A category analogue of the density topology

by

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Abstract. In this paper we introduce the concept of an I-density point of a set for an arbitrary σ-ideal I. In the case of the σ-ideal of null sets it reduces to the notion of a density point and in the case of the σ-ideal of sets of the first category it gives a new notion, which seems to be quite delicate and which can be considered as a starting point to the study of category analogues of approximate continuity, differentiability and so on.

Our paper is a continuation of the previous research concerning similarities and differences between measure and category and contributes, in some sense, to the excellent Oxtoby's book [5].

We start with some remarks concerning convergence in measure. Let (X, S, m) be a finite or σ-finite measure space. It is well known that a sequence $\{f_n\}_{n\in N}$ of S-measurable real functions defined on X converges in measure to a function f if and only if every subsequence $\{f_{n_m}\}_{m\in N}$ of $\{f_n\}_{n\in N}$ contains a subsequence $\{f_{n_{m_p}}\}_{p\in N}$ which converges a.e. to f. This fact allows us to introduce a generalization of the notion of convergence in measure in the following way (see [9], [10], [11], [12]): Let (X, S) be a measurable space and let $I\subset S$ be a proper σ-ideal of sets. We say that some property holds I-almost everywhere (in abbr. I-a.e.) if and only if the set of points which do not have this property belongs to I. We say that a sequence $\{f_n\}_{n\in N}$ of S-measurable real functions defined on X converges with respect to I to some S-measurable real function f defined on X if and only if every subsequence $\{f_{n_m}\}_{m\in N}$ of $\{f_n\}_{n\in N}$ contains a subsequence $\{f_{n_{m_p}}\}_{p\in N}$ which converges to f I-a.e. We shall use the notation

$$f_n \xrightarrow[n\to\infty]{I} f.$$

Now let $X = R$ (the real line), let S be the σ-algebra of Lebesgue measurable sets and m the Lebesgue measure. The point 0 is a *density point* of a set $A \in S$ if and only if

$$\lim_{h\to 0^+} [(2h)^{-1}\cdot m(A \cap [-h, h])] = 1.$$

Observe that this condition is fulfilled if and only if

$$\lim_{n\to\infty} [(2n)^{-1}\cdot m(A \cap [-\tfrac{1}{n}, \tfrac{1}{n}])] = 1.$$

The last limit can be described in terms of convergence in measure in the following way: 0 is a *density point* of A if and only if the sequence $\{\chi_{(n\cdot A)\cap[-1,1]}\}_{n\in N}$ of characteristic functions (where $n\cdot A = \{nx: x\in A\}$) converges in measure to 1 on the interval $[-1, 1]$. This fact is the basis for the following definition, where $X = R$, S is a σ-algebra of subsets of R invariant with respect to linear transformations and $I\subset S$ is a σ-ideal also invariant with respect to linear transformations.

DEFINITION 1. We say that 0 is an *I-density point* of a set $A\in S$ if and only if $\chi_{(n\cdot A)\cap[-1,1]} \xrightarrow[n\to\infty]{I} 1$.

We say that x_0 is an I-density point of $A\in S$ if and only if 0 is an I-density point of $A-x_0 = \{x-x_0: x\in A\}$. We say that x_0 is an *I-dispersion point* of $A\in S$ if and only if x_0 is an I-density point of $R-A$. Observe that 0 is an I-dispersion point of A if and only if

$$\chi_{(n\cdot A)\cap[-1,1]} \xrightarrow[n\to\infty]{I} 0.$$

Similarly one can define right- and left-hand I-density points. We can take some interval $[-a, a]$, $a > 0$, instead of $[-1, 1]$. For $a < 1$ this follows immediately from Definition 1, for $a > 1$ this is a consequence of Theorem 1.

In the sequel we shall consider only sets having the Baire property as the σ-algebra S and for I we shall always take the family of meager sets. Under these assumptions we have:

THEOREM 1. *A point x_0 is an I-density point of a set $B\in S$ if and only if for every increasing sequence $\{t_n\}_{n\in N}$ of positive real numbers tending to infinity there exists a subsequence $\{t_{n_m}\}_{m\in N}$ such that $\chi_{(t_{n_m}\cdot(B-x_0))\cap[-1,1]} \xrightarrow[m\to\infty]{} 1$ I-a.e.*

Proof. The "only if" part is immediate.

The "if" part is a consequence of the following lemma (cf. [6]):

LEMMA. *If A is an open set and the sequences $\{i_n\}_{n\in N}$ and $\{j_n\}_{n\in N}$ have the following properties: $i_n > 0$, $j_n > 0$ for each $n\in N$, $\lim_{n\to\infty} i_n = \infty$, $\lim_{n\to\infty} j_n = \infty$, $\lim_{n\to\infty} j_n/i_n = 1$ and if $\chi_{(i_n\cdot A)\cap[-1,1]} \xrightarrow[n\to\infty]{} 0$ I-a.e., then also $\chi_{(j_n\cdot A)\cap[-1,1]} \xrightarrow[n\to\infty]{} 0$ I-a.e.*

Remark 1. From the above theorem it follows that the sequence $\{n\}_{n\in N}$ in Definition 1 does not play a distinguished role. However, we shall always use this sequence instead of $\{t_n\}_{n\in N}$ for the sake of simplicity.

Let us introduce the following notation: $\Phi(A) = \{x\in R: x$ is an I-density point of $A\}$ for $A\in S$; $A\sim B$ always means that $A\triangle B\in I$.

THEOREM 2. *For every $A, B\in S$*

(1) $\Phi(A)\sim A$

(2) *if* $A\sim B$, *then* $\Phi(A) = \Phi(B)$

(3) $\Phi(\emptyset) = \emptyset$, $\Phi(R) = R$

(4) $\Phi(A\cap B) = \Phi(A)\cap\Phi(B)$.

Proof. Conditions (2) and (3) follow immediately from the definition. Now we prove (1). The set A has the Baire property, so there exists an open set G and two meager sets P_1, P_2 such that $A = (G-P_1)\cup P_2$. If $x\in G$, then obviously $x\in\Phi(A)$, so $G\subset\Phi(A)$. Hence $A-\Phi(A)\subset A-G\subset P_2\in I$. Observe also that $\Phi(A)-A\subset P_1\cup \mathrm{Fr}(G)\in I$. Indeed, let $x\notin P_1\cup\mathrm{Fr}(G)$ and suppose that $x\notin A$. We shall show that $x\notin\Phi(A)$. If $x\notin A$ and $x\notin P_1$, then $x\notin G$, so $x\in\mathrm{Int}(R-G)$ and $x\in\Phi(R-G)$. But $(R-G)\triangle(R-A) = G\triangle A\in I$, and in virtue of (2) $x\in\Phi(R-A)$. This means that $x\notin\Phi(A)$. From the above reasoning we have $\Phi(A)-A\in I$, and finally $\Phi(A)\triangle A\in I$. This ends the proof of (1). To prove 4) first observe that if, $C\subset D$, $C, D\in S$, then obviously $\Phi(C)\subset\Phi(D)$. Hence $\Phi(A\cap B)\subset\Phi(A)\cap\Phi(B)$, because $A\cap B\subset A$ and $A\cap B\subset B$. Now suppose that $x_0\in\Phi(A)\cap\Phi(B)$. Then let $\{n_m\}_{m\in N}$ be an increasing sequence of natural numbers. From the assumption that $x_0\in\Phi(A)$ it follows that there exists a subsequence $\{n_{m_p}\}_{p\in N}$ such that $\chi_{(n_{m_p}(A-x_0))\cap[-1,1]} \xrightarrow[p\to\infty]{} 1$ I-a.e. From the assumption that $x_0\in\Phi(B)$ it follows that there exists a subsequence $\{n_{m_{p_r}}\}_{r\in N}$ such that $\chi_{(n_{m_{p_r}}\cdot(B-x_0))\cap[-1,1]} \xrightarrow[r\to\infty]{} 1$ I-a.e. Hence we conclude that $\chi_{(n_{m_{p_r}}\cdot(A\cap B-x_0))\cap[-1,1]} \xrightarrow[r\to\infty]{} 1$ I-a.e., which means that $x_0\in\Phi(A\cap B)$. Thus $\Phi(A)\cap\Phi(B)\subset\Phi(A\cap B)$ and we are done.

From the above theorem it follows that the operation Φ coincides with the so called "lower density" (see [5], Th. 22.4). Put $\mathcal{T}_I = \{\Phi(A)-N: A\in S, N\in I\}$.

THEOREM 3. *$\mathcal{T}_I$ is a topology on the real line.*

This theorem is a consequence of Proposition 1, Section 1, Chapter 5 from [8].

Remark 2. Observe that $\mathcal{T}_I = \{A\in S: A\subset\Phi(A)\}$. Indeed, if $A\in S$ and $A\subset\Phi(A)$, then $A = \Phi(A)-(\Phi(A)-A)$ and $N = \Phi(A)-A\in I$ from Theorem 2, Claim (1), and so $A\in\mathcal{T}_I$. Conversely, if $B\in\mathcal{T}_I$, then $B = \Phi(A)-N$ for some $A\in S$ and $N\in I$. We have $A\sim\Phi(A)\sim B$, so $A\sim B$. In virtue of Theorem 2, Claim (2), $\Phi(A) = \Phi(B)$. This means that $B\subset\Phi(B)$ and obviously $B\in S$.

DEFINITION 2. We call $\mathcal{T}_I$ the *I-density topology.*

Remark 3. There exists an open set $E = \bigcup_{n=1}^{\infty}(a_n, b_n)$, where $\{b_n\}_{n\in N}$ tends decreasingly to zero, $a_{n+1} < b_{n+1} < a_n$ for every $n\in N$ such that 0 is an I-dispersion point of E (cf. [6]).

From the above remark it follows that the notion of an I-density point is rather delicate and different from the notion of a residual point.

Obviously $\mathcal{T}_I$ is stronger than the natural topology on the real line, and so it is Hausdorff topology. Let us list some other properties of $\mathcal{T}_I$.

THEOREM 4. *Every countable subset of R is $\mathcal{T}_I$-closed. Consequently, $(R, \mathcal{T}_I)$ is not a separable space and every compact subspace of $(R, \mathcal{T}_I)$ is finite.*

The proof is immediate.

THEOREM 5. *$(R, \mathcal{T}_I)$ is not a regular (T_3) space.*

Proof. It suffices to observe that if Q is the set of rational numbers, then 0 and $Q-\{0\}$ cannot be separated by $\mathcal{T}_I$-open sets.

Now we shall study some basic properties of continuous functions from $(R, \mathcal{T}_I)$ into R equipped with the natural topology.

DEFINITION 3. We say that a function $f\colon R \to R$ is *I-approximately continuous at x_0* if and only if for every $\varepsilon > 0$ the set $f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$ has x_0 as an I-density point.

DEFINITION 4. We say that a function $f\colon R \to R$ is *I-approximately continuous* if and only if for every interval (y_1, y_2) the set $f^{-1}((y_1, y_2))$ belongs to $\mathcal{T}_I$.

COROLLARY 1. *From Theorem 5 it follows that the topology $\mathcal{T}_I$ in R is not the coarsest topology for which every I-approximately continuous function is continuous.*

From the above definitions we obtain immediately the following theorem:

THEOREM 6. *A function $f\colon R \to R$ is I-approximately continuous if and only if it is I-approximately continuous at every point.*

THEOREM 7. *A function $f\colon R \to R$ has the Baire property if and only if it is I-approximately continuous I-a.e.*

Proof. Suppose that f has the Baire property. Then there exists a residual set $E \subset R$ such that the restriction $f|E$ is continuous. Then f is I-approximately continuous at every point of E, i.e. it is I-approximately continuous I-a.e.

Now suppose that $f\colon R \to R$ is I-a.e. I-approximately continuous. Let $a \in R$, $E = \{x\colon f(x) < a\}$, and let A be the set where f is I-approximately continuous. Since $E - A$ is meager, it suffices to show that $E \cap A$ has the Baire property. If $x \in E \cap A$, there exists a set $E(x)$ having x as a point of I-density and having the Baire property such that $E(x) \subset E$; further, we can take $E(x) \subset A$. Then $E \cap A = \bigcup_{x \in E \cap A} E(x)$. Suppose that $E \cap A$ does not have the Baire property. There exists a pair of sets K and H such that $K \subset E \cap A \subset H$, K is of type G_δ, H is of type F_σ, $H - K \in S - I$ and for each $B \in S$, if $B \subset H - (E \cap A)$ or $B \subset (E \cap A) - K$, then $B \in I$ (compare [2], Th. 1.6, p. 25). The set $(E \cap A) - K$ is of the second category (in the opposite case E would have the Baire property), so there exists a point $x_0 \in (E \cap A) - K$ which is an I-density point of $H - K$; this follows from the fact that $(E \cap A) - K \subset H - K$ and the set of points belonging to $H - K$ which are not I-density points of $H - K$ belongs to I (according to Theorem 2). The point x_0 is an I-density point of $E(x_0)$, hence it is also an I-density point of $E(x_0) - K = E(x_0) \cap (H - K)$. It follows that $E(x_0) - K \in S - I$ and $(E \cap A) - K \supset E(x_0) - K$, a contradiction. Thus $E \cap A$ has the Baire property and the proof is finished.

THEOREM 8. *If a function $f\colon R \to R$ is I-approximately continuous, then f is of the first class of Baire and has the Darboux property.*

Proof. We shall need the following lemma:

LEMMA 1. *If 0 is an I-density point of a set A, then for every natural number n there exists a number $\delta_n > 0$ such that for every h with $0 < h < \delta_n$ and for every natural number k fulfilling the inequality $-n \leqslant k \leqslant n-1$ we have*

$$A \cap [\tfrac{k}{n} \cdot h, \tfrac{k+1}{n} \cdot h] \neq \emptyset.$$

Proof of the lemma. Suppose that this is not the case, i.e. that there exists a natural number n_0 such that for every $\delta = m^{-1}$ (m natural) there exist h_m, $0 < h_m < m^{-1}$, and a natural number k_m, $-n_0 \leqslant k_m \leqslant n_0 - 1$, such that $A \cap [\frac{k_m}{n_0} \cdot h_m, \frac{k_m+1}{n_0} \cdot h_m] = \emptyset$. Obviously we can find a subsequence $\{k_{m_p}\}_{p \in N}$, which is constant. For simplicity we shall suppose that the whole sequence $\{k_m\}_{m \in N}$ is constant, $k_m = k_0$ for each natural m, and that $k_0 \geqslant 0$ (the case $k_0 \leqslant -1$ needs only a slight modification). For each natural m let n_m be the greatest natural number such that $n_m \cdot h_m \leqslant 1$. Then we have $\lim_{m \to \infty} n_m \cdot h_m = 1$, because $h_m \xrightarrow[m \to \infty]{} 0$. Hence we have for sufficiently large m $(n_m \cdot A) \cap \left[\frac{k_0}{n_0}, \frac{k_0 + \frac{1}{2}}{n_0}\right] = \emptyset$ and there is no subsequence $\{n_{m_p}\}_{p \in N}$ such that $\chi_{(n_{m_p} \cdot A) \cap [-1,1]} \xrightarrow[p \to \infty]{} 1$ I-a.e. —a contradiction. The lemma is proved.

Proof of the theorem. Observe that the lemma is true for x_0 instead of 0 and for the intervals $[x_0 + \frac{k}{n} h, x_0 + \frac{k+1}{n} \cdot h]$, where $k \in [-n, n-1]$ for some $n \in N$. In this situation we say that A is relatively n-dense on $[x_0 - h, x_0 + h]$.

Now suppose that f is not in the first class of Baire. According to the theorem of Preiss [7], which is an immediate consequence of a classical theorem (cf. [4], p. 395), there exists a perfect set F and two real numbers a, b $(a < b)$ such that the sets $T_1 = \{x\colon f(x) < a\}$ and $T_2 = \{x\colon f(x) > b\}$ are both dense in F, (i.e. $\overline{T_1 \cap F} \supset F$ and $\overline{T_2 \cap F} \supset F$).

Let $P_i \subset T_i$ be a countable set, dense in T_i (thus also dense in F) for $i = 1, 2$. Suppose that $P_1 = \{x_1, x_2, \ldots\}$. For any natural n take a number $\delta_n > 0$ such that T is relatively n-dense in $[x_n - \delta_n, x_n + \delta_n]$ (and in every shorter interval concentric with it). The existence of δ_n follows from the lemma. Obviously we may suppose that $\delta_n \xrightarrow[n \to \infty]{} 0$. The set V_1 of points belonging to infinitely many intervals of the form $[x_n - \delta_n, x_n + \delta_n]$ is residual in F. Similarly we construct the set V_2 (for T_2), which is also residual in F. Hence $V_1 \cap V_2$ is residual in F; in particular, it is non-empty.

Now we shall show that if $\bar{x} \in V_1$, then $\bar{x}$ is not an I-dispersion point of T_1. Let $\{m_k\}_{k \in N}$ be an increasing sequence of natural numbers such that $\bar{x} \in [x_{m_k} - \delta_{m_k}, x_{m_k} + \delta_{m_k}]$. For simplicity of notation assume that $\bar{x} = 0$. Suppose that infinitely many x_{m_k}'s are positive (in the other case the proof is similar).

Modifying the subsequence $\{m_k\}_{k \in N}$ we can suppose that all x_{m_k}'s are positive. Then the piece of the interval $[x_{m_k} - \delta_{m_k}, x_{m_k} + \delta_{m_k}]$ lying to the right from zero is greater than the piece lying to the left. For sufficiently big k let n_k be the greatest natural number such that $n_k \cdot (x_{m_k} + \delta_{m_k}) < 1$. We have $n_k \cdot (x_{m_k} + \delta_{m_k}) \xrightarrow[k \to \infty]{} 1$, since $x_{m_k} + \delta_{m_k} \xrightarrow[k \to \infty]{} 0$. Let $E_k = n_k \cdot (T_1 \cap [0, x_{m_k} + \delta_{m_k}])$. The set T_1 is relatively m_k-dense on $[x_{m_k} - \delta_{m_k}, x_{m_k} + \delta_{m_k}]$.

Observe also that there exists an open (in the natural topology) set $G \subset T_1$

which is dense in T_1. Indeed, $T_1 = \{x: f(x) < a\} = \bigcup_{n=1}^{\infty} \{x: f(x) \leqslant a-\frac{1}{n}\}$. Obviously $\{x: f(x) \leqslant a-\frac{1}{n}\}$ has the Baire property for each n, and so $\{x: f(x) \leqslant a-\frac{1}{n}\} = G_n \triangle P_n$ where G_n is open and P_n is of the first category. We shall show that $G_n \subset \{x: f(x) \leqslant a-\frac{1}{n}\}$ for each n. Fix n and suppose that this is not the case. Then there exists a point $x_n \in G_n - \{x: f(x) \leqslant a-\frac{1}{n}\}$, so $f(x_n) > a-\frac{1}{n}$. Let $\varepsilon > 0$ be a number such that $f(x_n)-\varepsilon > a-\frac{1}{n}$. Since f is I-approximately continuous at x_n, we have $\{x: f(x) > f(x_n)-\varepsilon\} \in \mathcal{T}_I$, x_n is an I-density point of this set and obviously $\{x: f(x) > f(x_n)-\varepsilon\} \cap \{x: f(x) \leqslant a-\frac{1}{n}\} = \emptyset$. Let $\{x: f(x) > f(x_n)-\varepsilon\} = \hat{G}_n \triangle \hat{P}_n$, where $\hat{G}_n$ is open and $\hat{P}_n \in I$. By the above argument we have $(\hat{G}_n \triangle \hat{P}_n) \cap (G_n \triangle P_n) = \emptyset$, whence it follows immediately that $\hat{G}_n \cap G_n = \emptyset$. But x_n is an I-density point of $\hat{G}_n \triangle \hat{P}_n$, and so in every neighbourhood of x_n there are points of $\hat{G}_n$. If we take the component of G_n containing x_n in its interior, we arrive at a contradiction. Thus $G_n \subset \{x: f(x) \leqslant a-\frac{1}{n}\}$. Hence $G = \bigcup_{n=1}^{\infty} G_n \subset T_1$. It suffices to show that G is dense in T_1. Let $x_0 \in T_1$. There exists n_x such that $x_0 \in \{x: f(x) \leqslant a-\frac{1}{n_x}\}$. Now it is easy to see that in every neighbourhood of x_0 there are some points of G_{n_x+1}. This proves that G is dense in T_1.

From the two facts, namely, the relative m_k-density of T_1 on $[x_{m_k}-\delta_{m_k}, x_{m_k}+\delta_{m_k}]$ and the existence of open set G with the above properties we infer that for every increasing sequence $\{k_p\}_{p \in N}$ of natural numbers the union $\bigcup_{p=1}^{\infty} E_{k_p}$ contains an open set dense in $[0, 1]$. Hence the set $\limsup_{p\to\infty} E_{k_p}$ is residual on $[0, 1]$ and $\{\chi_{E_{k_p}}\}_{p \in N}$ does not tend to zero I-a.e. We have proved the existence of a sequence $\{n_k\}_{k \in N}$ without a subsequence $\{n_{k_p}\}_{p \in N}$ for which $\chi_{E_{k_p}} \xrightarrow[p\to\infty]{} 0$ I-a.e. This means that 0 is not an I-dispersion point of T_1 (because $E_k \subset n_k \cdot T_1$).

Similarly one can prove that if $\bar{x} \in T_2$, then $\bar{x}$ is not an I-dispersion point of T_2.

Let now $x_0 \in V_1 \cap V_2$ and let $\varepsilon > 0$ fulfils also the inequality $\varepsilon < \dfrac{b-a}{3}$. Let $T = f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$. A point x_0 is an I-density point of T and it is an I-dispersion point neither of T_1 nor of T_2. Hence $T \cap T_1 \neq \emptyset$ and $T \cap T_2 \neq \emptyset$. Let $x' \in T \cap T_1$, $x'' \in T \cap T_2$. We have $|f(x')-f(x_0)| < \varepsilon$ and $|f(x'')-f(x_0)| < \varepsilon$, and simultaneously $f(x') < a$ and $f(x'') > b$. Hence $|f(x')-f(x'')| < 2\varepsilon < b-a$ and simultaneously $f(x'')-f(x') > b-a$, a contradiction. Consequently f is of the first class of Baire.

The fact that f has the Darboux property follows immediately from Th. 1.1 (2) in [1].

Remark 5. An analogous theorem holds for the density topology (see [3]).

Corollary 2. *Every interval $[a, b]$ is a connected set in $(R, \mathcal{T}_I)$.*

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Received 29 July 1983