For $s \leqslant s_0$, δ's are chosen with Davenport's method or with the method in [5]. In all cases, (10.12) can be used for values of $s < s_0$ also to get better bounds for $U_s^{(k)}(N)$. However, these improvements do not seem to be sufficient to get better estimates for $G(k)$. Of the δ's and α's obtained by Davenport's method or as in [5], only those that are required for our purposes will be indicated.

Before estimating $U_s^{(k)}(N)$ for $7 \leqslant k \leqslant 10$, we note the following:

The sets $\mathscr{U}$ are considered as in Lemma 9.1 with $s = 10$, 12, 17 and 22 for $k = 7$, 8, 9 and 10 respectively (using $\delta_i \leqslant 1/2^{k-2}$ for $i > s$). (For $k = 9$ and 10, $\mathscr{U}$ may be considered with smaller values of s.)

The expressions $P^{\lambda_1+\ldots+\lambda_s}$ occurring in the estimates in the Fundamental Lemma will be replaced by $P^{\kappa}U$ (cf. (9.2) and (9.4)).

In the estimates of all the I_j's (of the form (9.9)), the number σ can be taken to be $(1/2^{k-1})$. For, with the values of α_j (estimated iteratively), it can be verified that (with $j = r+s$) α_j satisfies (9.10) for $1 \leqslant r \leqslant k-2$, and (9.14) for $r \geqslant k-1$.

11. Estimation of $U_s^{(7)}(N)$. The next lemma follows (with Davenport's method) as in Lemma 9.10 in [12].

LEMMA 11.1. *In estimating* $U_s^{(7)}(N)$, *Lemma* 4.2 *allows the choice of* δ_i *with*

$$\delta_{10} = 0.0323, \quad \delta_{11} = 0.0269, \quad \delta_{12} = 0.0224, \quad \delta_{13} = 0.0184, \quad \delta_{14} = 0.0156. \tag{11.1}$$

Also, $U_s^{(7)}(N) > N^{\alpha_s-\varepsilon}$ (*for* $10 \leqslant s \leqslant 15$) *with*

$$0.82781 < \alpha_{10} < 0.82785, \quad \alpha_{i+1} = (1/7)+(6+\delta_i)\alpha_i/7 \quad (10 \leqslant i \leqslant 14). \tag{11.2}$$

(The δ's are close to, but slightly less than the best possible values allowable by the method.)

Now, the iterative use of (10.12) gives the following:

LEMMA 11.2. *In addition to the premises of Lemma* 11.1, *let*

$$\delta_{15} = 0.025, \quad \delta_{16} = 0.0195, \quad \delta_{17} = 0.0185, \quad \delta_{18} = 0.0175, \quad \delta_{19} = 0.0168, \quad \delta_{20} = 0.0162. \tag{11.3}$$

Further suppose that

(11.4)
$$\lambda_{20} = \lambda_{20}^{(20)} = (6+\delta_{20})/7, \quad \lambda_i = \lambda_i^{(20)} = (6+\delta_i)\lambda_{i+1}^{(20)}/7 \quad (11 \leqslant i \leqslant 19);$$

also that the set $\mathscr{U}$ *is defined with* $s = 10$ *as in Lemma* 9.1. *Then, with* u_i, $u_j \in \mathscr{U}$, x_i, $y_i \in (P_i, 2P_i)$ *for* $11 \leqslant i \leqslant 20$, x, $y \in (P, 2P)$, *the number of solutions of*

$$x^7 + (\sum_{i=11}^{20} x_i^7) + u_i = y^7 + (\sum_{i=11}^{20} y_i^7) + u_j$$

is

$$\ll P^{1+\lambda_{11}+\ldots+\lambda_{20}+\varepsilon} U,$$

and $U_{21}^{(7)}(N) > N^{\alpha_{21}-\varepsilon}$, *where*

(11.5)
$$\alpha_{21} > 0.98305.$$

[Here, the approximate values of the α's used in the computations with (10.12) are as follows: (These values are slightly less than the precise values occurring in the method, and for estimating the δ_i's with (10.12), slightly larger values (differing in the 5-th of 6-th decimals) have to be used. The same remarks will apply to other values of k also.)

$\alpha_{11} = 0.856227$, $\alpha_{12} = 0.880055$, $\alpha_{13} = 0.900005$, $\alpha_{14} = 0.916654$, $\alpha_{15} = 0.930603$, $\alpha_{16} = 0.94384$, $\alpha_{17} = 0.954491$, $\alpha_{18} = 0.963514$, $\alpha_{19} = 0.971134$, $\alpha_{20} = 0.977587$.]

12. Estimation of $U_s^{(8)}(N)$. As in Lemma 9.11 of [12], we have the following:

LEMMA 12.1. *In estimating* $U_s^{(8)}(N)$, *Lemma* 4.2 *allows the choice of* δ_i *with*

(12.1)
$$\delta_{12} = 0.0188, \quad \delta_{13} = (1/64), \quad \delta_{14} = 0.014,$$
$$\delta_{15} = 0.0121, \quad \delta_{16} = 0.0104, \quad \delta_{17} = 0.0089.$$

Also, $U_s^{(8)}(N) > N^{\alpha_s-\varepsilon}$ *(for* $12 \leqslant s \leqslant 18$*) with*

(12.2)
$$0.8276 < \alpha_{12} < 0.8277, \quad \alpha_{i+1} = (1/8)+(7+\delta_i)\alpha_i/8 \quad (12 \leqslant i \leqslant 17).$$

With these, the use of (10.12) gives the following:

[The approximate values of the α's are:

$\alpha_{13} = 0.851093$, $\alpha_{14} = 0.87136$, $\alpha_{15} = 0.888963$, $\alpha_{16} = 0.904186$, $\alpha_{17} = 0.917337$, $\alpha_{18} = 0.928688$, $\alpha_{19} = 0.938877$, $\alpha_{20} = 0.947748$, $\alpha_{21} = 0.955463$, $\alpha_{22} = 0.962163$, $\alpha_{23} = 0.967992$, $\alpha_{24} = 0.973062$, $\alpha_{25} = 0.977462$, $\alpha_{26} = 0.981286$, $\alpha_{27} = 0.984617$, $\alpha_{28} = 0.987505$, $\alpha_{29} = 0.990016$.]

LEMMA 12.2. *In addition to* (12.1), *let*

(12.3)
$$\begin{cases} \delta_{18} = 0.011, & \delta_{19} = 0.0105, \quad \delta_{20} = 0.01, \quad \delta_{21} = 0.0095, \\ \delta_{22} = 0.00915, & \delta_{23} = 0.00885, \quad \delta_{24} = 0.0085, \quad \delta_{25} = 0.00825, \\ \delta_{26} = 0.0081, & \delta_{27} = 0.00785, \quad \delta_{28} = 0.0077, \quad \delta_{29} = 0.00755; \end{cases}$$

(12.4)
$$\lambda_{29} = \lambda_{29}^{(29)} = (7+\delta_{29})/8, \quad \lambda_i = \lambda_i^{(29)} = (7+\delta_i)\lambda_{i+1}^{(29)}/8 \quad (13 \leqslant i \leqslant 28);$$

and the set $\mathscr{U}$ *be defined as in Lemma* 9.1 *with* $s = 12$. *Then, with* u_i, $u_j \in \mathscr{U}$; x_i, $y_i \in (P_i, 2P_i)$ *for* $13 \leqslant i \leqslant 29$; x, $y \in (P, 2P)$, *the number of solutions of*

$$x^8 + (\sum_{i=13}^{29} x_i^8) + u_i = y^8 + (\sum_{i=13}^{29} y_i^8) + u_j$$

is

$$\ll P^{1+\lambda_{13}+\ldots+\lambda_{29}+\varepsilon} U,$$

and $U_{30}^{(8)}(N) > N^{\alpha_{30}-\varepsilon}$, *where*

(12.5)
$$\alpha_{30} > 0.9922.$$

13. Estimation of $U_s^{(9)}(N)$. For $k = 9$, we combine Davenport's method and the method in [5] as indicated in § 7 of [13]. We take

(13.1)
$$\begin{cases} \delta_{17} = 0.00736, \quad \delta_{18} = 0.0065, \quad \delta_{19} = 0.00574, \quad \delta_{20} = 0.00506, \\ \delta_{21} = 0.00446, \quad \delta_{22} = 0.00393, \quad \delta_{23} = 0.00347; \\ 0.88201 < \alpha_{17} < 0.88202, \quad \alpha_{i+1} = (1/9)+(8+\delta_i)\alpha_i/9 \quad (17 \leqslant i \leqslant 23). \end{cases}$$

By using (10.12), we can now take

(13.2)
$$\begin{cases} \delta_{24} = 0.006, \quad \delta_{25} = 0.005, \quad \delta_{26} = 0.0047, \quad \delta_{27} = 0.00455, \\ \delta_{28} = 0.0044, \quad \delta_{29} = 0.0043, \quad \delta_{30} = 0.0042. \end{cases}$$

[The approximate values of the α's are:

$\alpha_{18} = 0.895841$, $\alpha_{19} = 0.908061$, $\alpha_{20} = 0.918855$, $\alpha_{21} = 0.928387$, $\alpha_{22} = 0.936803$, $\alpha_{23} = 0.944233$, $\alpha_{24} = 0.950793$, $\alpha_{25} = 0.956894$, $\alpha_{26} = 0.962214$, $\alpha_{27} = 0.966914$, $\alpha_{28} = 0.971078$, $\alpha_{29} = 0.974766$, $\alpha_{30} = 0.978035$.]

LEMMA 13.1. *Let the* δ_i*'s be chosen as above, and*

(13.3)
$$\lambda_{30} = \lambda_{30}^{(30)} = (8+\delta_{30})/9, \quad \lambda_i = \lambda_i^{(30)} = (8+\delta_i)\lambda_{i+1}^{(30)}/9 \quad (18 \leqslant i \leqslant 29).$$

Also, let $\mathscr{U}$ *be defined with* $s = 17$ *in Lemma* 9.1, *and* u_i, $u_j \in \mathscr{U}$. *Then, with* x_i, $y_i \in (P_i, 2P_i)$ *for* $18 \leqslant i \leqslant 30$; x, $y \in (P, 2P)$, *the number of solutions of*

$$x^9 + (\sum_{i=18}^{30} x_i^9) + u_i = y^9 + (\sum_{i=18}^{30} y_i^9) + u_j$$

is

$$\ll P^{1+\lambda_{18}+\ldots+\lambda_{30}+\varepsilon} U$$

and $U_{31}^{(9)}(N) > N^{\alpha_{31}-\varepsilon}$ *where*

(13.4)
$$\alpha_{31} > 0.98093.$$

We also have (as required later) $U_{25}^{(9)}(N) > N^{\alpha_{25}-\varepsilon}$, *where*

$$\alpha_{25} > 0.95689. \tag{13.5}$$

14. Estimation of $U_s^{(10)}(N)$. For $k = 10$ also, we proceed as in § 7 of [13], and take

$$\begin{cases}\delta_{22} = 0.00308, & \delta_{23} = 0.00275, \quad \delta_{24} = 0.00247, \quad \delta_{25} = 0.00221,\\ \delta_{26} = 0.00198, & \delta_{27} = 0.00177, \quad \delta_{28} = 0.00159, \quad \delta_{29} = 0.00142;\\ 0.912285 < \alpha_{22} < 0.91229, & \alpha_{i+1} = (1/10)+(9+\delta_i)\alpha_i/10 \quad (22 \leqslant i \leqslant 29).\end{cases} \tag{14.1}$$

Now the use of (10.12) allows the choice

$$\delta_{30} = 0.0032, \quad \delta_{31} = \delta_{32} = \delta_{33} = \delta_{34} = 0.0022, \quad \delta_{35} = 0.0021. \tag{14.2}$$

[Here, the approximate values of the α's are:
$\alpha_{23} = 0.921337$, $\alpha_{24} = 0.929456$, $\alpha_{25} = 0.936739$, $\alpha_{26} = 0.943272$, $\alpha_{27} = 0.949131$, $\alpha_{28} = 0.954385$, $\alpha_{29} = 0.959098$, $\alpha_{30} = 0.963324$, $\alpha_{31} = 0.9673$, $\alpha_{32} = 0.970781$, $\alpha_{33} = 0.973916$, $\alpha_{34} = 0.976738$, $\alpha_{35} = 0.979278$].

LEMMA 14.1. *Let the δ_i's be as above, and*

$$\lambda_{35} = \lambda_{35}^{(35)} = (9+\delta_{35})/10, \quad \lambda_i = \lambda_i^{(35)} = (9+\delta_i)\lambda_{i+1}^{(35)}/10 \quad (23 \leqslant i \leqslant 34). \tag{14.3}$$

Further let $\mathcal{U}$ be defined with $s = 22$ in Lemma 9.1, and $u_i, u_j \in \mathcal{U}$. Then, with $x_i, y_i \in (P_i, 2P_i)$ for $23 \leqslant i \leqslant 35$, and $x, y \in (P, 2P)$, the number of solutions of

$$x^{10}+\Big(\sum_{i=23}^{35} x_i^{10}\Big)+u_i = y^{10}+\Big(\sum_{i=23}^{35} y_i^{10}\Big)+u_j$$

is

$$\ll P^{1+\lambda_{23}+\ldots+\lambda_{35}+\varepsilon}U,$$

and $U_{36}^{(10)}(N) > N^{\alpha_{36}-\varepsilon}$, where

$$\alpha_{36} > 0.98155. \tag{14.4}$$

Also, $U_{31}^{(10)}(N) > N^{\alpha_{31}-\varepsilon}$, with

$$\alpha_{31} > 0.9673. \tag{14.5}$$

15. Proof of Theorem 1 for the cases $k = 7$ and 8. Let the λ_i's and the corresponding f_i's (and the set $\mathcal{U}$) be defined according to Lemmas 11.2 and 12.2 for $k = 7$ and 8 respectively, so that (with $P = N^{1/k}$)

$$f = f(\alpha) = \sum_{P<x<2P} e(\alpha x^k), \quad f_i = f_i(\alpha) = \sum_{P_i<x<2P_i} e(\alpha x^k), \quad U(\alpha) = \sum_{u_i \in \mathcal{U}} e(\alpha u_i). \tag{15.1}$$

As in the proof of Lemma 4.8, let the unit interval $(Q^{-1}, 1+Q^{-1})$ (with $Q = P^{k-1+\delta_0}$) be divided into $\mathfrak{m}$ and m (so that, the basic intervals $\mathfrak{m}_{a,q}$'s are defined with $1 \leqslant q \leqslant P^{k/2^{k-1}}$, $|\alpha - a/q| \leqslant q^{-1}Q^{-1}$). Then (as already shown in Section 4), for each k,

$$f(\alpha) \ll P^{(1-1/2^{k-1}+\delta_0)} \quad \text{if} \quad \alpha \in m. \tag{15.2}$$

Case (a): $k = 7$. Let

$$r_7(N) = \int_{Q^{-1}}^{1+Q^{-1}} f^8(\alpha)\{f_{11}(\alpha)\ldots f_{20}(\alpha)f(\alpha)\}^2 U^2(\alpha)e(-N\alpha)\,d\alpha. \tag{15.3}$$

From Lemma 11.2,

$$\int_{Q^{-1}}^{1+Q^{-1}} |f_{11}\ldots f_{20}f|^2|U(\alpha)|^2\,d\alpha \ll (P_{11}\ldots P_{20}P)^2U^2P^{-7\alpha_{21}+\delta_0}$$

(using (9.2), (9.3), (9.4) and (9.7)). Hence, with the estimate for $f^8(\alpha)$ by using (15.2) with $k = 7$), the contribution to $r_7(N)$ from m is

$$\begin{aligned} &\ll P^{8(1-1/64+\delta_0)}(P_{11}\ldots P_{20}P)^2U^2P^{-7\alpha_{21}+\delta_0}\\ &\ll P^{-7-\delta_0}P^8(P_{11}\ldots P_{20}P)^2U^2\end{aligned} \tag{15.4}$$

since from (11.5), $7\alpha_{21}+(8/64) > 7$.

Since (from [9]) $\Gamma(7) = 4$ (and the singular series considered with $(2k+1)$ kth powers is convergent), we consider the estimates of $f^{10}f_{20}^2f_{19}^2f_{18} - g^{10}g_{20}^2g_{19}^2g_{18}$ over $\mathfrak{m}$ (in order to obtain a satisfactory estimate for the integral of the error term over $\mathfrak{m}$). Since (4.30) and (4.31) easily hold for $18 \leqslant i \leqslant 20$, the integral over $\mathfrak{m}$ is estimated in the usual way (see [13]). Accordingly, from (15.3) and (15.4), it follows that $r_7(N) \gg P(P_{11}\ldots P_{20}P)^2U^2$, proving that $G(7) \leqslant 50$.

Case (b): $k = 8$. Let

$$r_8(N) = \int_{Q^{-1}}^{1+Q^{-1}} f^8(\alpha)\{f_{13}(\alpha)\ldots f_{29}(\alpha)f(\alpha)\}^2U^2(\alpha)e(-N\alpha)\,d\alpha. \tag{15.5}$$

From Lemma 12.2,

$$\int_{Q^{-1}}^{1+Q^{-1}} |f_{13}\ldots f_{29}f|^2|U(\alpha)|^2\,d\alpha \ll (P_{13}\ldots P_{29}P)^2U^2P^{-8\alpha_{30}+\delta_0}.$$

Hence, using (15.2) with $k = 8$ (for $f^8(\alpha)$), the contribution to $r_8(N)$ from m is

$$\begin{aligned} &\ll P^{8(1-1/128+\delta_0)}(P_{13}\ldots P_{29}P)^2U^2P^{-8\alpha_{30}+\delta_0}\\ &\ll P^{-8-\delta_0}P^8(P_{13}\ldots P_{29}P)^2U^2,\end{aligned} \tag{15.6}$$

since from (12.5), $8\alpha_{30}+(8/128) > 8$.

Now, since $\Gamma(8) = 32$, over $\mathfrak{m}$, we consider
$$f^{10} f_{29}^2 \ldots f_{19}^2 - g^{10} g_{29}^2 \ldots g_{19}^2.$$
Since $\lambda_{19} > (7/8)^{11} > 1/5$, and $\mathfrak{m}_{a,q}$'s are defined with $q \leqslant P^{1/16}$, (4.30) and (4.31) are easily seen to hold for $19 \leqslant i \leqslant 29$. Thus, it follows from (15.5) and (15.6) that
$$r_8(N) \gg (P_{13} \ldots P_{29} P)^2 U^2,$$
proving that $G(8) \leqslant 68$.

16. Proof of Theorem 2. For $k = 9$ and 10, we use the method in [14]. Condition (14) in [14] is satisfied with

(a) $k = 9$, $s_1 = 25$, $s_2 = 31$, $\gamma_1 = \alpha_{25}$, $\gamma_2 = \alpha_{31}$ (cf. (13.4) and (13.5));

(b) $k = 10$, $s_1 = 31$, $s_2 = 36$, $\gamma_1 = \alpha_{31}$, $\gamma_2 = \alpha_{36}$ (cf. (14.4) and (14.5)).

(Note that (from Lemmas 13.1 and 14.1) there are sufficiently many f_i's for dealing with the basic intervals.) Accordingly,
$$G(9) \leqslant 2(31)+25 = 87 \quad \text{and} \quad G(10) \leqslant 2(36)+31 = 103,$$
proving Theorem 2.

17. Proof of Theorem 3 for the cases $k = 7$ and 8. Suppose that we get the estimate $G(k) \leqslant s_1 + 2s_2$ by using the fact that $\{\lambda_1, \ldots, \lambda_{s_2}\}$ form admissible exponents (for suitable λ's), and by using Weyl's inequality or Vinogradov's estimate (for Weyl's sum) for $\{f(\alpha)\}^{s_1}$ (in dealing with the minor arcs). Then, it is a standard deduction that
$$H(k) \leqslant s_1' + 2s_2 + 1, \quad \text{where} \quad s_1' = \begin{cases} s_1 & \text{if } s_1 \text{ is even,} \\ s_1 + 1 & \text{if } s_1 \text{ is odd.} \end{cases}$$
Here, we need only the following modifications:

With K defined as usual, as in the estimate of $U_s^{(k)}(N)$, method of Lemma 4.2 gives the following (with p's denoting primes):

The number of distinct integers $v_i \equiv s \pmod K$ that are $\leqslant N$, and representable in the form
$$v_i = \sum_{j=1}^{s} p_j^k,$$
where each prime $p_j \in (P_j, 2P_j)$, and $p_j \equiv 1 \pmod K$ is (with the same λ's as before) $\gg P^{\lambda_1 + \ldots + \lambda_s - \varepsilon}$. Here, we consider this set $\mathscr{U}^*$ in place of the set $\mathscr{U}$. (The cardinalities of $\mathscr{U}$ and $\mathscr{U}^*$ will differ by at most a factor $\ll P^\varepsilon$.) Now, the proof of the Fundamental Lemma can be adjusted as in Lemma 9.1, with $\mathscr{U}^*$ replacing $\mathscr{U}$. The rest of the argument is standard.

18. Estimates for $H(9)$ and $H(10)$. The method in this paper leads to substantial improvements on the known bounds for $H(k)$ for further values of k. This is illustrated here for $k = 9$ and 10. Hua (Theorem 14 in Chapter 9 of [12]) obtains the bounds

(18.1) $$H(9) \leqslant 103, \quad H(10) \leqslant 127.$$

Combining Davenport's method and the method in [5], the bounds obtainable for $G(9)$ and $G(10)$ are $G(9) \leqslant 96$, $G(10) \leqslant 121$ (see [1]); so that, these methods would give the bounds

(18.2) $$H(9) \leqslant 97, \quad H(10) \leqslant 123.$$

We can improve on these to get the following:

THEOREM 4. $H(9) \leqslant 91$, $H(10) \leqslant 115$.

Proof. (a) $k = 9$. In addition to the choice of the δ_i's indicated in Section 13, (10.12) allows us to take
$$\delta_{31} = 0.00416, \quad \delta_{32} = 0.00405, \quad \delta_{33} = 0.00398, \quad \delta_{34} = 0.00391,$$
$$\delta_{35} = 0.00386, \quad \delta_{36} = 0.0038, \quad \delta_{37} = 0.00376, \quad \delta_{38} = 0.00372,$$
to get
$$U_{39}^{(9)}(N) > N^{\alpha_{39} - \varepsilon} \quad \text{with} \quad \alpha_{39} = 0.99489.$$
[Here, the approximate values of the α's are: $\alpha_{31} = 0.980931$, $\alpha_{32} = 0.983503$, $\alpha_{33} = 0.985778$, $\alpha_{34} = 0.987794$, $\alpha_{35} = 0.989579$, $\alpha_{36} = 0.99116$, $\alpha_{37} = 0.99256$, $\alpha_{38} = 0.993801$.]

These lead to the estimate $G(9) \leqslant 12 + 2(39) = 90$ (by using Weyl's inequality for $f^{12}(\alpha)$); so that $H(9) \leqslant 91$.

(b) $k = 10$. With δ_i $(1 \leqslant i \leqslant 35)$ as in Section 14, we use (10.12) to take
$$\delta_{36} = 0.0021, \quad \delta_{37} = \delta_{38} = \delta_{39} = 0.002, \quad \delta_{40} = 0.00198, \quad \delta_{41} = 0.00195,$$
$$\delta_{42} = 0.00193, \quad \delta_{43} = 0.00191, \quad \delta_{44} = 0.00189, \quad \delta_{45} = 0.00188, \quad \delta_{46} = 0.00186,$$
$$\delta_{47} = 0.00185, \quad \delta_{48} = 0.00184, \quad \delta_{49} = 0.00183, \quad \delta_{50} = 0.00182,$$
leading to the estimate $U_{51}^{(10)}(N) > N^{\alpha_{51} - \varepsilon}$ with $\alpha_{51} = 0.99767$, and (with Weyl's inequality for $f^{12}(\alpha)$), $G(10) \leqslant 12 + 2(51) = 114$; so that $H(10) \leqslant 115$.

[The approximate values of the α's are:
$\alpha_{36} = 0.981555$, $\alpha_{37} = 0.983604$, $\alpha_{38} = 0.985435$, $\alpha_{39} = 0.987089$, $\alpha_{40} = 0.988577$, $\alpha_{41} = 0.989914$, $\alpha_{42} = 0.991115$, $\alpha_{43} = 0.992194$, $\alpha_{44} = 0.993163$, $\alpha_{45} = 0.994034$, $\alpha_{46} = 0.994817$, $\alpha_{47} = 0.995519$, $\alpha_{48} = 0.99615$, $\alpha_{49} = 0.996718$, $\alpha_{50} = 0.997227$.]

19. Indication of other results. In [12] (§ 2 of Ch. 12), $h(k)$ is defined to be the least positive integer s such that almost all positive integers $\equiv s \pmod K$ are sums of s kth powers of primes (K depending on k).

For $5 \leqslant k \leqslant 10$, the following bounds are indicated:

$$h(5) \leqslant 13,\ h(6) \leqslant 20,\ h(7) \leqslant 28,\ h(8) \leqslant 40,\ h(9) \leqslant 52,\ h(10) \leqslant 64.$$

(With the methods in [3] and [5], one can get $h(8) \leqslant 38$, $h(9) \leqslant 49$, $h(10) \leqslant 62$.)

With the methods in this paper (as in the proof of Theorem 3), these can be, improved to:

$$h(5) \leqslant 12,\ h(6) \leqslant 18,\ h(7) \leqslant 26,\ h(8) \leqslant 35,\ h(9) \leqslant 46,\ h(10) \leqslant 58.$$

The methods can be extended to additive problems where the summands are integral-valued polynomials.

Solubility of Diophantine inequalities, and Waring's problem with mixed powers are other additive problems to which the method may be applied.

20. The case $k = 12$: Corrigendum and addendum to [14]. In [14], the bounds (for α_{40} and α_{47}) given in the tables for $k = 12$ fall slightly short of satisfying (14) (required in the proof that $G(12) \leqslant 134$). This gap can be filled in different ways, and quite easily by one application of the method in this paper. As indicated in [13], the methods in [3] and [5] give $U_{46}^{(12)}(N) > N^{\alpha_{46}-\varepsilon}$ with $\alpha_{46} = 0.983527$. Now computing δ_i for $36 \leqslant i \leqslant 45$ (given by these methods), it is possible to take $\delta_{46} = 0.000824$ in (10.12) (with $k = 12$, $s = 45$). Accordingly, $\alpha_{47} = (1/12)+(11+\delta_{46})\alpha_{46}/12 > 0.984967$, which is sufficient for our purposes.

[The approximate values of the α's (using the methods in [3] and [5] as in [13]) are:

$\alpha_{36} = 0.960301$, $\alpha_{37} = 0.963644$, $\alpha_{38} = 0.966705$, $\alpha_{39} = 0.969508$, $\alpha_{40} = 0.972075$, $\alpha_{41} = 0.974427$, $\alpha_{42} = 0.97658$, $\alpha_{43} = 0.978552$, $\alpha_{44} = 0.980358$, $\alpha_{45} = 0.982012$.]

21. Addendum to [13]. While it does not affect the proofs, the use of the phrase 'Admissible exponents' in Lemma 18, should be modified as in Lemma 9.1 in this paper when Davenport's method is used with $1/2^l < \delta \leqslant 1/2^{l-1}$.

22. Addendum to [15] (and [16]). Here again, while the proofs are sufficient, the following adjustments have to be made (since Davenport's method was used with $1/2^l < \delta \leqslant 1/2^{l-1}$):

(a) In the proof of Theorem 1, after using Theorem 3 (and before using Davenport's method), introduce the sets $\mathscr{U}$ as in Lemma 9.1 in this paper. The integrals of the error terms over the basic intervals can be estimated separately (similar to Lemma 26) without using Lemma 3. With these, the proof is precisely the same as before. (This is now improved to $N = \sum_{s=1}^{20} x_s^{s+1}$ in [17].)

(b) In the proof of Theorem 2 (with the definitions as in [15]), (1) apply Theorem 3 to the entire set K_3 (without Davenport's method for $k = 6$). This also gives $\beta_1 > 1/2$ (cf. (16)); (2) after using Theorem 3 for the set K_2 (and before using Davenport's method with $k = 5$ and 4), introduce the set $\mathscr{U}^*$ as in Section 17 in this paper. (Here, for $k = 4$, $l = 2$ and $\delta < 1/4$.) The singular series has to considered with $k \in [K_3 \cup \{2, 3, 4, 5, 24\}]$ (instead of with $2 \leqslant k \leqslant 24$), and this does not make any significant difference.

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DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY
BEAVER CAMPUS, MONACA, PA 15061.

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