Exotic ANR's via null decompositions of Hilbert cube manifolds

by

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Abstract. The following theorem is proved.

THEOREM. *For each Hilbert cube manifold M^Q, there exists a family $\{G_\lambda\colon \lambda \text{ in } \Lambda\}$ of upper semicontinuous decompositions having the cardinality of continuum such that*: (a) *For each λ in Λ, the nondegenerate elements of G_λ form a null collection of arcs and the associated decomposition space M^Q/G_λ is an* ANR; (b) *any two decomposition spaces are topologically distinct, i.e., the family of* ANR*'s M^Q/G_λ's has the cardinality of continuum*; (c) *for each λ in Λ, M^Q/G_λ does not contain any proper subset of dimension $\geqslant 2$ which is an* FANR; *and* (d) *for each λ in Λ, $M^Q/G_\lambda \times I$ is homeomorphic to $M^Q \times I$ where $I = [-1, 1]$. Moreover, the family of decomposition can be constructed so that, in addition, M^Q/G_λ is rigid (the only self-homeomorphism is the identity) for each λ in Λ. This result generalizes our earlier similar result for finite dimensional manifolds.*

§ 1. **Introduction.** By AR (ANR) we mean compact metric absolute (neighborhood) retract. Bing and Borsuk [5] gave an example of an upper semicontinuous decomposition of the 3-cell whose nondegenerate elements form a null collection of arcs such that: the associated decomposition space is a 3-dimensional AR which does not contain any disk. This work of Bing and Borsuk is extended in several directions in various papers; see [22, 23, 24, 25, 26, 28]. As a sample, we state the following result which appears in [25]:

THEOREM. *For each topological n-manifold with $3 \leqslant n < \infty$, there exists an upper semicontinuous decomposition whose nondegenerate elements form a null collection of arcs such that: the associated decomposition space is an n-dimensional* ANR (*or* AR *when the n-manifold is the n-cell*) *which does not contain any proper subset which is an* FANR *of dimension $\geqslant 2$.*

A compact metric fundamental absolute neighborhood retract is denoted by FANR; this terminology is due to Borsuk [8]. The class of FANR sets includes ANR sets and compacta shape equivalent to ANR sets; see Borsuk [8] for a sharper statement.

The purpose of this paper is to extend the theorem given above. Indeed, this theorem remains valid in its entirety when the phrase "Q-manifold" is substituted for the phrase "topological n-manifold with $3 \leqslant n < \infty$"; see § 5 for a sharper and more specific statement. This work is based on T. Lay's thesis [21] and our earlier work [25]. We have set things up so that we may apply "the Backing-up Technique" given in [25]. This has helped in keeping the length under control, however, this required a careful review of some crucial results from [21]. In the review, we have also included some proofs whenever they were necessary for latter development, clarification, and motivation. The reader interested in the cell-like decompositions of the Hilbert cube manifolds whose every element is nondegenerate may consult [21] which proceeds in the spirit of [14].

The main results of this paper were announced in [3]. This work is dedicated to the memory of late Professor K. Borsuk whose mathematics is a source of inspiration. Finally, I wish to thank Steve Armentrout and R. J. Daverman for helpful conversations.

§ 2. Notation and terminology. There are two types of manifolds that appear in this paper. They are: (a) finite dimensional n-manifolds or simply n-manifolds, and (b) infinite dimensional Hilbert cube manifolds or Q-manifolds where Q denotes the Hilbert cube. We often denote by M^n an n-manifold which may have nonempty boundary. *All n-manifolds considered here will be orientable.* We often denote a Q-manifold by M^Q. A good reference for Q-manifold theory is Chapman [10]. We equate Q with $\prod_{i\geqslant 1} I_i$ where $I_i = [-1, 1]$, and we often factor Q as, $Q = I^n \times Q_{n+1}$, where $I^n = \prod_{i=1}^{n} I_i$ and $Q_{n+1} = \prod_{i>n} I_i$. A *disk with holes* is a possibly disconnected compact planar 2-manifold with boundary. All maps will be continuous. We denote by $\operatorname{Int} A$ and $\operatorname{Bd} A$, the interior of A and the boundary of A, respectively. A 2-cell is often called a disk. We denote the compact n-ball, consisting of all the vectors in the real Euclidean n-space E^n having length $\leqslant 1$, by B^n. The boundary sphere of B^n is denoted by S^{n-1}. A collection C of subsets of a metric space is called *null* provided for each $\varepsilon > 0$ there are at most finitely many sets in C each of whose diameter is larger than ε. The term "PL" will mean "piecewise linear".

§ 3. Wild Cantor sets in Q-manifolds.

(3.1) General comments. The combined works of Antoine [1] and Blankinship [6] give an example of a wild Cantor set in E^n for each $n \geqslant 3$. Wong [27] gave a similar example of a wild Cantor set in the Hilbert cube Q. Recently, Daverman–Edwards [13] have given a general technique for producing wild Cantor sets in an n-manifold with $n \geqslant 3$. An extension of this technique for Q-manifolds is given by Lay [21].

We are mainly interested in the wild Cantor set construction in Q-manifolds given by Lay [21]. The section is devoted to a quick review of some useful results from [21] with a view to our specific application.

(3.2) I-essential maps and a property (CR). Consider the product $M \times B^2$ where M is a topological space (a manifold in our application) and B^2 is the 2-disk. We shall identify M with the subset $M \times \{0\}$ of $M \times B^2$ whenever the context is

clear where 0 denotes a point in $\operatorname{Int} B^2$. A map $h\colon H \to M \times B^2$ from a disk with holes is called *I-essential* ("*interior essential*") if $h(\operatorname{Bd} H) \subset M \times \operatorname{Bd} B^2$ and there is no map $h'\colon H \to M \times \operatorname{Bd} B^2$ which agrees with h along $\operatorname{Bd} H$. A subset N of $M \times \operatorname{Int} B^2$ satisfies *Property* (CR) in $M \times B^2$ if the image of any I-essential map $h\colon H \to M \times B^2$ from a disk with holes meets N. Note that the "core" $M \times \{0\}$ has (CR) in $M \times B^2$; the geometric idea is to replace this core by some other subset without destroying this property, and hence, the name (CR) stands for *core replacement* in this sense. Property (CR) is called Property ($*$) in [21].

(3.3) A Core Replacement Lemma. *Suppose M denotes a compact, orientable, PL n-manifold with or without boundary. Suppose C is a Cantor set in $M \times B^2$. Then, for each $\varepsilon > 0$, there exists a compact, orientable, PL n-manifold N with or without boundary such that*:

(a) *$N \times B^2$ is contained in $(M \times \operatorname{Int} B^2) - C$,*
(b) *N has Property* (CR), *and*
(c) *each component of N has diameter less than ε.*

Note. The products $N \times B^2$ and $M \times B^2$ are not ambiently related. More specifically, the product $N \times B^2$ is abstract and the B^2 factor in $N \times B^2$ is not contained in the B^2 factor in $M \times B^2$. This is similar to the construction of Antoine's necklace where the product structures on the successive manifolds (union of disjoint tori) are not related. As a rule, interpret all products to be abstract unless to the contrary is clear from the context.

This lemma is extremely useful for producing wild Cantor sets (missing some preassigned Cantor sets) in $M \times B^2$. Daverman and Edwards first proved this lemma for a closed n-manifold M. The extended version given above is due to Lay [21]; see Chapters II–III of [21].

The geometric meaning of this lemma can be easily understood as follows. The reader may draw a figure as in the construction of Antoine necklace [1, 5, 6]. Note that the curve $J = \{m\} \times \operatorname{Bd} B^2$ is essential in the complement of M. This lemma guarantees that J is also essential in the complement of $N_1 = N$ since N_1 satisfies (CR). Now apply the lemma to $N_1 \times B^2$ to find N_2 and conclude that J remains essential in the complement of N_2. Continue in this manner to find $N_1, N_2, \ldots$ such that J is essential in the complement of each N_i, $1 \leqslant i < \infty$, and the intersection of $N_1 \times B^2, N_2 \times B^2, \ldots$ is a zero dimensional compact set C' (use (c) to make the diameters of the components of N_i go to zero as i goes to ∞). By discarding isolated points if necessary assume C' is a Cantor set. The key fact is that C' satisfies (CR) in $M \times B^2$; in particular, the curve J is essential in $(M \times B^2) - C'$, and thus, C' is wild. For a complete proof see [21].

The following lemma due to Lay [21] shows how the preceding lemma can be used to construct wild Cantor sets in Q-manifolds (compare Wong [27]).

(3.4) Lemma (A construction of Wild Cantor sets in Q-manifolds). *Every compact Q-manifold M^Q contains a wild Cantor set.*

For the purpose of notation as well as other reasons, it is instructive to sketch a proof of this lemma.

Proof. By the Triangulation Theorem (cf. [10]), M^Q can be factored $M^Q = M\times B^2\times Q$. Put $J = \{m\}\times \operatorname{Bd} B^2\times\{q\}$ where m and q belong to M and Q, respectively.

Step (0): Put $N_0 = M$ and $n(0) = 1$.

Step (1): Choose $n(1) > n(0)$ such that $\operatorname{diam} Q_{n(1)+1} < \frac{1}{2^1}$. Use (3.3) to find a compact orientable PL $(n+n(1))$-manifold N_1 satisfying (CR) and having a PL product neighborhood $N_1\times B^2$ contained in $N_0\times I_1\times I_2\times\ldots\times I_{n(1)}\times \operatorname{Int} B^2$.

Step (2): Choose $n(2) > n(1)$ such that $\operatorname{diam} Q_{n(2)+1} < \frac{1}{2^2}$. Use (3.3) to find a compact orientable PL $(n+n(2)$-manifold N_2 satisfying (CR) and having a PL product neighborhood $N_2\times B_2$ contained in $N_1\times(I_{n(1)+1}\times\ldots\times I_{n(2)})\times B^2$.

Continue in this manner (the inductive step is clear). Put $S_i = N_i\times Q_{n(i)+1}$ and let C denote the intersection of the nested sequence $S_1\times B^2$, $S_2\times B^2,\ldots$ We refer to the collection of components of $S_i\times B^2$ by ith *stage* of the construction. Now C is a 0-dimensional compact metric space (review (c) of (3.3)) which, if constructed more carefully, is a Cantor set or it contains a Cantor set with the desired properties. At any rate, assume (without loss of generality) C is a Cantor set. Let $f\colon D\to M^Q$ be a map from a 2-disk such that f restricted to $\operatorname{Bd} D$ is a homeomorphism onto J. Since f is I-essential, $f(D)$ meets $S_i\times B^2$ for each $i\geqslant 1$, and thus, $f(D)$ meets C. This proves that the fundamental group $\pi_1(M^Q - C)$ of the complement of C is nontrivial (since the loop J essential). Thus, C is wild. ■

4. Construction of decompositions.

(4.1.) Linking in Q-manifolds. Suppose X and Y are two disjoint compact subsets of a Q-manifold M^Q. We say *X links Y* if each neighborhood of X contains a loop in $(M^Q - Y)$ which is essential in $(M^Q - Y)$. The following useful lemma due to Lay [21] is a generalization to Q-manifolds of a finite dimensional result given in [24]; see [24] for more discussions on linking in manifolds. An embedding of a manifold M^n into another manifold N^m is said to be *proper* if it carries boundary and interior of M^n into boundary and interior of N^m, respectively.

(4.1.1) A Linking Lemma. *Suppose $M\times Q$ is a compact Q-manifold where M is a compact connected* PL *n-manifold with $n\geqslant 3$. Suppose A is a closed subset of $M\times Q$ such that there exists an essential map $h\colon A\to S^1$. Then there exists an integer k and a compact $(n+k-2)$-manifold N properly embedded in $M\times I^k$ such that*:

(a) *N has a* PL *product neighborhood $N\times B^2$ in $M\times I^k$,*

(b) *$N\times B^2\times Q_{k+1}$ is contained in $(M\times Q)-A$, and*

(c) *A links $N\times Q_{k+1}$.*

We present a proof of this crucial lemma since it is elementary and not lengthy. Our proof is a slightly modified version of the proof given in [21]. Since [21] may not be as readily available, we give sufficient details.

Proof. Since S^1 is an ANR, there exists an extension of h to a neighborhood of A. Choose a compact Q-manifold nbd. V of A and an extension $H\colon V\to S^1$ such that H is essential. Let a and b be two distinct points of S^1. Write S^1 as the union of two arcs L, R whose intersection is the set $\{a, b\}$ consisting of their common endpoints. Choose an integer m such that the images, $p_m(H^{-1}(a))$, $p_m(H^{-1}(b))$, of the disjoint compact sets, $H^{-1}(a)$, $H^{-1}(b)$, are disjoint under the projection map $p_m\colon M\times Q\to M\times I^m$. Choose a compact, bicollared, properly embedded PL $(n+m-1)$-manifold N_1 contained in $M\times I^m$ which separates the disjoint compact subsets $p_m(H^{-1}(a))$ and $p_m(H^{-1}(b))$.

Note that the intersection of the three sets $H^{-1}(L)$, $H^{-1}(R)$, and $N_1\times Q_{m+1}$ is empty. Choose an integer $k > m$ such that the intersection of three sets $p_k(H^{-1}(L))$, $p_k(H^{-1}(R))$, and $p_k(N_1\times Q_{m+1})$ under the projection $p_k\colon M\times Q\to M\times I^k$, is empty. Choose a compact, properly embedded, and bicollared PL $(n+k-2)$-manifold N in $N_1\times I_{m+1}\times\ldots\times I_k$ separating the compact subsets $[p_k(H^{-1}(L))\cap p_k(N_1\times Q_{m+1})]$ and $[p_k(H^{-1}(R)\cap p_k(N_1\times Q_{m+1})]$. Observe that the collars can be used to produce a PL product neighborhood $N\times B^2$ in $M\times I^k$. Also, observe that $N\times Q_{k+1}$ does not meet A. Thus a suitable thickening $N\times B^2\times Q_{k+1}$ of $N\times Q_{k+1}$ is contained in $(M\times Q)-A$. The only remaining step is to show that A links $N\times Q_{k+1}$. This is done below.

By the Bridge Theorems of Hu [17; p. 59], find the nerve L of a suitable open cover of A such that h is homotopic to the composite $d\circ c$ where $c\colon A\to L$ is a canonical map and $d\colon L\to S^1$ is a suitable map. Observe that d is essential. This implies that the restriction $d'\colon L^1\to S^1$ of d to the 1-skeleton L^1 of L is also essential since S^1 is aspherical (i.e., the higher homotopy groups of S^1 vanish). Find a map $e\colon L^1\to V$ such that $H\circ e$ is homotopic to d'. Use the local path connectedness of V to construct e or this follows from shape theory [8]. It is easy to see that there exists a map $g\colon S^1\to L^1$ such that $H\circ e\circ g$ is essential. We next show that the map $f = e\circ g\colon S^1\to V$ determines a loop which is essential in the complement of $N\times Q_{k+1}$.

Let $F\colon B^2\to M\times Q$ be an extension of f (observe that our proof is finished if there is no extension F). Consider $F = (F_1, F_2)$ where $F_1 = p_k\circ F$. Adjust F_1 a little bit so that the new map which is again denoted by F satisfies: (a) the restriction of $H\circ F$ to $S^1 = \operatorname{Bd} B^2$ is still essential, and (b) $F^{-1}(N_1\times Q_{m+1})$ is a finite collection of disjoint simple closed curves and spanning arcs. Since $H\circ F$ restricted to S^1 is essential and $N_1\times Q_{m+1}$ separates $H^{-1}(a)$ from $H^{-1}(b)$, there exists at least one spanning arc α such that $HF(\operatorname{Bd}\alpha)$ meets both L and R. Now $F(\alpha)$, and hence, $F(B^2)$ meets $N\times Q_{k+1}$ since $N\times Q_{k+1}$ separates the sets $[H^{-1}(L)\cap(N_1\times Q_{m+1})]$ and $[H^{-1}(R)\cap(N_1\times Q_{m+1})]$. This finishes the proof. ■

(4.2) A countable collection of thickened codimension two manifolds. Suppose $M^Q = M^n\times Q$ denote a compact and connected Q-manifold where M^n is a PL n-manifold with $n\geqslant 3$. For each integer $k\geqslant 1$, choose a countable collection $\mathscr{C}_k$ of compact PL $(n+k)$-manifolds such that: (a) each manifold is properly embedded in $M^n \times I^k$, (b) each is of the form $S_k \times B^2$ where S_k is an $(n+k-2)$-manifold called the *surface* of the original $(n+k)$-manifold, and (c) for each $\varepsilon > 0$ and for each proper embedding $e\colon S \times B^2 \to M^n \times I^k$, there exists a manifold $S_k \times B^2$ in $\mathscr{C}_k$ and a surjective PL homeomorphism $e'\colon S \times B^2 \to S_k \times B^2$ such that the distance between the points $e(x)$ and $e'(x)$ is less than ε for each x in $S \times B^2$. Define $\hat{\mathscr{C}}_k = \{\mathscr{C} \times Q_{k+1}\colon \mathscr{C}$ belongs to $\mathscr{C}_k\}$. Let $\mathscr{C}$ denote the union of the collections $\hat{\mathscr{C}}_1, \hat{\mathscr{C}}_2, \ldots$ Enumerate elements of $\mathscr{C}$ as $M_1, M_2, \ldots$ so that each element of $\mathscr{C}$ is repeated infinitely many times.

(4.3) Dyadic arcs. Recall $M^Q = M^n \times Q$ is a compact and connected Q-manifold where M^n is a PL n-manifold with $n \geqslant 3$. We will be brief and rely on [25] for details. A collection $\{X_i\colon 1 \leqslant i \leqslant n\}$ consisting of disjoint subsets of a metric space X is called a *δ-chain* provided each X_i has diameter less than δ and each of the sets $(X_1 \cup X_2), (X_2 \cup X_3), \ldots, (X_{n-1} \cup X_n), (X_n \cup X_1)$ has diameter less than 2δ; see [25] for some discussion relevant to this notion. We inductively proceed to construct a null sequence of arcs one for each element M_i of $\mathscr{C}$ (see (4.2) for a definition of $\mathscr{C}$). These arcs are called "dyadic arcs" for geometric reasons.

Step 1. Proceed as in Lemma (3.4) to construct a wild Cantor set C in M_1. Let $\{M(1,1), M(1,2), \ldots, M(1,k(1))\}$ denote the first stage in the construction of C (see proof of Lemma (3.4) for the definition of ith stage). We construct this first stage with additional care so that it is a $\frac{1}{2}$-chain (in M_1). By a suitable ramification process, see Cannon–Daverman ([9]; page 376]) or [25], we construct a "parallel" first stage and a "parallel" Cantor set C'. The word "parallel" indicates the geometric meaning, i.e., C and C' are disjoint and isotopic by an isotopy that carries stages of C to the disjoint stages of C'; see [25] for more discussions. For each i, $1 \leqslant i \leqslant k(1)$, put $C(1,i) = C \cap M(1,i)$ and $C'(1,i) = C' \cap M(1,i)$. We say $C(1,i)$ and $C'(1,i)$ are two *parallel* Cantor sets in $M(1,i)$. Let $A(1,i)$ be an arc inside $M(1,i)$ which contains $C(1,i)$ so that $A(1,i)$ is PL modulo $C(1,i)$. Similarly, find an arc $A'(1,i)$ in $M(1,i)$ containing $C'(1,i)$ so that $A(1,i)$ and $A'(1,i)$ are parallel. Observe that $\{A(1,i) \cup A'(1,i)\colon 1 \leqslant i \leqslant k(1)\}$ is $\frac{1}{2}$-chain inside M_1 and we call it *$\frac{1}{2}$-chain of parallel arcs substituting for* M_1. For each i, $1 \leqslant i \leqslant k(1)$, construct a *dyadic arc* $L(1,i)$ containing $A(1,i)$ and $A'(1,i)$; the details are as in [25] and we omit them. Observe that $\{L(1,i)\colon 1 \leqslant i \leqslant k(1)\}$ is a 1-chain in M_1 and we call it *1-chain of dyadic arcs substituting for* M_1.

Step 2. Suppose for each $i = 1, 2, \ldots, n-1$, a $\dfrac{1}{2^{i-1}}$-chain of dyadic arcs substituting for M_i has been constructed. Proceed as in Step 1 to construct a $\dfrac{1}{2^{n-1}}$-chain of dyadic arcs $\{L(n,1), \ldots, L(n,k(n))\}$ substituting for M_n satisfying the additional requirement that the totality of arcs constructed thus far belonging to the chains substituting for $M_1, \ldots, M_n$ are pairwise disjoint. Since this additional requirement can be easily satisfied by using Lemmas (3.3) and (3.4), our inductive step follows.

(4.4) A decomposition. We now describe an upper semicontinuous decomposition G of the Hilbert cube manifold $M^Q = M^n \times Q$ (see (4.2) for a definition of M^Q). The nondegenerate elements of G is the collection

$$\{L(n,i)\colon 1 \leqslant n < \infty,\ 1 \leqslant i \leqslant k(n)\}$$

of dyadic arcs constructed in (4.3). Since this collection of dyadic arcs is null, it follows that the decomposition is upper semicontinuous. Let $p\colon M^Q \to M^Q/G$ denote the natural projection onto the decomposition space. The main properties of this decomposition space will be given in § 5.

(4.5) A family of decompositions. The following result appears in [16]:

There exists a family $\{A_\lambda\colon \lambda$ belongs to $\Lambda\}$ of arcs having the cardinality of continuum such that the fundamental groups of the complements $(M^Q - A_\lambda)$ and $(M^Q - A_\mu)$ are non-isomorphic whenever $\lambda \neq \mu$.

By suitably including countably many disjoint arcs from this family, in addition to, the null collection of dyadic arcs given in (4.4), and keeping all the arcs disjoint, we construct a new decomposition of M^Q whose nondegenerate elements is the null collection consisting of the totality of pairwise disjoint arcs. This method yields a family of topologically distinct decomposition spaces having the cardinality of continuum with additional properties; see [25] and [16] for more details. A more formal statement of these results appears in § 5.

§ 5. Main results.

The main result of this paper is Theorem (5.1). Although, the more general Theorem (5.2) also holds, we do not attempt its proof at present since it is more technical and lengthy.

(5.1) Theorem. *For each compact and connected Hilbert cube manifold M^Q, there exists a collection $\{G_\lambda\colon \lambda$ in $\Lambda\}$ of upper semicontinuous decompositions of M^Q having the cardinality of continuum such that the following hold:*

(a) *For each λ in Λ, the nondegenerate elements of G_λ form a null collection of arcs and the associated decomposition space M^Q/G_λ is an infinite dimensional* ANR.

(b) *Any two decomposition spaces M^Q/G_λ and M^Q/G_μ are topologically distinct, i.e., family of* ANR's *M^Q/G_λ's has the cardinality of continuum.*

(c) *For each λ in Λ, the decomposition space M^Q/G_λ does not contain any proper subset of dimension $\geqslant 2$ which is an* FANR *(the term* FANR *is explained below).*

(d) *For each λ in Λ, the* ANR *M^Q/G_λ is a Q-manifold factor. More precisely, $M^Q/G_\lambda \times I$ is homeomorphic to $M^Q \times I$ for each λ in Λ.*

Moreover, the construction of the decomposition can be performed with extra care so that the following additional conclusion holds:

(e) *For each λ in Λ, the decomposition space M^Q/G_λ is rigid, i.e., it has no self-homeomorphism other than the identity.*

(5.2) Remark. Theorem 5.1 also remains valid when M^Q is noncompact, i.e., the assumption of compactness is not essential, however, it is a technical convenience.

(5.3) Remarks on FANR sets. The notion of FANR (pronounced "*fundamental absolute neighborhood retract*") is a generalization of the notion of ANR; see Borsuk [8]. An FANR will always be compact and metrizable; this is consistent with [8]. Any compact metric space having the shape of an ANR is an FANR; see Borsuk [8] for a sharper result.

(5.4) Proof of Theorem (5.1). Apply the Triangulation Theorem from [10] to represent M^Q as a product of the form $M^n \times Q$ where M^n is a PL n-manifold with $n \geq 3$. Let G be the decomposition of M^Q described in § 4; see (4.4). It suffices to prove that M^Q/G satisfies the properties (a)–(e). The collection of similar decompositions $\{G_\lambda: \lambda \text{ in } \Lambda\}$ satisfying the conclusions of Theorem (5.1) has the cardinality of continuum; this follows from (4.5).

The nondegenerate elements of the decomposition G form a null collection of arcs. This follows from the construction; see (4.4). Since the decomposition is null and its nondegenerate elements are arcs, it follows from a theorem of Kozlowski [19] that the decomposition space M^Q/G is an ANR.

Our proof of part (c) is essentially the one given in [25]. The key ingredients are the dyadic arcs. The dyadic arcs allow us to apply the "Backing-up Technique" discussed in [25]. Thus, the entire line of argument given in [25] is available in this context. This proves (c). Part (d) follows from [29] and we omit details.

We rely on the results proved in [16] to construct rigid decomposition spaces satisfying (e). The details of this type of construction already appear in [16, 25, 26] and we omit them. This finishes our proof of Theorem (5.1). ■

(5.5) Concluding remarks. Each ANR given above, satisfying the conclusions of Theorem (5.1) or (5.2), contains movable continua of dimension greater than or equal to 2; a proof of this fact appears in [3]. The conclusion (c) of Theorem (5.1) can be sharpened as follows: (c′) For each λ in Λ, M^Q/G_λ does not contain any proper closed subset of dimension ≥ 2 which satisfies $UV^1(sl)$; see [25] for related discussions.

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