The maximal normal subspace of the inverse image of a normal space of sequences by a non-negative matrix transformation

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Abstract. Suppose λ is a normal space of sequences and A is an infinite matrix with non-negative entries. The spaces of sequences $\text{nor-}A^{-1}(\lambda)$ and $A^{-1}(\lambda)$, respectively, consist of those sequences $\{x_n\}$ such that A maps $\{|x_n|\}$ or $\{x_n\}$ into λ . When λ is a t.v.s. with a certain sort of topology, $A^{-1}(\lambda)$ and $\text{nor-}A^{-1}(\lambda)$ become t.v.s. in a natural way. Under certain conditions both $A^{-1}(\lambda)$ and $\text{nor-}A^{-1}(\lambda)$ inherit various t.v.s. properties of λ viz. normality, local convexity, completeness, etc. The differences between the behaviours of $A^{-1}(\lambda)$ and $\text{nor-}A^{-1}(\lambda)$ and some inclusion relations between φ , $A^{-1}(\lambda)$ and $\text{nor-}A^{-1}(\lambda)$ are considered.

0. Introduction. Let ω be the vector space of all real or complex numbers, φ the subspace of ω consisting of eventually zero sequences. All operations on sequences will be coordinatewise. All sequences $x \equiv \{x_n\}$ will be indexed from 1. Z^+ would denote the set of natural numbers. For $0 < p \leq \infty$, l_p will have its usual meaning and $\|\cdot\|_p$ is the usual norm or quasinorm; c will denote the space of convergent sequences, and c_0 the space of null sequences.

If $a \in \omega$, $S \subseteq \omega$, then $aS = \{ab \mid b \in S\}$. For $a \in \omega$, the *normal hull* $N(a)$ of a is given by

$$N(a) = \{b \in \omega \mid |b_n| \leq |a_n|, n \in Z^+\}.$$

If $S \subseteq \omega$, $N(S)$, the *normal hull* of S is given by

$$N(S) = \bigcup_{a \in S} N(a).$$

S is said to be *normal* if $N(S) = S$.

In what follows $A = (a_{mn})$ ($m, n = 1, 2, \dots$) will be an infinite matrix of scalars. The domain of A , denoted by $\mathcal{D}(A)$, is

$$\mathcal{D}(A) = \{x \in \omega \mid \sum_{n=1}^{\infty} a_{mn} x_n \text{ converges for } m \in Z^+\}.$$

For $x \in \mathcal{D}(A)$, Ax , the A -transform of x , is given by

$$(1) \quad (Ax)_m = \sum_{n=1}^{\infty} a_{mn} x_n.$$

If $S \subseteq \mathcal{D}(A)$,

$$(2) \quad AS = \{Ax \mid x \in S\}.$$

If $S \subseteq \omega$

$$(3) \quad A^{-1}(S) = \{x \in \mathcal{D}(A) \mid Ax \in S\}.$$

When A^{-1} , a matricial inverse of A exists, then we write

$$(4) \quad A^{-1}S = \{A^{-1}x \mid x \in S\}$$

whenever $S \subseteq \mathcal{D}(A^{-1})$. The presence or omission of parentheses in (3) and (4) should be noted with care.

We now introduce the

DEFINITION 1. If $S \subseteq \omega$ and A is an infinite matrix,

$$(5) \quad \text{nor-}A^{-1}(S) = \{x \in \omega \mid |x| \in A^{-1}(S)\} = \{x \in \omega \mid |x| \in \mathcal{D}(A) \text{ and } A|x| \in S\}.$$

The reason for the notation "nor" is that under the circumstances in which we shall be interested, $\text{nor-}A^{-1}(S)$ is the *largest* normal subset of $A^{-1}(S)$.

If λ is a topological vector space with some topology T , $A^{-1}(\lambda)$ naturally becomes a topological vector space with the induced topology $A^{-1}(T)$, the open sets of which are $A^{-1}(U)$, $U \in T$.

It will be shown in Section 2, Theorem 2.2, that under certain circumstances, T induces a topology on $\text{nor-}A^{-1}(\lambda)$ which is not coarser than the relative topology on $\text{nor-}A^{-1}(\lambda)$ as a subspace of $(A^{-1}(\lambda), A^{-1}(T))$. If we call that topology $\text{nor-}A^{-1}(T)$, the matrix A generates from a t.v.s. (λ, T) , two other t.v.s. $(A^{-1}(\lambda), A^{-1}(T))$ and $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$.

The main object of this paper is to examine to what extent $(A^{-1}(\lambda), A^{-1}(T))$ and $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ share the topological vector space properties of (λ, T) .

If A is the identity matrix, $\lambda = l_p$ ($p > 0$), and T is the usual norm topology in l_p , then both $A^{-1}(\lambda)$ and $\text{nor-}A^{-1}(\lambda)$ reduce to l_p . If, on the other hand $\lambda = l_p$ ($p > 0$), A is the Cesàro matrix $C = (c_{nk})$ with

$$c_{nk} = \begin{cases} 1/n & (k \leq n); \\ 0 & (k > n); \end{cases} \quad Cx = \left(\frac{1}{n} \sum_{k=1}^n x_k \right)_n,$$

then $\text{nor-}C^{-1}(l_p)$ is the sequence space ces_p studied by Shiue [10] and Leibowitz [6]. Their studies showed that:

- (i) ces_p is trivial, consisting of zero sequence alone, if $0 < p \leq 1$.
- (ii) For $1 < p \leq \infty$, ces_p contains l_p as a proper linear subspace as a consequence of Hardy's inequality [1], p. 239.
- (iii) ces_p is a normed linear subspace of ω with the norm defined by

$$\|x\|_{\text{ces}_p} = \|C|x|\|_p.$$

- (iv) ces_p is complete and separable with this norm.

Our investigations generalize the study of ces_p and bring out the special properties of the sequence space l_p and the Cesàro matrix C which played an important role in obtaining the conclusions stated above. Starting with a matrix A and a subspace λ of ω , satisfying suitable hypotheses, we will concentrate on a subspace of ω derived from A and λ as ces_p was derived from C and l_p . In Section 1 suitable hypotheses on A and λ will be introduced and certain fundamental features of the derived space $\text{nor-}A^{-1}(\lambda)$ will be noted. Section 2 deals with the inheritance by $\text{nor-}A^{-1}(\lambda)$ of topological vector space properties of λ . In Section 3 and 4 we consider some related questions.

1. In this section we build up results which will be used in the study of $A^{-1}(\lambda)$ and $\text{nor-}A^{-1}(\lambda)$.

LEMMA 1.1. *The convex hull of a normal set of sequences is normal.*

Proof. For $S \subseteq \omega$, define

$$L(S) = \{tx + (1-t)y \mid x, y \in S \text{ and } t \in [0, 1]\}.$$

Since the convex hull of S is $\bigcup_{n=1}^{\infty} L^n(S)$ and a union of normal sets is normal, it suffices to show that $L(S)$ is normal whenever S is.

Suppose $x, y \in S$, $t \in [0, 1]$, and, for all n ,

$$|z_n| \leq |tx_n + (1-t)y_n|.$$

Define $a, b \in \omega$ by $a_n = b_n = 0$ if $tx_n + (1-t)y_n = 0$, and $a_n = z_n x_n (tx_n + (1-t)y_n)^{-1}$, $b_n = z_n y_n (tx_n + (1-t)y_n)^{-1}$ otherwise. Then $a \in N(x) \subseteq S$ and $b \in N(y) \subseteq S$, so $z = ta + (1-t)b \in L(S)$. Thus $L(S)$ is normal.

Our next result justifies the name " $\text{nor-}A^{-1}(\lambda)$ ".

PROPOSITION 1.2. *Suppose A is an infinite matrix with non-negative entries, and λ is a normal subspace of ω . Then*

- (a) $\text{nor-}A^{-1}(\lambda)$ is normal;
- (b) $\text{nor-}A^{-1}(\lambda) \subseteq A^{-1}(\lambda)$;
- (c) $\text{nor-}A^{-1}(\lambda)$ is the largest normal set of sequences contained in $A^{-1}(\lambda)$;
- (d) $\text{nor-}A^{-1}(\lambda)$ is a subspace of ω .

Proof. (a) If $x \in \text{nor-}A^{-1}(\lambda)$ and $y \in N(x)$, then the non-negativity of

entries of A implies $A|y| \in N(A|x|)$. The conclusion follows from the normality of λ .

(b) Again by the non-negativity of a_{mn} , if $x \in \text{nor-}A^{-1}(\lambda)$, then $Ax \in N(A|x|)$.

(c) It is obvious that every normal set of sequences in $A^{-1}(\lambda)$ must be contained in $\text{nor-}A^{-1}(\lambda)$.

(d) Scalar multiplication is immediate. If $x, y \in \text{nor-}A^{-1}(\lambda)$, then from the non-negativity of a_{mn} and the triangle inequality we can conclude $A|x+y| \in N(A|x| + A|y|)$.

This completes the proof.

EXAMPLE 1.3 (a). Let $\beta = \{(-1)^n\}_{n \geq 1}$ and let λ be the linear span of β and c_0 . Let A be the identity matrix. Then $\text{nor-}A^{-1}(\lambda) = c_0$ and (a)–(d) hold, but λ is not normal. This example shows that the assumption λ is normal, is not necessary for the truth of the conclusion of Proposition 1.2.

EXAMPLE 1.3 (b). Let C , be the Cesàro matrix and $\lambda = c$, the class of convergent sequences. Propositions 1.2 (a)–(d) do not hold.

Proof. If $e = \{1, 1, \dots\}$, then $e \in \text{nor-}C^{-1}(c)$. But there are sequences of 0's and 1's that C does not map into c and hence (a) is not true.

To see that (b) fails, note that one can construct a sequence of ± 1 's which C does not map into c .

The failure of (c) depends upon that of (a).

Finally to see that (d) fails, let x be a sequence of ± 1 's such that $x + e$, a sequence of 2's and 0's, is not mapped by C into c .

For each positive integer n , let e_n denote the n -th coordinate sequence, with 1 in the n -th entry and zeros elsewhere.

PROPOSITION 1.4. Suppose $A = (a_{mn})$ has non-negative entries, and λ is a normal subspace of ω . Then

$$\varphi \cap \text{nor-}A^{-1}(\lambda) = \text{span}\{e_n | \text{the } n\text{-th column of } A \text{ is in } \lambda\}.$$

Proof. Ae_n is the n -th column of A ; it is in λ if and only if $e_n \in \text{nor-}A^{-1}(\lambda)$. The conclusion follows from (a) and (d) of Proposition 1.2.

COROLLARY 1.5. Under the hypothesis of Proposition 1.4, $\text{nor-}A^{-1}(\lambda)$ is trivial if and only if no column of A is in λ ; on the other hand, $\varphi \subseteq \text{nor-}A^{-1}(\lambda)$ if and only if every column of A is in λ .

EXAMPLE 1.6. $A = C$ and $\lambda = l_1$. Then $\text{nor-}A^{-1}(\lambda)$ is trivial. On the other hand, $e_1 - e_2 \in C^{-1}(l_1)$, so $\varphi \cap A^{-1}(\lambda)$ is non-empty.

Henceforward, A will be a matrix with non-negative entries, and λ will be a normal subspace of ω . There are additional hypotheses with which A and λ will be burdened occasionally, and among them are the following:

$H(1)$: A maps $A^{-1}(\lambda)$ one-to-one into λ .

$H(2)$: A maps $A^{-1}(\lambda)$ one-to-one onto λ .

$H(3)$: A and λ satisfy $H(2)$, and the inverse linear transformation, from λ onto $A^{-1}(\lambda)$, is matricial.

Of these, the most important will be $H(2)$. We shall not take up the question of deciding when A and λ satisfy any of the $H()$, but it is worth noting that if A is lower triangular matrix with positive entries on the main diagonal, then $H(3)$ holds, for any normal subspace λ of ω .

2. If λ is topological vector space with some topology T , $A^{-1}(\lambda)$ naturally becomes a topological vector space with the induced topology $A^{-1}(T)$, the open sets of which are $A^{-1}(U)$, $U \in T$. (This topology will not be Hausdorff if $H(1)$ does not hold.) In case $H(2)$ holds, $(A^{-1}(\lambda), A^{-1}(T))$ is isomorphic, as a topological vector space, to (λ, T) .

Under certain natural circumstances, T will induce a topology on $\text{nor-}A^{-1}(\lambda)$ no coarser than the relative topology on $\text{nor-}A^{-1}(\lambda)$ as a subspace of $(A^{-1}(\lambda), A^{-1}(T))$. This topology will be described in Theorem 2.2, after some preliminaries.

If (E, T) is a topological vector space, and ζ a function from E into the non-negative real numbers, we will say " ζ determines T " if and only if the sets

$$U_\alpha = \{x \in E \mid \zeta(x) < \alpha\}, \quad \alpha > 0,$$

constitute a neighbourhood base at the origin for T . If ζ and τ both determine T , ζ and τ will be called equivalent.

In what follows we shall need the essentials of norms and quasinorms [5], paranorms [7] and pseudonorms [9].

We shall use the following without restatement:

(i) Every quasinormed or Hausdorff paranormed space is a metric space;

(ii) A Hausdorff paranormed or quasinormed space is equivalently pseudonormable (see Theorem 6.1, [9], p. 28);

(iii) A paranorm g determining a Hausdorff t.v.s. topology on a vector space E for which each set

$$\{x \in E \mid g(x) \leq \alpha\} \quad (\alpha > 0)$$

is balanced is, in fact, a pseudonorm.

If μ is a normal subspace of ω , and (μ, Γ) is a topological vector space, the topology Γ will be called *normal* if and only if (μ, Γ) possesses a neighbourhood base at the origin consisting of normal sets. A function γ from μ into the non-negative reals will be called *normal* if and only if each of the sets $\{x \in \mu; \gamma(x) \leq \alpha\}$, $\alpha > 0$, is normal. (Equivalently, $y \in N(x) \subseteq \mu$ implies $\gamma(y) \leq \gamma(x)$.) Note that if γ is a normal function, and determines a t.v.s. topology Γ on μ , then Γ is normal.

LEMMA 2.1. Suppose μ is a normal subspace of ω , and Γ is a normal, Hausdorff topology on μ . If Γ is determined by a norm, quasinorm, or paranorm ϱ , then there exists an equivalent norm, quasinorm, or pseudonorm (respectively) $\hat{\varrho}$, which is normal.

Proof. Suppose ϱ is a quasinorm, and let $U = \{x \in \mu; \varrho(x) \leq 1\}$. By hypothesis, there is a normal (and thus balanced) neighbourhood of the origin $V \subseteq U$.

The Minkowski functional $\hat{\varrho}$ of V , defined by $\hat{\varrho}(x) = \inf \{r > 0; x \in rV\}$, is a quasinorm on μ equivalent to ϱ . It is easy to see that $\hat{\varrho}$ is normal.

If ϱ is a norm, U is convex; let W be a normal neighbourhood of the origin contained in U , and let V be the convex hull of W . Then $V \subseteq U$, and V is normal, by Lemma 1.1. The Minkowski function $\hat{\varrho}$ of V is a norm equivalent to ϱ , satisfying the conclusion of the lemma.

When ϱ is a paranorm, the proof is more exotic, and will not be given here entirely. Since Γ is a metric, and a normal topology, there is a basis $\{V_n\}$ of neighbourhoods of the origin, each normal, satisfying $V_{n+1} + V_{n+1} \subseteq V_n$, $n = 1, 2, \dots$. Now we appeal to the proof of Theorem 6.1, [9], and leave it to the reader to verify that the pseudonorm $\hat{\varrho}$ (called " $|\cdot|$ " in the reference) produced through these V_n satisfies the conclusion of the lemma. The salient point is that if S and T are normal subsets of ω , then so is $S + T$. The paranorm ϱ enters the construction not at all, except to insure that Γ is a metric topology.

THEOREM 2.2. Suppose T is a normal topology on λ , and $\mathcal{U}$ is a neighbourhood base at the origin in (λ, T) consisting of normal sets. Then the collection $\text{nor-}A^{-1}(\mathcal{U})$ of the sets

$$\text{nor-}A^{-1}(U) = \{x \in \text{nor-}A^{-1}(\lambda); A|x| \in U\}, \quad U \in \mathcal{U},$$

each normal, is a neighbourhood base at the origin for a topology $\text{nor-}A^{-1}(T)$ with which $\text{nor-}A^{-1}(\lambda)$ becomes a topological vector space. The topology $\text{nor-}A^{-1}(T)$ is at least as fine as the relative topology induced by $A^{-1}(T)$ on $\text{nor-}A^{-1}(\lambda)$. Furthermore,

(a) if (λ, T) is Hausdorff, and A has no zero columns, then $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ is Hausdorff;

(b) if (λ, T) is locally bounded, so is $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$;

(c) if (λ, T) is locally convex, then so is $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$;

(d) if T is a metric, quasinorm, or norm topology on λ , and A has no zero columns, then $\text{nor-}A^{-1}(T)$ is, respectively, a metric, quasinorm, or norm topology on $\text{nor-}A^{-1}(\lambda)$;

(e) if ϱ is a normal norm, quasinorm, or pseudonorm determining T , and A has no zero columns, then the functional $\hat{\varrho}$ on $\text{nor-}A^{-1}(\lambda)$ defined by $\hat{\varrho}(x) = \varrho(A|x|)$ is, respectively, a normal norm, quasinorm, or pseudonorm, and determines the topology $\text{nor-}A^{-1}(T)$ on $\text{nor-}A^{-1}(\lambda)$.

Proof. Since the proof is straightforward we shall sketch it and omit the details.

To see that $\text{nor-}A^{-1}(T)$ is a t.v.s. topology, refer to [9], Corollary 1.2, for a standard theorem describing when a collection of sets in a vector space comprise a neighbourhood base at the origin for a t.v.s. topology on that vector space. We are not, here, requiring that a t.v.s. topology be Hausdorff.

To see that $\text{nor-}A^{-1}(T)$ is at least as fine as the relative topology induced by $A^{-1}(T)$, note that the sets $A^{-1}(U)$, $U \in \mathcal{U}$, are a neighbourhood base at the origin in $(A^{-1}(\lambda), A^{-1}(T))$, and that the normality of $U \in \mathcal{U}$ implies $\text{nor-}A^{-1}(U) \subseteq A^{-1}(U) \cap \text{nor-}A^{-1}(\lambda)$ just because $Ax \in N(A|x|)$. Here, as above, the non-negativity of the entries of A enters strongly.

Proof of (c) can be completed with the help of Lemma 1.1.

Assertion (d) follows from (e), Lemma 2.1, and the observation that every metric t.v.s. is pseudonormable.

The proofs of (a) and (e) are routine.

Remark. The idea behind the hypothesis, in (a), (d), and (e), that A have no zero columns, is that if A has no zero columns, and x is a non-zero sequence in $\text{nor-}A^{-1}(\lambda)$, then $A|x|$ is a non-zero element of λ , again appealing to the non-negativity of the entries of A . Note that if A and λ satisfy $H(1)$, then A certainly has no zero columns.

After Theorem 2.2 it is reasonable to ask if there are natural circumstances under which the inheritance by $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ of the features of (λ, T) goes the other way; for instance, when will the local convexity of the derived space $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ imply that of (λ, T) ? Clearly no very general implication of that type holds, because $\text{nor-}A^{-1}(\lambda)$ may so easily be trivial, as in the case of the $\text{ces}_p = \text{nor-}C^{-1}(l_p)$, $0 < p \leq 1$. The purpose of the following example is to drive the question of inheritance by λ of the properties of $\text{nor-}A^{-1}(\lambda)$ deeper into obscurity; if there are "natural circumstances" under which such inheritance occurs, they will not be so easy to find

EXAMPLE 2.3. Let p and q be positive numbers such that $p < 1 < q < 1 + p$. Note that

$$\sum_{n \geq m} n^{-q} > \int_m^{\infty} x^{-q} dx = \frac{m^{1-q}}{q-1} > (m-1)^{-p}$$

for sufficiently large m . Consequently, there is a sequence $\{m_k\}$ of positive integers such that $m_{k+1} > m_k + 2$ and

$$m_k^{-p} \leq \sum_{m_k < n < m_{k+1}} n^{-q} \quad \text{for each } k.$$

Let $M = \{m_k \mid k = 1, 2, \dots\}$, define $p_n = p$ if $n \in M$, $p_n = q$ otherwise, and

set $l(p_n) = \{x \in \omega; \sum_n |x_n|^{p_n} < \infty\}$; $l(p_n)$ is a direct sum of l_p and l_q ; its topology is determined by the quasinorm ϱ defined by

$$\varrho(x) = \left(\sum_{n \notin M} |x_n|^q \right)^{1/q} + \left(\sum_{n \in M} |x_n|^p \right)^{1/p}.$$

It is clear that $l(p_n) \subseteq l_q$.

For a matrix, take C , the Cesàro matrix. We claim $\text{nor-}C^{-1}(l(p_n)) = \text{ces}_q$. Clearly $\text{nor-}C^{-1}(l(p_n)) \subseteq \text{nor-}C^{-1}(l_q) = \text{ces}_q$.

Suppose $x \in \text{ces}_q$; then

$$\sum_n \left(\frac{1}{n} \sum_{k=1}^n |x_k| \right)^q < \infty.$$

Set $a_n = \sum_{k=1}^n |x_k|$, so $\{a_n\}$ is non-negative and non-decreasing. We want to show that $\sum_n (n^{-1} a_n)^{p_n} < \infty$. Clearly suffices to show that $\sum_{n \in M} (n^{-1} a_n)^p < \infty$.

Now, by choice of $\{m_k\}$,

$$(m_k^{-1} a_{m_k})^p \leq a_{m_k}^p \sum_{m_k < n < m_{k+1}} n^{-q} \leq \sum_{m_k < n < m_{k+1}} n^{-q} \max(1, a_n^q)$$

since the a_n 's are non-decreasing and $p < q$. Thus $\sum_{n \in M} (n^{-1} a_n)^p$ converges, by comparison with $\sum_k \sum_{m_k < n < m_{k+1}} n^{-q} \max(1, a_n^q)$.

So ces_q has now two topologies, one induced upon it by l_q , the other by $l(p_n)$. The first is a norm topology. Both are metric, and, by a result to be proven, Theorem 2.16, ces_q is complete with either. They are comparable (the norm topology being no finer than the other), and are thus the same. So $\text{nor-}C^{-1}(l(p_n))$ is locally convex, in fact normed, but $l(p_n)$ is not locally convex ([8], Theorem 1).

We consider another example:

EXAMPLE 2.4. Let $\lambda = l_\infty$ and A be the matrix whose every entry is 1. In this case, $\text{nor-}A^{-1}(l_\infty)$ is l_1 , and the norm $\hat{\varrho} = \|A|\cdot|\|_\infty$ is the usual l_1 norm, $\|\cdot\|_1$.

The main thing about this example is that, although $\text{nor-}A^{-1}(l_\infty)$ is a wholesome space with a fine topology, $H(1)$ is far from being satisfied. Indeed, A is a rank 1 map from $A^{-1}(l_\infty)$ into l_∞ .

Let $\langle \cdot, e_n \rangle$ denote the n -th coordinate projection on ω . That is, $\langle x, e_n \rangle = x_n$ for each $x \in \omega$.

Our next result obtains conditions on a topology of a t.v.s. such that coordinate projections are continuous.

PROPOSITION 2.5. Suppose μ is a normal subspace of ω , with normal

topology Γ . Then $\langle \cdot, e_n \rangle$ is continuous on (μ, Γ) for each n if and only if (μ, Γ) is Hausdorff.

Proof. Suppose (μ, Γ) is Hausdorff. If $x_n = 0$ for all $x \in \mu$, then $\langle \cdot, e_n \rangle$ is the zero linear functional on μ , and thus continuous. If $x_n \neq 0$ for some $x \in \mu$, then, since μ is normal and closed under scalar multiplication, $e_n \in \mu$. Since (μ, Γ) is Hausdorff and normal there exists a normal neighbourhood U of the origin in (μ, Γ) such that $e_n \notin U$. Then $x \in U$ implies $|x_n| = |\langle x, e_n \rangle| < 1$, since $|x_n| \geq 1$ would imply $e_n \in U$, by the normality of U . Thus $\langle \cdot, e_n \rangle$ is continuous on (μ, Γ) .

On the other hand, suppose each of the $\langle \cdot, e_n \rangle$ is continuous on (μ, Γ) . Then Γ is at least as fine as the relative topology of coordinatewise convergence on μ (the topology gotten from ω by considering ω a countable product of copies of the scalar field), and is therefore a Hausdorff topology. (The implication in this direction requires only that μ be a subset of ω and Γ some topology on it.)

COROLLARY 2.6. *If T is a normal topology on λ , (λ, T) is Hausdorff, and A has no zero columns, then the coordinate projections are continuous on $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$.*

The proof follows from Theorem 2.2 and Proposition 2.5.

Let P_0 denote the zero map from ω into ω , I the identity map, and define P_n by $P_n(x) = \sum_{k=1}^n \langle x, e_k \rangle e_k$. Note that, for any $x \in \omega$, and any n , $P_n(x), (I - P_n)(x) \in N(x)$, and if $y \in N(x)$ then $P_n(y) \in N(P_n(x))$, and $(I - P_n)(y) \in N((I - P_n)(x))$.

If μ is a subspace of ω such that $P_n(\mu) \subseteq \mu$ for each n , and μ is a t.v.s. with a topology Γ , the space (μ, Γ) is called *AK* (for abschnitt-konvergenz, sectionwise convergence) if and only if $(I - P_n)(x)$ converges to zero in (μ, Γ) as $n \rightarrow \infty$, for each $x \in \mu$. If $\varphi \subseteq \mu$ and the coordinate projections are continuous on (μ, Γ) , then (μ, Γ) is *AK* if and only if the sequence $\{e_n\}$ is a Schauder basis for (μ, Γ) .

Our next theorem shows that *AK* is inherited by $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ from (λ, T) . Precisely, we have

THEOREM 2.7 ([4]). *Suppose T is a normal topology on λ and (λ, T) is *AK*. Then $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ is *AK*.*

By Example 2.4, $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ may be *AK* even when (λ, T) is not so.

The last property of (λ, T) whose inheritance by $\text{nor-}A^{-1}(\lambda)$ will be investigated is completeness. For the rest of the paper T will be understood to be a normal topology on λ .

PROPOSITION 2.8. *In a normal subspace of ω , with a normal topology, the closure of a normal set is normal.*

Proof. Suppose (μ, Γ) is a normal subspace of ω with a normal topology, S is a normal subset of μ , $(y^{(\alpha)})$ is a net in S converging to y in (μ, Γ) , and $x \in N(y)$.

For each n , let $a_n = x_n y_n^{-1}$ if $y_n \neq 0$, $a_n = 0$, otherwise. Let $a = \{a_n\}$. Then $|a_n| \leq 1$ for all n , and $ay = x$;

$$ay^{(\alpha)} \in N(y^{(\alpha)}) \subseteq S$$

for each α , and

$$ay^{(\alpha)} - y = a(y^{(\alpha)} - y) \in N(y^{(\alpha)} - y).$$

Since Γ is a normal topology, it follows that $\{ay^{(\alpha)}\}$ converges to x in (μ, Γ) , so $x \in \bar{S}$, the closure of S .

Let $\mathcal{P}$ denote the topology of coordinatewise convergence on ω . If μ is a subspace of ω , by $(\mu, \mathcal{P})$ we mean μ considered with the relative topology induced by $\mathcal{P}$. Note that $\mathcal{P}$ is a normal topology on ω , so the topology induced by it on any normal subspace of ω is normal. If μ is a subspace of ω and Γ is a t.v.s. topology on μ , we will say (μ, Γ) is *locally coordinatewise closed*, or "LCC" for the sake of brevity, if and only if there is a neighbourhood base at the origin in (μ, Γ) comprised of sets which are closed in $(\mu, \mathcal{P})$. The property of being an LCC space seems to be pretty widespread; indeed, we have not found (μ, Γ) both μ and Γ normal, such that (μ, Γ) is not LCC, nor any non-LCC (μ, Γ) on which the coordinate projections are continuous. Beyond the observations that $(l_\infty, \|\cdot\|_\infty)$ is obviously LCC, and that any subspace of an LCC space is LCC with the relative topology, the best we can offer toward a characterization of LCC spaces is the following

PROPOSITION 2.9. *Suppose μ is a normal subspace of ω , Γ a normal topology on μ , and (μ, Γ) is AK. Then (μ, Γ) is LCC.*

Proof. There are neighbourhood bases at the origin in (μ, Γ) comprised of normal sets, and of sets closed in (μ, Γ) (as in all topological vector spaces), so by Proposition 2.8 there is a neighbourhood base $\mathcal{U}$ at the origin in (μ, Γ) comprised of sets which are simultaneously normal and closed in (μ, Γ) . We claim each of these sets is closed in $(\mu, \mathcal{P})$. Suppose $U \in \mathcal{U}$, and $x \in \mu \setminus U$. Then $(x + V) \cap U = \emptyset$ for some $V \in \mathcal{U}$. Let $W \in \mathcal{U}$ be such that $W + W \subseteq V$. Since (μ, Γ) is AK, there is an integer n such that $P_n x - x \in W$.

Let $M = \{k \mid e_k \in \mu\}$, and note that $y_k = 0$ for $k \notin M$, $y \in \mu$, since μ is a normal subspace of ω . Since (μ, Γ) is a t.v.s., $\sum_{\substack{k \in M \\ 1 \leq k \leq n}} c_k e_k \in W$ for all c_k

sufficiently small. It follows that if x is in the closure of U in $(\mu, \mathcal{P})$, then there exists $y \in U$ such that $P_n y - P_n x \in W$. Then

$$P_n y = P_n y - P_n x + P_n x - x + x \in W + W + x \subseteq x + V.$$

On the other hand, $P_n y \in N(y) \subseteq U$, since U is normal, which contradicts $(x + V) \cap U = \emptyset$.

THEOREM 2.10. *Suppose (λ, T) is LCC. Then $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ is LCC.*

Proof. Since $T, \mathcal{P}$ are normal topologies, and (λ, T) is LCC, by Proposition 2.8 there is a neighbourhood base $\mathcal{U}$ at the origin in (λ, T) comprised of sets which are normal, and closed in $(\lambda, \mathcal{P})$.

For each $U \in \mathcal{U}$, we shall show that $\text{nor-}A^{-1}(U) = \{x \in \text{nor-}A^{-1}(\lambda) \mid A|x| \in U\}$ is closed in $(\text{nor-}A^{-1}(\lambda), \mathcal{P})$. This will suffice to show that $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ is LCC.

Suppose $U \in \mathcal{U}$ and $x \in \text{nor-}A^{-1}(\lambda) \setminus \text{nor-}A^{-1}(U)$. Then $A|x| \in \lambda \setminus U$. Since U is closed in $(\lambda, \mathcal{P})$, there is a positive integer t , and $\varepsilon > 0$, such that

$$(6) \quad \left| \sum_n a_{mn} |x_n| - y_m \right| > 3\varepsilon, \quad m = 1, \dots, t, \quad y \in U.$$

Since U is normal, this implies

$$(7) \quad \sum_n a_{mn} |x_n| > |y_m| + 3\varepsilon, \quad m = 1, \dots, t, \quad y \in U,$$

for if, for some $y \in U$, and some $m \in \{1, \dots, t\}$, $0 \leq \sum_n a_{mn} |x_n| \leq |y_m| + 3\varepsilon$, then the sequence $z = \{z_m\}$ obtained from y by replacing y_m by $\max(0, \sum_n a_{mn} |x_n| - 3\varepsilon)$ is in $N(y) \subseteq U$, and $\left| \sum_n a_{mn} |x_n| - z_m \right| \leq 3\varepsilon$, contrary to (6).

Now, from (7), clearly there exists k such that $\sum_{n=1}^k a_{mn} |x_n| > |y_m| + 2\varepsilon$, $m = 1, \dots, t$, $y \in U$, and thus there exists $\delta > 0$ such that $|x_n - x'_n| < \delta$, $n = 1, \dots, k$ implies

$$(8) \quad \sum_n a_{mn} |x'_n| \geq \sum_{n=1}^k a_{mn} |x'_n| > |y_m| + \varepsilon, \quad m = 1, \dots, t, \quad y \in U.$$

Now (8) implies that the $\mathcal{P}$ neighbourhood $\{x \in \omega \mid |x_n - x'_n| < \delta, n = 1, 2, \dots, k\}$ of x contains no elements of $\text{nor-}A^{-1}(U)$. Since the choice of x was arbitrary, $\text{nor-}A^{-1}(U)$ is closed in $(\text{nor-}A^{-1}(\lambda), \mathcal{P})$.

Remark 2.11. $(A^{-1}(\lambda), A^{-1}(T))$ may fail to be LCC even if (λ, T) is LCC. For instance, in Example 2.4, $(A^{-1}(\lambda), A^{-1}(T))$ is $\{x \in \omega; \sum_{n=1}^{\infty} x_n \text{ converges}\}$, equipped with the single seminorm $p(x) = \left| \sum_{n=1}^{\infty} x_n \right|$. The closure in $(A^{-1}(\lambda), \mathcal{P})$ of any $A^{-1}(T)$ neighbourhood of zero is all of $A^{-1}(\lambda)$. We do not know if $(A^{-1}(\lambda), A^{-1}(T))$ inherits the property of being LCC from (λ, T) in case $H(2)$ holds, or even $H(3)$. However, if A is row finite, meaning each row of A is an element of φ , then (λ, T) LCC implies $(A^{-1}(\lambda), A^{-1}(T))$ is LCC, just because mapping by A preserves coordinate-wise convergence in ω .

LEMMA. 2.12. *Suppose μ is a normal subspace of ω , Γ is a normal*

topology on μ , and (μ, Γ) is Hausdorff and LCC. Then if $\{y^{(\alpha)}\}$ is a Cauchy net in (μ, Γ) , and convergent to zero in $(\mu, \mathcal{P})$, then $\{y^{(\alpha)}\}$ converges to zero in (μ, Γ) .

Proof. By Proposition 2.5, the fact that (μ, Γ) is Hausdorff implies that Γ is finer than the topology induced on μ by $\mathcal{P}$. The conclusion now follows from Lemma 34.2, [12], p. 355.

We are now ready to state and prove our result on completeness.

THEOREM 2.13. *Suppose (λ, T) is Hausdorff, LCC, and complete, and A has no zero columns. Then $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ is complete.*

Proof. By Theorem 2.2, $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ is Hausdorff, and its topology is normal, and by Theorem 2.10 it is LCC. So we are in a position to apply Lemma 2.12 with $(\mu, \Gamma) = (\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$, which we shall presently do.

Suppose $\{x^{(\alpha)}\}$ is a Cauchy net in $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$. For any $x, y \in \omega$, $|x| - |y| \in N(x - y)$, so, since $\text{nor-}A^{-1}(T)$ is a normal topology, it follows that $\{|x^{(\alpha)}|\}$ is a Cauchy net in $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$.

The coordinate projections are continuous on $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ by Proposition 2.5. Hence $\{x^{(\alpha)}\}$ converges to some x in $(\omega, \mathcal{P})$; and consequently $\{|x^{(\alpha)}|\}$ converges to $|x|$ in $(\omega, \mathcal{P})$.

A maps $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ continuously into (λ, T) , so $\{A|x^{(\alpha)}|\}$ is Cauchy in (λ, T) . Then $A|x^{(\alpha)}|$ converges to some y in (λ, T) , and thus converges coordinatewise to y (Proposition 2.5 and the hypothesis that (λ, T) is Hausdorff). That is, $\{\sum_n a_{mn}|x_n^{(\alpha)}|\}_\alpha$ converges to y_m for each m . Since $\{|x^{(\alpha)}|\}$

converges coordinatewise to $|x|$, it follows that $\sum_n a_{mn}|x_n| \leq y_m$ for each m .

Thus $A|x| \in N(y) \subseteq \lambda$, so $x \in \text{nor-}A^{-1}(\lambda)$.

Now set $y^{(\alpha)} = x^{(\alpha)} - x$. Then $\{y^{(\alpha)}\}$ is a net in $\text{nor-}A^{-1}(\lambda)$, Cauchy with respect to $\text{nor-}A^{-1}(T)$, and convergent to zero coordinatewise; by Lemma 2.12, $\{y^{(\alpha)}\}$ converges to zero in $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$. That means that $\{x^{(\alpha)}\}$ converges to x .

Remarks. The requirement of Hausdorffness in Lemma 2.12, and the requirement that A have no zero columns in Theorem 2.13, can be deleted, by suitably modifying the proof. But we do not think it worthwhile since these are natural restrictions occurring in practice.

As indicated earlier, we have no example of a non-LCC (λ, T) (nor can we prove that λ and T normal, (λ, T) Hausdorff implies (λ, T) is LCC); so, *a fortiori*, we have no example to show that the hypothesis that " (λ, T) be LCC" cannot be deleted from the preceding theorem. However, Proposition 2.8 and preceding remarks yield the following, useful

COROLLARY 2.14. *Suppose either $(\lambda, T) = (l_\infty, \|\cdot\|_\infty)$ or else (λ, T) is AK,*

Hausdorff, and complete. Suppose that A has no zero columns. Then $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ is complete.

EXAMPLE 2.15. Let $(\lambda, T) = (l_\infty, \mathcal{P})$, and let A be the matrix whose every entry is 1. Then (λ, T) is not complete but $(\text{nor-}A^{-1}\lambda, \text{nor-}A^{-1}(T))$ is complete.

Proof. In this case $\text{nor-}A^{-1}(\lambda) = l_1$ and $\text{nor-}A^{-1}(T)$ is the topology generated by the l_1 norm. The assertion is now clear.

3. In this section we make some observations on the inclusion among φ , λ , $A^{-1}(\lambda)$ and $\text{nor-}A^{-1}(\lambda)$. The results about density of φ are considered in [3].

Our first result gives a condition to be satisfied by the columns of the matrix A so that neither $A^{-1}(\lambda)$ nor $\text{nor-}A^{-1}(\lambda)$ is trivial. In fact, we have

PROPOSITION 3.1. *The following are equivalent:*

- (a) $\varphi \subseteq A^{-1}(\lambda)$;
- (b) $\varphi \subseteq \text{nor-}A^{-1}(\lambda)$;
- (c) *each column of A is in λ .*

Proof follows from Propositions 1.2 (c) and 1.4.

Remark 3.2. If (λ, T) is complete, metric and LCC, and A has no zero columns, then the inclusion

$$\varphi \cap \text{nor-}A^{-1}(\lambda) \subseteq \text{nor-}A^{-1}(\lambda)$$

is proper if and only if infinitely many columns of A are in λ ; for, under this last assumption, $\text{nor-}A^{-1}(\lambda)$ is infinite dimensional by Proposition 1.4, and $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ is a complete metric space by Theorems 2.2 and 2.13. If, on the other hand, finitely many columns of A only are in λ , then clearly,

$$\text{nor-}A^{-1}(\lambda) = \varphi \cap \text{nor-}A^{-1}(\lambda) = \text{span} \{e_n \mid \text{the } n\text{-th column of } A \in \lambda\}.$$

The fact derivable from Hardy's inequality, that

$$l_p \subseteq \text{ces}_p \quad (1 < p \leq \infty)$$

reappears in a general formulation in the following manner:

PROPOSITION 3.3. *The following are equivalent:*

- (a) $\lambda \subseteq A^{-1}(\lambda)$,
- (b) $\lambda \subseteq \text{nor-}A^{-1}(\lambda)$,
- (c) *A maps λ into λ .*

Proof. Since λ is normal, (a), (b) are equivalent by Proposition (1.2) (b) and (c). Since, clearly $\lambda \subseteq A^{-1}(\lambda)$ is equivalent to $A(\lambda) \subseteq \lambda$, the proposition follows.

This leads to the following problem whose answer we do not know.

PROBLEM 3.4. Suppose A maps λ one-to-one and onto λ and $\varphi \subseteq \lambda$. Does it follow that

$$\lambda = \text{nor-}A^{-1}(\lambda) = A^{-1}(\lambda)?$$

Remark. It certainly does if A and λ satisfy $H(1)$, but that is not of any help.

In Proposition 1.2 we showed that $\text{nor-}A^{-1}(\lambda) \subseteq A^{-1}(\lambda)$. Our next result deals with the equality $A^{-1}(\lambda) = \text{nor-}A^{-1}(\lambda)$. Strangely enough the decision gets involved with several concepts. Precisely, we have

THEOREM 3.5. Suppose (λ, T) is normed, complete and AK , matrix A and the vector space of sequences λ satisfy $H(2)$, and every column of A is in λ . Then

$$A^{-1}(\lambda) = \text{nor-}A^{-1}(\lambda)$$

if and only if the columns of A constitute an unconditional Schauder basis of (λ, T) .

Proof. We note that whenever A and λ satisfy $H(2)$, A has no zero columns. It should also be noted that view of Theorem 17.1, [11] (p. 499), whenever a normal subspace μ of ω with $\varphi \subseteq \mu$, is a Banach space with a normal norm ζ , then (μ, ζ) is AK , i.e. the $\{e_n\}$ constitute a Schauder basis of (μ, ζ) , if and only if the $\{e_n\}$ constitute an unconditional Schauder basis. Suppose now that $\text{nor-}A^{-1}(\lambda) = A^{-1}(\lambda)$. By $H(2)$, $(A^{-1}(\lambda), A^{-1}(T))$ is normed and complete. Further by Theorem 2.2, Theorem 2.7, and Corollary 2.14, $(\text{nor-}A^{-1}(\lambda), \text{nor-}A^{-1}(T))$ is a Banach space with a normal norm, and AK . By previous remarks, $\text{nor-}A^{-1}(T)$ and $A^{-1}(T)$ are the same topology. Thus the $\{e_n\}$ constitute an unconditional Schauder basis of $(A^{-1}(\lambda), A^{-1}(T))$. Since $A: A^{-1}(\lambda) \rightarrow \lambda$ is an isomorphism, the columns $\{Ae_n\}$ of A are an unconditional Schauder basis of (λ, T) .

Now by the same theorem of [11] referred to earlier, it is easy to see that if μ is a subspace of ω , not necessarily normal, $\varphi \subseteq \mu$, and ζ is a norm on μ such that (μ, ζ) is Banach and the $\{e_n\}$ are an unconditional Schauder basis of (μ, ζ) , then μ is in fact normal. [For $x \in \mu$, $y \in N(x)$, observe that the sequence $\{P_n y\}_n$, a sequence in μ since $\varphi \subseteq \mu$, is Cauchy, by 7° of Theorem 17.1, [11]; the sequence then converges to something in μ , and that something must be y because, by the best known result in basis theory, the coordinate projections are continuous on (μ, ζ) .] Thus, if the columns of A are an unconditional basis in (λ, T) , then $\{e_n\}$ are an unconditional basis of $(A^{-1}(\lambda), A^{-1}(T))$, so $A^{-1}(\lambda)$ is normal, which implies that

$$A^{-1}(\lambda) = \text{nor-}A^{-1}(\lambda)$$

by Proposition 1.2 (c).

Remark. The "if" assertion in Theorem 3.5 does not require that (λ, T)

be AK . The "only if" part certainly needs it; a case to justify our assertion is

$$\lambda = l_\infty, \quad A \text{ is the identity matrix.}$$

We are led to the following

PROBLEM 3.6. *Supposing (λ, T) is a Banach space, AK , $H(2)$ holds, and all the columns of A are in λ ; if $(A^{-1}(\lambda), A^{-1}(T))$ is AK , does it follow that $A^{-1}(\lambda) = \text{nor-}A^{-1}(\lambda)$?*

4. It is worth making a remark that $A(\text{nor-}A^{-1}(\lambda))$ may fail to be a normal subspace of λ , even when $H(3)$ holds. We give the following

EXAMPLE 4.1. Let $A = C$, the Cesàro matrix, $\lambda = l_\infty$; note that $e = C(e) \in C(\text{ces}_\infty)$. Let $\{b_n\}$ be $\{1, 0, 1, 0, \dots\}$. Clearly $b \in N(e)$, but it is easy to see that $C^{-1}(b) \notin \text{ces}_\infty$.

In fact, $\|x - b\|_\infty < \frac{1}{4}$ implies $C^{-1}(x) \notin \text{ces}_\infty$. Thus $C(\text{ces}_\infty)$ is not dense in l_∞ , so ces_∞ is not dense in $C^{-1}(l_\infty)$.

For an example in which $H(3)$ holds, $\text{nor-}A^{-1}(\lambda)$ is dense in $A^{-1}(\lambda)$, and $A(\text{nor-}A^{-1}(\lambda))$ is not normal, we consider

EXAMPLE 4.2. Let $A = C$, the Cesàro matrix, $\lambda = l_p$ ($1 < p < \infty$). Set

$$y = \{0, \log 2, \log 3 - \log 2, \log 4 - \log 3, \dots\}.$$

Then y is non-negative, and

$$Cy = \left\{ \frac{\log n}{n} \right\} \in l_p, \quad \text{so } y \in \text{ces}_p.$$

Now, set

$$x = \left\{ 0, \frac{\log 2}{2}, 0, \frac{\log 4}{4}, \dots \right\} \in N(Cy).$$

Then

$$C^{-1}x = \{0, \log 2, -\log 2, \log 4, -\log 4, \dots\};$$

the $(2m+1)$ -th entry of $C|C^{-1}x|$ is

$$\begin{aligned} (2m+1)^{-1} 2 \sum_{k=1}^m \log k &\geq 2(2m+1)^{-1} \int_1^m \log x \, dx \\ &= 2(2m+1)^{-1} (m(\log m - 1) + 1) \geq \frac{\log m}{2}, \end{aligned}$$

for sufficiently large m .

Thus $C|C^{-1}x| \notin l_p$, so $C^{-1}x \notin \text{ces}_p$ and consequently $x \notin C(\text{ces}_p)$.

Added in proof. It is now known that there are normal Banach spaces of sequences, with normal norms, which are not LCC. However, Richard Haydon and Mireille Levy, in a private communication, have proven that the conclusion of Theorem 2.13 holds when λ is normed, whether λ is LCC or not.

References

- [1] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge 1934.
- [2] P. D. Johnson Jr. and R. N. Mohapatra, *Inequalities involving lower triangular matrices*, Proc. London Math. Soc. (3) 41 (1980), 83–131.
- [3] —, —, *Density of finitely non-zero sequences in some sequence spaces*, Math. Japon. 24.3 (1979), 253–262.
- [4] —, —, *Sectional convergence in spaces obtained as inverse images of sequence spaces under matrix transformations*, ibidem 24.2 (1979), 179–185.
- [5] G. Köthe, *Topologische Lineare Räume*, Springer-Verlag, 1960.
- [6] G. M. Leibowitz, *A note on the Cesàro sequence spaces*, Tamkang J. Math. 2 (1971), 151–157.
- [7] I. J. Maddox, *Elements of Functional Analysis*, Cambridge 1970.
- [8] — and W. J. Roles, *Absolute convexity in certain topological linear spaces*, Proc. Camb. Phil. Soc. 66 (1969), 541–545.
- [9] H. H. Schaeffer, *Topological Vector Spaces*, MacMillan, 1966.
- [10] J. S. Shille, *On the Cesàro sequence spaces*, Tamkang J. Math. 1 (1970), 19–25.
- [11] I. Singer, *Bases in Banach Spaces I*, Springer-Verlag, 1970.
- [12] F. Trèves, *Topological Vector Spaces, Distributions, and Kernels*, Academic Press, New York–London 1967.

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Reçu par la Rédaction le 18. 05. 1979