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Admissible variations and the local maximum theorems for pairs of vector functions and for generalized Bieberbach–Eilenberg, bounded and Grunsky–Shah functions

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Abstract. Let $C_{m,n}$ denotes the class of all pairs $\{F, G\}$ of functions $F = [F_1, \ldots, F_m]$: $\Delta \mapsto \mathbf{C}^m$ and $G = [G_1, \ldots, G_n]$: $\Delta \mapsto \mathbf{C}^n$, $\Delta = \{z \in \mathbf{C}: |z| < 1\}$, where $F_1, \ldots, F_m$, $G_1, \ldots, G_n$ – analytic and univalent mappings in Δ such that $F_k(z) \neq F_j(\zeta)$, $k \neq j$, $k, j = 1, \ldots, m$, when $m \geqslant 2$, $G_k(z) \neq G_j(\zeta)$, $k \neq j$, $k, j = 1, \ldots, n$, when $n \geqslant 2$, $F_k(z) G_j(\zeta) \neq 1$, $k = 1, \ldots, m$, $j = 1, \ldots, n$, for all $(z, \zeta) \in \Delta \times \Delta$ ([8], [10]).

Let $F = [F_1, \ldots, F_m]$: $\Delta \mapsto \mathbf{C}^m$, where $F_1, \ldots, F_m$ are analytic and univalent mappings in Δ satisfying, for all $(z, \zeta) \in \Delta \times \Delta$, in the case $m \geqslant 2$ the condition: $F_k(z) \neq F_j(\zeta)$ for $k \neq j$, $k, j = 1, \ldots, m$. We say that: $F \in C_m^1$ if $F_k(z) F_j(\zeta) \neq 1$ for $(z, \zeta) \in \Delta \times \Delta$ and $k, j = 1, \ldots, m$; $F \in C_m^2$ if $|F_k(z)| < 1$ for $z \in \Delta$ and $k = 1, \ldots, m$; $F \in C_m^3$ if $F_k(z) \overline{F_j(\zeta)} \neq -1$ for $(z, \zeta) \in \Delta \times \Delta$ and $k, j = 1, \ldots, m$ ([2], [4], [7], [9]).

In the present paper there have been constructed general admissible variations of the pairs $\{F, G\}$ (functions F) belonging to non-compact classes $C_{m,n}$, $m, n \geqslant 1$ (C_m^l, $m \geqslant 1$, $l = 1, 2, 3$), and given, in the form of differential-functional equations, necessary conditions for a pair $\{F, G\}$ (a function F) to be a local maximum for $\operatorname{Re} J$ ($\operatorname{Re} J^l$), where J (J^l) is a functional having on $C_{m,n}$ (C_m^l) a complex derivative in the sense of Gâteaux.

Introduction and basic definitions. Let $\mathbf{C}$ be the open complex plane, $\Delta = \{z \in \mathbf{C}: |z| < 1\}$, and m, n be any positive integers.

Let $\mathscr{A}_{m,n} = (\mathscr{A}_{m,n}, \tau)$ be a topological vector space (abbreviated t.v.s.) over the field $\mathbf{C}$, where $\mathscr{A}_{m,n}$ is a vector space (v.s.) whose elements are pairs $\{F, G\}$ of vector functions $F = [F_1, \ldots, F_m]$: $\Delta \mapsto \mathbf{C}^m$ and $G = [G_1, \ldots G_n]$: $\Delta \mapsto \mathbf{C}^n$, with $F_1, \ldots, F_m$, $G_1, \ldots, G_n$ are analytic mappings in Δ, in which there have been defined a mapping $(\{F, G\}, \{\check{F}, \check{G}\}) \mapsto \{F + \check{F}, G + \check{G}\}$ of the product $\mathscr{A}_{m,n} \times \mathscr{A}_{m,n}$ in $\mathscr{A}_{m,n}$ as well as a mapping $(\lambda, \{F, G\}) \mapsto \{\lambda F, \lambda G\}$ of the product $\mathbf{C} \times \mathscr{A}_{m,n}$ in $\mathscr{A}_{m,n}$, and τ is the topology on $\mathscr{A}_{m,n}$ defined as follows: if the elements of the sequence $(\{F^p, G^p\})$ and $\{F, G\}$ belong to $\mathscr{A}_{m,n}$, then $\{F^p, G^p\} \to \{F, G\}$ if and only if $F_k^p \to F_k$, $k = 1, \ldots, m$, and $G_k^p \to G_k$, $k = 1, \ldots, n$, uniformly on each compact subset of Δ. A set Y is called *open* in the t.v.s. $\mathscr{A}_{m,n}$ if $Y = \mathscr{A}_{m,n} \setminus X$, where X is a some subset of $\mathscr{A}_{m,n}$ identical with the set of all pairs $\{F, G\} \in \mathscr{A}_{m,n}$ for which there exist sequences $(\{F^p, G^p\}) \subset X$ such that $\{F^p, G^p\} \to \{F, G\}$. By a neighbourhood of

$\{F, G\} \in \mathscr{A}_{m,n}$ we mean any open set in the t.v.s. $\mathscr{A}_{m,n}$ containing $\{F, G\}$.

Let $C_{m,n} = C_{m,n}(A_0, B_0)$, see [8], [9], stand for the class of all pairs $\{F, G\}$ of vector functions $F = [F_1, \ldots, F_m]$: $\Delta \mapsto \boldsymbol{C}^m$ and $G = [G_1, \ldots, G_n]$: $\Delta \mapsto \boldsymbol{C}^n$ of the form

$$F(z) = A_0 + A_1 z + \ldots + A_k z^k + \ldots, \qquad G(z) = B_0 + B_1 z + \ldots + B_k z^k + \ldots,$$
$$A_k = [a_{k1}, \ldots, a_{km}], \qquad B_k = [b_{k1}, \ldots, b_{kn}], \qquad k = 0, 1, 2, \ldots,$$

where $F_1, \ldots, F_m$, $G_1, \ldots, G_n$ are analytic and univalent mappings in Δ such that

$$F_k(z) \neq F_j(\zeta), \quad k \neq j, \ k, j = 1, \ldots, m, \text{ when } m \geqslant 2,$$
$$G_k(z) \neq G_j(\zeta), \quad k \neq j, \ k, j = 1, \ldots, n, \text{ when } n \geqslant 2,$$
$$F_k(z) G_j(\zeta) \neq 1, \quad k = 1, \ldots, m, \ j = 1, \ldots, n,$$

for all $(z, \zeta) \in \Delta \times \Delta$. Note that the class $C_{m,n}$ is not compact; for if $\{F^p, G^p\} \in C_{m,n}$ for $p = 1, 2, \ldots$ and $\{F^p, G^p\} \to \{F, G\}$, then, in virtue of the Montel theorem and the corollary from the Hurwitz theorem, we find that either $\{F, G\} \in C_{m,n}$ or $F_k(z) = a_{0k}$ for some $k = 1, \ldots, m$ or $G_k(z) = b_{0k}$ for some $k = 1, \ldots, n$.

Let $\mathscr{A}_m = (\mathscr{A}_m, \tau)$ be a t.v.s over the field $\boldsymbol{C}$, where $\mathscr{A}_m$ is the v.s. of all functions $F = [F_1, \ldots, F_m]$: $\Delta \mapsto \boldsymbol{C}^m$ such that $F_1, \ldots, F_m$ are analytic mappings in Δ, in which there have been defined a mapping $(F, \check{F}) \mapsto F + \check{F}$ of the product $\mathscr{A}_m \times \mathscr{A}_m$ in $\mathscr{A}_m$ and a mapping $(\lambda, F) \mapsto \lambda F$ of the product $\boldsymbol{C} \times \mathscr{A}_m$ in $\mathscr{A}_m$, while τ is a topology of locally uniform convergence.

Let $F = [F_1, \ldots, F_m]$: $\Delta \mapsto \boldsymbol{C}^m$ be a function of the form

$$F(z) = A_0 + A_1 z + \ldots + A_k z^k + \ldots, \qquad A_k = [a_{k1}, \ldots, a_{km}], \qquad k = 0, 1, 2, \ldots,$$

where $F_1, \ldots, F_m$ are analytic and univalent mappings in Δ satisfying, for $m \geqslant 2$ and for all $(z, \zeta) \in \Delta \times \Delta$, the condition

$$F_k(z) \neq F_j(\zeta) \quad \text{for} \quad k \neq j, \ k, j = 1, \ldots, m.$$

We shall say that:

$F \in C_m^1 = C_m^1(A_0)$ if $F_k(z) F_j(\zeta) \neq 1$ for $(z, \zeta) \in \Delta \times \Delta$ and $k, j = 1, \ldots, m$,

$F \in C_m^2 = C_m^2(A_0)$ if $|F_k(z)| < 1$ for $z \in \Delta$ and $k = 1, \ldots, m$,

$F \in C_m^3 = C_m^3(A_0)$ if $F_k(z) \overline{F_j(\zeta)} \neq -1$ for $(z, \zeta) \in \Delta \times \Delta$ and $k, j = 1, \ldots, m$.

The classes C_m^l, $l = 1, 2, 3$, generalizing the classes of Bieberbach–Eilenberg, bounded and Grunsky–Shah functions, respectively, were introduced in a

somewhat different form by Gromova and Lebedev [4] and examined by means of methods of Löwner type as well as those of Grunsky–Nehari type in [1], [2], [7]–[9]. These classes are not compact.

By a variation of $\{F, G\} \in C_{m,n}$ ($F \in C_m^l$, $l = 1, 2, 3$) we mean a continuous mapping $\varepsilon \mapsto \{F^*(\varepsilon), G^*(\varepsilon)\}$ ($\varepsilon \mapsto F^*(\varepsilon)$) of the interval $\langle 0; \lambda\rangle$ or of the interval $\langle -\lambda, \lambda\rangle$, $\lambda > 0$, in $\mathscr{A}_{m,n}$ ($\mathscr{A}_m$), such that $\{F^*(0), G^*(0)\} = \{F, G\}$ ($F^*(0) = F$). The variation is called *admissible* if, for all ε sufficiently close to zero, $\{F^*(\varepsilon), G^*(\varepsilon)\} \in C_{m,n}$ ($F^*(\varepsilon) \in C_m^l$).

Let J (J^l, $l = 1, 2, 3$) be continuous functionals defined on $\mathscr{A}_{m,n}$ ($\mathscr{A}_m$). $\{F, G\} \in C_{m,n}$ ($F \in C_m^l$) such that

$$\operatorname{Re} J(\{F^*, G^*\}) \leqslant \operatorname{Re} J(\{F, G\}) \quad (\operatorname{Re} J^l(F^*) \leqslant \operatorname{Re} J^l(F))$$

for all $\{F^*, G^*\}$ (F^*) belonging to an intersection of $C_{m,n}$ (C_m^l) with some neighbourhood of $\{F, G\}(F)$ in the t.v.s. $\mathscr{A}_{m,n}$ ($\mathscr{A}_m$) is called a *local maximum*.

In the present paper variational methods for the classes $C_{m,n}$, C_m^l, $l = 1, 2, 3$, $m, n \geqslant 1$ are given. In Section 1, general admissible variations of pairs $\{F, G\} \in C_{m,n}$ and of functions $F \in C_m^l$ were constructed. Section 2 included, in the form of differential-functional equations, necessary conditions for $\{F, G\} \in C_{m,n}$ ($F \in C_m^l$) to be a local maximum for $\operatorname{Re} J$ ($\operatorname{Re} J^l$) if J (J^l) has a complex Gâteaux derivative on $C_{m,n}$ (C_m^l).

1. General admissible variations

1.1. Admissible variations for pairs of vector functions. Domains $D_1, \ldots, D_m, E_1, \ldots, E_n \subset C$ will be said *to have the property of a pair* if they are simply connected, $a_{0k} \in D_k$, $k = 1, \ldots, m$, $b_{0k} \in E_k$, $k = 1, \ldots, n$, and if they satisfy the conditions:

(1) $$D_k \cap D_j = \emptyset, \quad k \neq j,\ k, j = 1, \ldots, m \text{ when } m \geqslant 2,$$

(2) $$E_k \cap E_j = \emptyset, \quad k \neq j,\ k, j = 1, \ldots, n \text{ when } n \geqslant 2,$$

(3) $$D_k \cap 1/E_j = \emptyset, \quad k = 1, \ldots, m,\ j = 1, \ldots, n,$$

where $1/E_j = \{w\colon 1/w \in E_j\}$, $j = 1, \ldots, n$. Vectors $U = [u_1, \ldots, u_m]$, $V = [v_1, \ldots, v_n]$ are called *admissible* with respect to the domains $D_1, \ldots, D_m$, $E_1, \ldots, E_n$ having the property of a pair if $u_k \neq u_j$ for $k \neq j$, $k, j = 1, \ldots, m$, $v_k \neq v_j$ for $k \neq j$, $k, j = 1, \ldots, n$, if $u_k \in D_k \setminus \{a_{0k}\}$ or $u_k \in \Omega$ for $k = 1, \ldots, m$, and if $v_k \in E_k \setminus \{b_{0k}\}$ or $1/v_k \in \Omega$ for $k = 1, \ldots, n$, where $\Omega = C \setminus \operatorname{cl}(D_1 \cup \ldots \cup D_m \cup (1/E_1) \cup \ldots \cup (1/E_n))$.

Let

$$\sum_{j=1}^{m} r_j + \sum_{j=1}^{n} s_j \leqslant 1 + \sum_{j=1}^{m} p_j + \sum_{j=1}^{n} q_j,$$

where r_j, s_j are any fixed positive integers, p_j, q_j are any fixed non-negative integers not greater than 1, and let

$$\Phi(w) = \Phi(U, V; w) = \prod_{j=1}^{m} \frac{(w-a_{0j})^{r_j}}{(w-u_j)^{p_j}} \prod_{j=1}^{n} \frac{(1-b_{0j}w)^{s_j}}{(1-v_j w)^{q_j}},$$

$$\Psi(w) = \Psi(U, V; w) = -w^2 \Phi(1/w),$$

with an additional assumption that $s_k = 0$ whenever $b_{0k} = 0$, and that $q_l = 0$ whenever $v_l = 0$, for some $k, l = 1, \ldots, n$.

We shall first prove

THEOREM 1.1.1. *If the domains $D_1, \ldots, D_m, E_1, \ldots, E_n$ have the property of a pair and vectors $U = [u_1, \ldots, u_m]$, $V = [v_1, \ldots, v_n]$ are admissible with respect to these domains, then, for all ε sufficiently close to zero, the domains $D_1^*, \ldots, D_m^*, E_1^*, \ldots, E_n^*$ such that*

$$\partial D_k^* = w_{D,\varepsilon}^*(\partial D_k), \quad k = 1, \ldots, m, \qquad \partial E_k^* = w_{E,\varepsilon}^*(\partial E_k), \quad k = 1, \ldots, n, \tag{4}$$

where

$$w_{D,\varepsilon}^*(w) = w + \varepsilon\Phi(w), \qquad w_{E,\varepsilon}^*(w) = 1/w_{D,\varepsilon}^*(1/w) = w + \varepsilon\Psi(w) + o(\varepsilon), \tag{5}$$

have the property of a pair.

Here and throughout the paper, $o(\varepsilon)$ indicates that $o(\varepsilon)/\varepsilon \to 0$ almost uniformly as $\varepsilon \to 0$.

Proof. Let r be a sufficiently small positive number such that if $u_k \in D_k$, then $U_k = \{w\colon |w-u_k| \leqslant r\} \subset D \setminus \{a_{0k}\}$ or if $u_k \in \Omega$, then $U_k \subset \Omega$ for all $k = 1, \ldots, m$, and if $v_k \in E_k$, then $V_k = \{w\colon |w-v_k| \leqslant r\} \subset E_k \setminus \{b_{0k}\}$ or if $1/v_k \in \Omega$, then $1/V_k \subset \Omega$ for all $k = 1, \ldots, n$, with that $U_k \cap U_j = \emptyset$ if $U_k, U_j \subset \Omega$, $k \neq j$, and $V_k \cap V_j = \emptyset$ if $1/V_k, 1/V_j \subset \Omega$, $k \neq j$, and $U_k \cap (1/V_j) = \emptyset$ if $U_k, 1/V_j \subset \Omega$; let W be a domain contained in cl C such that $\partial W = \partial(U_1 \cup \ldots \cup U_m \cup (1/V_1) \cup \ldots \cup (1/V_n))$.

We shall demonstrate that the mapping $w_{D,\varepsilon}^*$ is univalent in W. Assume that it is not the case, i.e., let $w_{D,\varepsilon}^*(w) = w_{D,\varepsilon}^*(\omega)$ for some distinct points w, ω of the domain W. Hence follows the equality

$$1 + \varepsilon T(w, \omega) = 0, \tag{6}$$

where

$$T(w, \omega) = \begin{cases} \dfrac{\Phi(w) - \Phi(\omega)}{w - \omega}, & w \neq \omega, \\ \Phi'(\omega), & w = \omega, \end{cases} \qquad (w, \omega) \in W \times W.$$

Since

$$\Phi(w) = \gamma_1 w + \gamma_0 + \sum_{j=1}^{m} \frac{\alpha_j}{(w-u_j)^{p_j}} + \sum_{j=1}^{n} \frac{\beta_j}{(1-v_j w)^{q_j}}$$

for some $\gamma_1, \gamma_0, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in C$ and $|w-u_j| \geqslant r$ and $|1-wv_j| \geqslant r/(r+|v_j|)$ for all $w \in \operatorname{cl} W$, then

$$|T(w, \omega)| \leqslant |\gamma_1| + \sum_{j=1}^{m} \frac{|\alpha_j|}{r^{2p_j}} + \sum_{j=1}^{n} |\beta_j v_j| (1+|v_j|/r)^{2q_j}.$$

Consequently, T is an analytic and bounded mapping in $\operatorname{cl} W \times \operatorname{cl} W$, whence it immediately follows that equality (6) does not hold for all ε sufficiently close to zero, which contradicts the assumption. The contradiction obtained proves the univalence of $w^*_{D,\varepsilon}$ in W from which it follows directly, in virtue of the inclusion $\partial(D_1 \cup \ldots \cup D_m \cup (1/E_1) \cup \ldots \cup (1/E_n)) \subset W$, that the domains $D_1^*, \ldots, D_m^*, E_1^*, \ldots, E_n^*$ with boundaries defined in (4) satisfy conditions (1) and (2).

We shall show that these domains satisfy condition (3) as well. Indeed, if it were not the case, let w and $1/\omega$ be elements of the sets $W \cap D_k$ and $W \cap (1/E_j)$, respectively, such that $w^*_{D,\varepsilon}(w) w^*_{E,\varepsilon}(\omega) = 1$. Then we would have $1+\varepsilon T(w, 1/\omega) = 0$, which does not hold for all ε sufficiently close to zero, despite our supposition; because w and $1/\omega$ belong to disjoint sets. We have thus come to a contradiction. It still remains to notice that $w^*_{D,\varepsilon}(a_{0k}) = a_{0k} \in D_k^*$, $k = 1, \ldots, m$, and $w^*_{E,\varepsilon}(b_{0k}) = b_{0k} \in E_k^*$, $k = 1, \ldots, n$.

For all $k = 1, \ldots, m$ such that $p_k \neq 0$ and for all $k = 1, \ldots, n$ such that $q_k \neq 0$, let

$$\Phi_k(w) = \Phi_k(U, V; w) = \frac{(w-u_k)^{p_k}}{(w-a_{0k})^2} \Phi(w)$$

and

$$\Psi_k(w) = \Psi_k(U, V; w) = \frac{(w-v_k)^{q_k}}{(w-b_{0k})^2} \Psi(w),$$

respectively.

We shall now prove

THEOREM 1.1.2. *If* $\{F, G\} \in C_{m,n}$ *and vectors* $U = [u_1, \ldots, u_m]$, $V = [v_1, \ldots, v_n]$ *are admissible with respect to the domains* $D_k = F_k(\Delta)$, $k = 1, \ldots, m$, $E_k = G_k(\Delta)$, $k = 1, \ldots, n$, *then, for all real* α, *the following variations are admissible*: $\varepsilon \mapsto \{F^*(\varepsilon), G^*(\varepsilon)\}$, $F^*(\varepsilon) = [F_1^*, \ldots, F_m^*]$, $G^*(\varepsilon) = [G_1^*, \ldots, G_n^*]$, $\varepsilon \geqslant 0$, *where for* $k = 1, \ldots, m$,

$$(7) \quad F_k^*(z) = F_k(z) + \varepsilon e^{i\alpha} \Phi \circ F_k(z) - \frac{1}{2} \varepsilon e^{i\alpha} z F_k'(z) \frac{z+\zeta_{1k}}{z-\zeta_{ik}} \left(\frac{F_k(\zeta_{1k}) - a_{0k}}{\zeta_{1k} F_k(\zeta_{1k})} \right)^2 \Phi_k \circ F_k(\zeta_{1k}) + \frac{1}{2} \varepsilon e^{-i\alpha} z F_k'(z) \frac{1+z\bar{\zeta}_{1k}}{1-z\bar{\zeta}_{1k}} \left[\left(\frac{F_k(\zeta_{1k}) - a_{0k}}{\zeta_{1k} F_k(\zeta_{1k})} \right)^2 \Phi_k \circ F_k(\zeta_{1k}) \right]^- + o(\varepsilon)$$

when $u_k = F_k(\zeta_{1k})$ *and* $p_k \neq 0$, *or*

$$F_k^*(z) = F_k(z)+\varepsilon e^{i\alpha}\,\Phi\circ F_k(z) \tag{8}$$

when $u_k\in\Omega$ *or* $p_k = 0$, *and, for* $k = 1, \ldots, n$,

$$\begin{aligned} G_k^*(z) = {} & G_k(z)+\varepsilon e^{i\alpha}\,\Psi\circ G_k(z)- \\ & -\tfrac{1}{2}\varepsilon e^{i\alpha}\,zG_k'(z)\frac{z+\zeta_{2k}}{z-\zeta_{2k}}\left(\frac{G_k(\zeta_{2k})-b_{0k}}{\zeta_{2k}\,G_k(\zeta_{2k})}\right)^2\Psi_k\circ G_k(\zeta_{2k})+ \\ & +\tfrac{1}{2}\varepsilon^{-i\alpha}\,zG_k'(z)\frac{1+z\bar{\zeta}_{2k}}{1-z\bar{\zeta}_{2k}}\left[\left(\frac{G_k(\zeta_{2k})-b_{0k}}{\zeta_{2k}\,G_k(\zeta_{2k})}\right)^2\Psi_k\circ G_k(\zeta_{2k})\right]^- +o(\varepsilon) \end{aligned} \tag{9}$$

when $v_k = G_k(\zeta_{2k})\neq 0$ *and* $q_k\neq 0$, *or*

$$G_k^*(z) = G_k(z)+\varepsilon e^{i\alpha}\,\Psi\circ G_k(z)+o(\varepsilon) \tag{10}$$

when $1/v_k\in\Omega$ *or* $q_k = 0$.

Here and throughout the paper, $(\ )^-$ *indicates the complex conjugate.*

Proof. Let D_k^*, $k = 1, \ldots, m$, E_k^*, $k = 1, \ldots, n$, be domains with boundaries defined in (4). The proof of the theorem will be finished if we show that the functions F_k^*, $k = 1, \ldots, m$, G_k^*, $k = 1, \ldots, n$ defined by formulae (7)–(10), respectively, have the property that $F_k^*(\Delta) = D_k^*$, $k = 1, \ldots, m$, $G_k^*(\Delta) = E_k^*$, $k = 1, \ldots, n$.

Assume that $u_k = F_k(\zeta_{1k})$ and $p_k\neq 0$ for some $k = 1, \ldots, m$. If r_k is a radius of the closed disc with centre at ζ_{1k} contained in Δ and if

$$\delta = \min_{|z-\zeta_{ik}|=r_k}|F_k(z)-F_k(\zeta_{1k})|,$$

then from the univalence of F_k follows that $|F_k(z)-F_k(\zeta_{1k})|>\delta$ for all $z\in\Delta\cap\{z\colon |z-\zeta_{1k}|>r_k\}$, so as, the function $\check{F}_k$, given by the formula

$$\check{F}_k(z,\varepsilon) = F_k(z)+\varepsilon e^{i\alpha}\,\Phi\circ F_k(z),\qquad \varepsilon\geqslant 0,$$

for all real α and for all sufficiently small ε, is analytic and univalent in $\{z\colon r_0<|z|<1\}$, where $r_0 = |\zeta_{1k}|+r_k$. In consequence,

$$F_k^*(z) = F_k(z)+\varepsilon h_k(z)+o(\varepsilon), \tag{11}$$

where

$$h_k(z) = e^{i\alpha}\,\Phi\circ F_k(z)-zF_k'(z)\{S_k(z)+c_k-[S_k(1/\bar{z})]^- -\bar{c}_k\},\qquad z\in\Delta,$$

with that S_k is the principal part of the expansion of the function H_k given by the formula

$$H_k(z) = e^{i\alpha}\frac{\Phi\circ F_k(z)}{zF_k'(z)}$$

in a Laurent series in the annulus $\{z\colon r_0<|z|<1\}$, while c_k is an arbitrary constant; the proof of formula (11) in the case when $c_k\neq 0$ is some

modification of the proof of this formula given in the case $c_k = 0$ by Goluzin [3]. A direct calculation yields

$$S_k(z) = \frac{res_{\zeta_{1k}} H_k(z)}{z-\zeta_{1k}} = \frac{e^{i\alpha}\zeta_{1k}}{z-\zeta_{1k}}\left(\frac{F_k(\zeta_{1k})-a_{0k}}{\zeta_{1k}F_k'(\zeta_{1k})}\right)^2 \Phi_k \circ F_k(\zeta_{1k})$$

and if

$$c_k = \tfrac{1}{2}e^{i\alpha}\left(\frac{F_k(\zeta_{1k})-a_{0k}}{\zeta_{1k}F_k'(\zeta_{1k})}\right)^2 \Phi_k \circ F_k(\zeta_{1k}),$$

then (11) may be condensed in the form of (7).

Formulae (8)–(10) are proved in an analogous way.

Remark 1. Two functions F and G are called an *Aharonov pair* $\{F, G\}$ if they are univalent in Δ, $F(z) = a_1 z + a_2 z^2 + \ldots$, $G(z) = b_1 z + b_2 z^2 + \ldots$, and $F(z)G(\zeta) \neq 1$ for all $(z, \zeta) \in \Delta \times \Delta$. If $m = n = 1$, $a_{01} = b_{01} = 0$, $r_1 = p_1 = 1$, $s_1 = q_1 = 0$ and if $c_1 = 0$, then Theorem 1.1.2 of the present paper is reduced to Theorem 3.1 of paper [5] concerning Aharonov pairs, where the Schiffer variational formula were used in the proof.

1.2. Admissible variations for generalized Bieberbach–Eilenberg, bounded and Grunsky–Shah functions. Domains $D_1^l, \ldots, D_m^l \subset C$, $l = 1, 2, 3$, l is fixed, are said to have the disjointness property if they are simply connected, $a_{0k} \in D_k^l$ for $k = 1, \ldots, m$,

$$D_k^l \cap D_j^l = \emptyset \quad \text{for } k \neq j \text{ and } k, j = 1, \ldots, m \text{ when } m \geqslant 2 \tag{12}$$

and, moreover,

$$D_k^1 \cap (1/D_j^1) = \emptyset \quad \text{for } k, j = 1, \ldots, m \text{ if } l = 1, \tag{13}$$

$$D_k^2 \subset \Delta \quad \text{for } k = 1, \ldots, m \text{ if } l = 2, \tag{14}$$

$$D_k^3 \cap (-1/\overline{D_k^3}) = \emptyset \quad \text{for } k, j = 1, \ldots, m \text{ if } l = 3, \tag{15}$$

where $-1/\overline{D_j^3} = \{-1/\bar{w}\colon w \in D_j^3\}$, $j = 1, \ldots, m$. A vector $U = [u_1, \ldots, u_m]$ is called *admissible* with respect to the domains $D_1^l, \ldots, D_m^l$ having the disjointness property if $u_k \neq u_j$ for $k \neq j$ and $k, j = 1, \ldots, m$, when $m \geqslant 2$, and if $u_k \in D_k^l \setminus \{a_{0k}\}$ or $u_k \in \Omega^l$ for $k = 1, \ldots, m$, where

$$\begin{aligned}
\Omega^l &= C \setminus \mathrm{cl} \bigcup_{k=1}^{m} (D_k^1 \cup (1/D_k^1)) && \text{if } l = 1,\\
&= \Delta \setminus \mathrm{cl} \bigcup_{k=1}^{m} D_k^2 && \text{if } l = 2,\\
&= C \setminus \mathrm{cl} \bigcup_{k=1}^{m} (D_k^3 \cup (-1/\overline{D_k^3})) && \text{if } l = 3.
\end{aligned}$$

Let

$$\sum_{j=1}^{m}(r_j+s_j) \leqslant 1+\sum_{j=1}^{m}(p_j+q_j),$$

where r_j, s_j are any fixed positive integers, p_j, q_j are any fixed non-negative integers not greater than 1, and let

$$\varphi^{l,\alpha}(w)=\varphi^{l,\alpha}(U;w)=e^{i\alpha}\prod_{j=1}^{m}\frac{(w-a_{0j})^{r_j}(1-a_{0j}w)^{s_j}}{(w-u_j)^{p_j}(1-u_jw)^{q_j}}, \quad \text{if } l=1,$$

$$=e^{i\alpha}\prod_{j=1}^{m}\frac{(w-a_{0j})^{r_j}(1-\bar{a}_{0j}w)^{s_j}}{(w-u_j)^{p_j}(1-\bar{u}_jw)^{q_j}}, \quad \text{if } l=2,$$

$$=e^{i\alpha}\prod_{j=1}^{m}\frac{(w-a_{0j})^{r_j}(1+\bar{a}_{0j}w)^{s_j}}{(w-u_j)^{p_j}(1+\bar{u}_jw)^{q_j}}, \quad \text{if } l=3,$$

$$\psi^{l,\alpha}(w)=-\varphi^{1,\alpha}(1/w) \quad \text{if } l=1,$$

$$=-[\varphi^{2,\alpha}(1/\bar{w})]^{-} \quad \text{if } l=2,$$

$$=-[\varphi^{3,\alpha}(-1/\bar{w})]^{-} \quad \text{if } l=3,$$

$$\Phi^{l,\alpha}(w)=\varphi^{l,\alpha}(w)+\psi^{l,\alpha}(w),$$

where α is real, with an additional assumption that $s_k=0$, where $a_{0k}=0$ for some $k=1,\ldots,m$, and $q_k=0$, where $u_k=0$ for some $k=1,\ldots,m$.

The following theorem holds.

THEOREM 1.2.3. *If domains $D_1^l,\ldots,D_m^l$, $l=1,2,3$, l is fixed, have the disjointness property and a vector $U=[u_1,\ldots,u_m]$ is admissible with respect to these domains, then, for any real α and for all ε sufficiently close to zero, the domains $D_1^{l*},\ldots,D_m^{l*}$ such that*

$$\partial D_k^{l*}=w_l^*(\partial D_k^l), \quad k=1,\ldots,m, \tag{16}$$

where

$$w_l^*(w)=we^{\varepsilon\Phi^{l,\alpha}(w)}=w+\varepsilon w\Phi^{l,\alpha}(w)+o(\varepsilon), \quad \varepsilon\geqslant 0, \tag{17}$$

have the disjointness property.

Proof. Assume that r is a sufficiently small positive number such that, if $u_k\in D_k^l$, then $U_k=\{w\colon |w-u_k|\leqslant r\}\subset D_k^l\setminus\{a_{0k}\}$ or, if $u_k\in\Omega^l$, then $U_k\subset\Omega^l$ for $k=1,\ldots,m$. Let W^l be a domain contained in cl C such that $\partial W^l=\partial(U_1\cup\ldots\cup U_m\cup(1/V_{11})\cup\ldots\cup(1/V_{1m}))$, where $V_{1k}=U_k$, $V_{2k}=\bar{U}_k$, $V_{3k}=-\bar{U}_k$. We shall consider a function $\Psi^{l,\alpha}$ defined in cl $W^l\times$cl W^l by the formula

$$\Psi^{l,\alpha}(\omega,w)=\frac{\Phi^{l,\alpha}(\omega)-\Phi^{l,\alpha}(w)}{\omega-w}, \quad \omega\neq w,$$

$$=\Phi^{l,\alpha}(w), \quad \omega=w.$$

It is an analytic and, what is more, bounded mapping on this set. Indeed, if $w \in W^l$, then also $w_l \in W^l$, where $w_1 = 1/w$, $w_2 = 1/\bar{w}$, $w_3 = -1/\bar{w}$. So, for all $w \in \operatorname{cl} W^l$, we have $|w-u_j| \geqslant r$, $|w_l-u_j| \geqslant r$, $|1-v_{lj}w| \geqslant r/(r+|v_{lj}|)$, $|1-v_{lj}w_l| \geqslant r/(r+|v_{lj}|)$, where $v_{1j} = u_j$, $v_{2j} = \bar{u}_j$, $v_{3j} = -\bar{u}_j$. Consequently, adopting the notation

$$\Phi^{l,\alpha}(w) = \gamma_{l1}w + \gamma_{l0} + \sum_{j=1}^{m}\left[\frac{\alpha_{lj}}{(w-u_j)^{p_j}} + \frac{\alpha'_{lj}}{(w-u_j)^{q_j}} + \frac{\beta_{lj}}{(1-v_{lj}w)^{q_j}} + \frac{\beta'_{lj}}{(1-v_{lj}w)^{p_j}}\right],$$

we find that

$$|\Psi^{l,\alpha}(\omega, w)| \leqslant |\gamma_{l1}| + \sum_{j=1}^{m}\left[\frac{|\alpha_{lj}|}{r^{2p_j}} + \frac{|\alpha'_{lj}|}{r^{2q_j}} + |\beta_{lj}v_{lj}|\left(1+\frac{|v_{lj}|}{r}\right)^{2q_j} + |\beta'_{lj}v_{lj}|\left(1+\frac{|v_{lj}|}{r}\right)^{2p_j}\right]$$

for all $(\omega, w) \in \operatorname{cl} W^l \times \operatorname{cl} W^l$.

We shall prove that the mappings w_l^* are univalent in W^l. Suppose it is not the case, i.e., let $w_l^*(\omega) = w_l^*(w)$ for some distinct points ω and w of the set W^l. Then, by applying the inequality $|1-\exp w| \leqslant |w|\exp|w|$, we obtain

$$\begin{aligned}|\omega - w| &= |\omega|\,|1-\exp[\varepsilon(\omega-w)\,\Psi^{l,\alpha}(\omega, w)]| \\ &\leqslant \varepsilon\,|\omega(\omega-w)\,\Psi^{l,\alpha}(\omega, w)|\exp[\varepsilon\,|(\omega-w)\,\Psi^{l,\alpha}(\omega, w)|].\end{aligned}$$

However, this inequality is impossible for all ε sufficiently close to zero. We have thus come to a contradiction.

It follows directly from the univalence of the mappings w_l^* in W^l that the domains $D_1^{l*}, \ldots, D_m^{l*}$ with boundaries (16) are simply connected; let us also notice that $a_{0k} \in D_k^{l*}$ for $k = 1, \ldots, m$, which follows from (17). These domains also satisfy, respectively, conditions (13)–(15). Suppose it is not so. Then $w_1^*(\omega)w_1^*(w) = 1$ or $w_2^*(\omega)\overline{w_2^*(w)} = 1$ or $w_3^*(\omega)\overline{w_3^*(w)} = -1$ for some points ω and w of the sets $W^l \cap D_k^l$ and $W^l \cap D_j^l$, respectively. Since

$$\Phi^{1,\alpha}(w) = -\Phi^{1,\alpha}(w_1), \quad \Phi^{2,\alpha}(w) = -[\Phi^{2,\alpha}(w_2)]^-, \quad \Phi^{3,\alpha}(w) = -[\Phi^{3,\alpha}(w_3)]^-,$$

we would have that

$$\begin{aligned}|\omega - w_l| &= |\omega\,[1-\exp\{\varepsilon(\omega-w_l)\,\Psi^{l,\alpha}(\omega, w_l)\}]| \\ &\leqslant \varepsilon\,|\omega(\omega-w_l)\,\Psi^{l,\alpha}(\omega, w_l)|\exp|\varepsilon(\omega-w_l)\,\Psi^{l,\alpha}(\omega, w_l)|\end{aligned}$$

that does not hold for all ε sufficiently close to zero.

This contradiction completes the proof.

For all $k = 1, \ldots, m$ such that $p_k \neq 0$, let

$$\varphi_k^{l,\alpha}(w) = (w-u_k)^{p_k}(w-a_{0k})^{-2}\,\varphi^{l,\alpha}(w)$$

and, for all $k = 1, \ldots, m$ such that $q_k \neq 0$, let

$$\psi_k^{l,\alpha}(w) = (w-u_k)^{q_k}(w-a_{0k})^{-2}\,\psi^{l,\alpha}(w).$$

We shall now prove a theorem which is a counterpart of Theorem 1.1.2.

THEOREM 1.2.4. *Let $l = 1, 2, 3$ be fixed. If $F \in C_m^l$ and a vector $U = [u_1, \ldots, u_m]$ is an admissible vector with respect to domains $D_k^l = F_k(\Delta)$, $k = 1, \ldots, m$, then, for all real α, the following variation is admissible: $\varepsilon \mapsto F^*(\varepsilon) = [F_1^*, \ldots, F_m^*]$, $\varepsilon \geqslant 0$, where, for $k = 1, \ldots, m$,*

$$(18)\quad F_k^*(z) = F_k(z) + \varepsilon F_k(z)\,\Phi^{l,\alpha} \circ F_k(z) - \\ -\tfrac{1}{2}\varepsilon z F_k'(z)\frac{z+\zeta_k}{z-\zeta_k}\left(\frac{F_k(\zeta_k)-a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 \times \\ \times F_k(\zeta_k)\,[\varphi_k^{l,\alpha} \circ F_k(\zeta_k) + \psi_k^{l,\alpha} \circ F_k(\zeta_k)] + \\ +\tfrac{1}{2}\varepsilon z F_k'(z)\frac{1+\bar{\zeta}_k z}{1-\bar{\zeta}_k z}\left\{\left(\frac{F_k(\zeta_k)-a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 \times \right. \\ \left. \times F_k(\zeta_k)\,[\varphi_k^{l,\alpha} \circ F_k(\zeta_k) + \psi_k^{l,\alpha} \circ F_k(\zeta_k)]\right\}^- + o(\varepsilon)$$

when $u_k = F_k(\zeta_k)$, $p_k \neq 0$ and $q_k \neq 0$, or

$$(19)\quad F_k^*(z) = F_k(z) + \varepsilon F_k(z)\,\Phi^{l,\alpha} \circ F_k(z) - \\ -\tfrac{1}{2}\varepsilon z F_k'(z)\frac{z+\zeta_k}{z-\zeta_k}\left(\frac{F_k(\zeta_k)-a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 F_k(\zeta_k)\,\varphi_k^{l,\alpha} \circ F_k(\zeta_k) + \\ +\tfrac{1}{2}\varepsilon z F_k'(z)\frac{1+\bar{\zeta}_k z}{1-\bar{\zeta}_k z}\left[\left(\frac{F_k(\zeta_k)-a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 F_k(\zeta_k)\,\varphi_k^{l,\alpha} \circ F_k(\zeta_k)\right]^- + o(\varepsilon)$$

when $u_k = F_k(\zeta_k)$, $p_k \neq 0$ and $q_k = 0$, or

$$(20)\quad F_k^*(z) = F_k(z) + \varepsilon F_k(z)\,\Phi^{l,\alpha} \circ F_k(z) - \\ -\tfrac{1}{2}\varepsilon z F_k'(z)\frac{z+\zeta_k}{z-\zeta_k}\left(\frac{F_k(\zeta_k)-a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 F_k(\zeta_k)\,\psi_k^{l,\alpha} \circ F_k(\zeta_k) + \\ +\tfrac{1}{2}\varepsilon z F_k'(z)\frac{1+\bar{\zeta}_k z}{1-\bar{\zeta}_k z}\left[\left(\frac{F_k(\zeta_k)-a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 F_k(\zeta_k)\,\psi_k^{l,\alpha} \circ F_k(\zeta_k)\right]^- + o(\varepsilon)$$

when $u_k = F_k(\zeta_k)$, $p_k = 0$ and $q_k \neq 0$, or

$$(21)\quad F_k^*(z) = F_k(z) + \varepsilon F_k(z)\,\Phi_k^{l,\alpha} \circ F_k(z) + o(\varepsilon)$$

when $u_k \in \Omega^l$.

Proof. We shall confine ourselves to the case where $u_k = F_k(\zeta_k)$, $p_k \neq 0$ and $q_k \neq 0$. Hence, if r_k is a radius of a closed disc with centre at ζ_k, contained Δ and $\delta = \min_{|z-\zeta_k|=r_k} |F_k(z) - F_k(\zeta_k)|$, then it follows from the univalence of F_k in Δ that $|F_k(z) - F_k(\zeta_k)| > \delta$ for all $z \in \Delta \cap \{z: |z-\zeta_k| > r_k\}$. Thus, for all real α and for all sufficiently small ε, the function $\hat{F}_k^l$ given by the formula

$$\hat{F}_k^l(z, \varepsilon) = F_k(z) + \varepsilon F_k(z)\,\Phi^{l,\alpha} \circ F_k(z) + o(\varepsilon), \qquad \varepsilon \geqslant 0,$$

will be analytic and univalent in $\{z\colon r_0 < |z| < 1\}$, where $r_0 = |\zeta_k| + r_k$, and, in consequence, if D_k^{l*} is a domain with boundary ∂D_k^{l*} defined by formula (16), in which $D_k^l = F_k(\Delta)$, then a mapping $F_k^*\colon \Delta \mapsto D_k^{l*}$, $F_k^*(0) = a_{0k}$, is given by the formula

$$F_k^*(z) = F_k(z) + \varepsilon h_k^l(z) + o(\varepsilon), \tag{22}$$

where

$$h_k^l(z) = F_k(z)\,\Phi^{l,\alpha} \circ F_k(z) - zF_k'(z)\{S_k^l(z) + c_k^l - [S_k^l(1/\bar{z})]^- - (c_k^l)^-\}, \qquad z \in \Delta,$$

with that S_k^l is the principal part of the expansion of the function H_k^l defined by the formula

$$H_k^l(z) = \frac{F_k(z)\,\Phi^{l,\alpha} \circ F_k(z)}{zF_k'(z)}$$

in a Laurent series in the annulus $\{z\colon r_0 < |z| < 1\}$, whereas c_k^l is an arbitrary constant. Since

$$S_k^l(z) = \frac{\zeta_k}{z - \zeta_k}\left(\frac{F_k(\zeta_k) - a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 F_k(\zeta_k)\,[\varphi_k^{l,\alpha} \circ F_k(\zeta_k) + \psi_k^{l,\alpha} \circ F_k(\zeta_k)],$$

formula (22) for

$$c_k^l = \tfrac{1}{2}\left(\frac{F_k(\zeta_k) - a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 F_k(\zeta_k)\,[\varphi_k^{l,\alpha} \circ F_k(\zeta_k) + \psi_k^{l,\alpha} \circ F_k(\zeta_k)],$$

will take the form (18).

The remaining cases are proved in an analogous way.

Remark 2. From (19) and (21), if $m = 1$, $a_{01} = 0$, $r_1 = s_1 = q_1 = 0$, $p_1 = 1$ and if $c_1^l = 0$, follow the results of Hummel and Schiffer ([5], Theorems 2.2 and 2.3), when $l = 1$ and the results of Jondro ([6], Theorems 2 and 3) when $l = 3$, concerning Bieberbach–Eilenberg functions and Grunsky–Shah ones, respectively.

2. The local maximum theorems

2.1. The local maximum theorems for pairs of vector functions. Let the functional J have a complex Gâteaux derivative on $C_{m,n}$, i.e., let the following asymptotic formula

$$J(\{F + \varepsilon F^0, G + \varepsilon G^0\}) = J(\{F, G\}) + \varepsilon\Big[\sum_{k=1}^{m} J_{1k}(F_k^0) + \sum_{k=1}^{n} J_{2k}(G_k^0)\Big] + o(\varepsilon) \tag{23}$$

hold, where $\{F, G\} \in C_{m,n}$, $\{F^0, G^0\} \in \mathscr{A}_{m,n}$, $J_{1k}(F_k^0) = J_{1k}(\{F, G\}; F_k^0)$, $k = 1, \ldots, m$, $J_{2k}(G^0) = J_{2k}(\{F, G\}; G_k^0)$, $k = 1, \ldots, n$, with that $J_{11}, \ldots, J_{1m}$, $J_{21}, \ldots, J_{2n}$ are continuous linear functionals on $F_1^0, \ldots, F_m^0$, $G_1^0, \ldots, G_n^0$, respectively, depending also on $\{F, G\}$.

Hence, if $\{F, G\} \in C_{m,n}$, and $u_k = F_k(\zeta_{1k})$ for $k = 1, \ldots, m$ and $v_k = G_k(\zeta_{2k})$

for $k = 1, \ldots, n$, then, for the pairs $\{F^*, G^*\}$ defined in Theorem 1.1.2, by making use of the fact that $\operatorname{Re}\{x\} = \operatorname{Re}\{\bar{x}\}$ we have

$$\operatorname{Re} J(\{F^*, G^*\})$$

$$= \operatorname{Re} J(\{F, G\}) + \varepsilon \operatorname{Re}\{e^{i\alpha}[\sum_{k=1}^{m} M_k \circ F_k(\zeta_{1k}) + \sum_{k=1}^{n} N_k \circ G_k(\zeta_{2k})]\} + o(\varepsilon),$$

where

(24) $M_k \circ F_k(\zeta_{1k})$

$$= J_{1k}[\Phi \circ F_k(z_{1k})] - \left(\frac{F_k(\zeta_{1k}) - a_{0k}}{\zeta_{1k} F_k'(\zeta_{1k})}\right)^2 Q_{1k}(\zeta_{1k})\, \Phi_k \circ F_k(\zeta_{1k}), \quad \text{when } p_k \neq 0,$$

$$= J_{1k}[\Phi \circ F_k(z_{1k})], \quad \text{when } p_k = 0,$$

(25) $N_k \circ G_k(\zeta_{2k})$

$$= J_{2k}[\Psi \circ G_k(z_{2k})] - \left(\frac{G_k(\zeta_{2k}) - b_{0k}}{\zeta_{2k} G_k'(\zeta_{2k})}\right)^2 Q_{2k}(\zeta_{2k})\, \Psi_k \circ G_k(\zeta_{2k}), \quad \text{when } q_k \neq 0,$$

$$= J_{2k}[\Psi \circ G_k(z_{2k})], \quad \text{when } q_k = 0,$$

with that

$$Q_{1k}(\zeta) = I_{1k}(\zeta) + \operatorname{Re} J_{1k}[z_{1k} F_k'(z_{1k})] + [I_{1k}(1/\bar{\zeta})]^-,$$

$$I_{1k}(\zeta) = J_{1k}[z_{1k} F_k'(z_{1k})\, \zeta/(z_{1k} - \zeta)],$$

$$Q_{2k}(\zeta) = I_{2k}(\zeta) + \operatorname{Re} J_{2k}[z_{2k} G_k'(z_{2k})] + [I_{2k}(1/\bar{\zeta})]^-,$$

$$I_{2k}(\zeta) = J_{2k}[z_{2k} G_k'(z_{2k})\, \zeta/(z_{2k} - \zeta)].$$

Since α can be arbitrary, we obtain for $\{F, G\}$ being a local maximum of $\operatorname{Re} J$

$$\sum_{k=1}^{m} M_k \circ F_k(\zeta_{1k}) + \sum_{k=1}^{n} N_k \circ G_k(\zeta_{2k}) = 0 \tag{26}$$

and, in particular, for all $k = 1, \ldots, m$ such that $p_k \neq 0$ and for all $k = 1, \ldots, n$ such that $q_k \neq 0$, we have, respectively,

$$\left(\frac{\zeta_{1k} F_k'(\zeta_{1k})}{F_k(\zeta_{1k}) - a_{0k}}\right)^2 P_{1k} \circ F_k(\zeta_{1k}) = Q_{1k}(\zeta_{1k}) \tag{27}$$

$$\text{and} \quad \left(\frac{\zeta_{2k} G_k'(\zeta_{2k})}{G_k(\zeta_{2k}) - b_{0k}}\right)^2 P_{2k} \circ G_k(\zeta_{2k}) = Q_{2k}(\zeta_{2k}),$$

where

(28) $P_{1k} \circ F_k(\zeta_{1k})$

$$= \{J_{1k}[\Phi \circ F_k(z_{1k})] + \sum_{\substack{j=1 \\ j \neq k}}^{m} M_j \circ F_j(\zeta_{1j}) + \sum_{j=1}^{n} N_j \circ G_j(\zeta_{2j})\}/\Phi_k \circ F_k(\zeta_{1k})$$

and

$$(29)\quad P_{2k}\circ G_k(\zeta_{2k})$$
$$= \{J_{2k}[\Psi\circ G_k(z_{2k})]+\sum_{\substack{j=1\\ j\neq k}}^{n} N_j\circ G_j(\zeta_{2j})+\sum_{j=1}^{m} M_j\circ F_j(\zeta_{1j})\}/\Psi_k\circ G_k(\zeta_{2k}).$$

Consequently, after changing in (26) the roles of $\zeta_{11}, \ldots, \zeta_{1m}, \zeta_{21}, \ldots, \zeta_{2n}$ and $z_{11}, \ldots, z_{1m}, z_{21}, \ldots, z_{2n}$, we obtain

THEOREM 2.1.5. *If the functional J has a complex Gâteaux derivative on $C_{m,n}$, as in* (23), *and if $\{F, G\}$ is a local maximum with respect to* Re J, *then for all $z_{11}, \ldots, z_{1m}, z_{21}, \ldots, z_{2n}\in\Delta\setminus 0$, satisfy the differential equation*

$$(30)\qquad \sum_{k=1}^{m} M_k\circ F_k(z_{1k})+\sum_{k=1}^{n} N_k\circ G_k(z_{2k})=0.$$

What is more, $\operatorname{Im} J_{1k}[z_{1k}F_k'(z_{1k})]=0$ *for* $z_{1k}\in\Delta$, $k=1, \ldots, m$, $\operatorname{Im} J_{2k}[z_{2k}G_k'(z_{2k})]=0$ *for* $z_{2k}\in\Delta$, $k=1, \ldots, n$, *and* $Q_{1k}(z)\leqslant 0$, $k=1, \ldots, m$, $Q_{2k}(z)\leqslant 0$, $k=1, \ldots, n$, *for* $z\in\partial\Delta$.

Proof. Let $\omega^*(z)=e^{i\varepsilon}z$, where ε is real, let $\omega^{**}(z)=K^{-1}\circ[K(z)/(1+\varepsilon)]=z+\varepsilon z p(z)+o(\varepsilon)$, $p(z)=(z+e^{i\alpha})/(z-e^{i\alpha})$, where $\varepsilon>0$, $K(z)=z/(1+e^{-i\alpha}z)^2$, α is real, and let $\varepsilon\mapsto\{F^*(\varepsilon), G^*(\varepsilon)\}$, $\varepsilon\mapsto\{F^{**}(\varepsilon), G^{**}(\varepsilon)\}$ be admissible variations defined as follows:

$$\begin{aligned} F_j^*(z)&=F_j(z), & F_j^{**}(z)&=F_j(z), & &j\neq k,\\ &=F_k\circ\omega^*(z), & &=F_k\circ\omega^{**}(z), & &j=k,\\ G_j^*(z)&=G_j(z), & G_j^{**}(z)&=G_j(z), & &j=1, \ldots, n, \end{aligned}\qquad j=1, \ldots, m,$$

for $k=1, \ldots, m$ and

$$\begin{aligned} F_j^*(z)&=F_j(z), & F_j^{**}(z)&=F_j(z), & &j=1, \ldots, m,\\ G_j^*(z)&=G_j(z), & G_j^{**}(z)&=G_j(z), & &j\neq k,\\ &=G_k\circ\omega^*(z), & &=G_k\circ\omega^{**}(z), & &j=k, \end{aligned}\qquad j=1, \ldots, n,$$

for $k=1, \ldots, n$. Then, since $F_k\circ\omega^*(z)=F_k(z)+i\varepsilon z F_k'(z)+o(\varepsilon)$, $G_k\circ\omega^{**}(z)=G_k(z)+\varepsilon z G_k'(z)p(z)+o(\varepsilon)$, we have

$$(31)\quad \operatorname{Re} J(\{F^*, G^*\})$$
$$\begin{aligned} &= \operatorname{Re} J(\{F, G\})+\varepsilon\operatorname{Re}\{iJ_{1k}[z_{1k}F_k'(z_{1k})]\}+o(\varepsilon), && k=1, \ldots, m,\\ &= \operatorname{Re} J(\{F, G\})+\varepsilon\operatorname{Re}\{iJ_{2k}[z_{2k}G_k'(z_{2k})]\}+o(\varepsilon), && k=1, \ldots, n, \end{aligned}$$

and

$$(32)\quad \operatorname{Re} J(\{F^{**}, G^{**}\})$$
$$\begin{aligned} &= \operatorname{Re} J(\{F, G\})+\varepsilon\operatorname{Re} J_{1k}[z_{1k}F_k'(z_{1k})p(z_{1k})]+o(\varepsilon), && k=1, \ldots, m,\\ &= \operatorname{Re} J(\{F, G\})+\varepsilon\operatorname{Re} J_{2k}[z_{2k}G_k'(z_{2k})p(z_{2k})]+o(\varepsilon), && k=1, \ldots, n. \end{aligned}$$

From (31), since ε is real, if follows that $\operatorname{Im} J_{1k}[z_{1k}F_k'(z_{1k})] = 0$, $z_{1k} \in \Delta$, $k = 1, \ldots, m$, and $\operatorname{Im} J_{2k}[z_{2k}G_k'(z_{2k})] = 0$, $z_{2k} \in \Delta$, $k = 1, \ldots, n$.

As $\varepsilon > 0$ we get from (32) that $\operatorname{Re} J_{1k}[z_{1k}F_k'(z_{1k})p(z_{1k})] \leqslant 0$, $k = 1, \ldots, m$, $\operatorname{Re} J_{2k}[z_{2k}G_k'(z_{2k})p(z_{2k})] \leqslant 0$, $k = 1, \ldots, n$. But $p(z) = 1 + 2e^{i\alpha}/(z - e^{i\alpha})$, and therefore

$$Q_{1k}(e^{i\alpha}) = \operatorname{Re} J_{1k}[z_{1k}F_k'(z_{1k})] + 2\operatorname{Re} I_{1k}(e^{i\alpha}) = \operatorname{Re} J_{1k}[z_{1k}F_k'(z_{1k})p(z_{1k})], \quad k = 1, \ldots, m,$$

$$Q_{2k}(e^{i\alpha}) = \operatorname{Re} J_{2k}[z_{2k}G_k'(z_{2k})] + 2\operatorname{Re} I_{2k}(e^{i\alpha}) = \operatorname{Re} J_{2k}[z_{2k}G_k'(z_{2k})p(z_{2k})], \quad k = 1, \ldots, n.$$

Consequently, $Q_{1k}(e^{i\alpha}) \leqslant 0$, $k = 1, \ldots, m$, and $Q_{2k}(e^{i\alpha}) \leqslant 0$, $k = 1, \ldots, n$, for $-\pi < \alpha \leqslant \pi$.

Let

$$X(U, V) = \sum_{k=1}^{m} J_{1k}[\Phi \circ F_k(z_{1k})] + \sum_{k=1}^{n} J_{2k}[\Psi \circ G_k(z_{2k})],$$

$$Y(F, G) = \boldsymbol{C} \setminus \Big[\bigcup_{k=1}^{m} F_k(\Delta) \cup \bigcup_{k=1}^{n} 1/G_k(\Delta)\Big].$$

There also holds the following

THEOREM 2.1.6. *If J and $\{F, G\}$ are such as in the preceding theorem, then, for all $k = 1, \ldots, m$ such that $p_k \neq 0$,*

$$\left(\frac{zF_k'(z)}{F_k(z) - a_{0k}}\right)^2 P_{1k} \circ F_k(z) = Q_{1k}(z) \quad \text{for } z \in \Delta \setminus \{0\}$$

and, for all $k = 1, \ldots, n$ such that $q_k \neq 0$,

$$\left(\frac{zG_k'(z)}{G_k(z) - b_{0k}}\right)^2 P_{2k} \circ G_k(z) = Q_{2k}(z) \quad \text{for } z \in \Delta \setminus \{0\}.$$

Here the function $P_{1k} \circ F_k$ (the function $P_{2k} \circ G_k$) is defined by equation (28) (equation (29)) in which $\zeta_{1k} = z$ ($\zeta_{2k} = z$), z is arbitrary, and $\zeta_{11}, \ldots, \zeta_{1k-1}, \zeta_{1k+1}, \ldots, \zeta_{1m}$, $\zeta_{21}, \ldots, \zeta_{2n}$ ($\zeta_{11}, \ldots, \zeta_{1m}, \zeta_{21}, \ldots, \zeta_{2k-1}, \zeta_{2k+1}, \ldots, \zeta_{2n}$) are fixed points of $\Delta \setminus \{0\}$.

If the mapping P_{1k} (P_{2k}) is analytic outside some isolated singularities and not identically zero, then the set $\partial F_k(\Delta)$ ($\partial G_k(\Delta)$) lies on the trajectory of the quadratic differential $P_{1k}(w)(w - a_{0k})^{-2}\,dw^2$ ($P_{2k}(w)(w - b_{0k})^{-2}\,dw^2$).

If the expression $X(U, V)$ is not identically zero with respect to the variables $u_1, \ldots, u_m, v_1, \ldots, v_n$, then the set $Y(F, G)$ has no interior points.

Proof. The inequality $Q_{1k}(z) \leqslant 0$ ($Q_{2k}(z) \leqslant 0$) for $z \in \partial\Delta$ implies the inequality $Q_{1k}(z)z^{-2}\,dz^2 \geqslant 0$ ($Q_{2k}(z)z^{-2}\,dz^2 \geqslant 0$) for $z \in \partial\Delta$, which proves that the set $\partial F_k(\Delta)$ ($\partial G_k(\Delta)$) lies on the trajectory $P_{1k}(w)(w - a_{0k})^{-2}\,dw^2 \geqslant 0$ ($P_{2k}(w)(w - b_{0k})^{-2}\,dw^2 \geqslant 0$).

Suppose that the set Int $Y(F, G)$ contains some disc Y_0, and let $u_1, \ldots, u_m, 1/v_1, \ldots, 1/v_n \in Y_0$. Then, in virtue of Theorem 1.1.2 (formulae (8) and (10)),

$$\operatorname{Re} J(\{F^*, G^*\}) = \operatorname{Re} J(\{F, G\}) + \varepsilon \operatorname{Re}[e^{i\alpha} X(U, V)] + o(\varepsilon).$$

Hence, since $\{F, G\}$ is a local maximum, and α is real, it follows that $X(U, V) = 0$ for all $u_1, \ldots, u_m, 1/v_1, \ldots, 1/v_n \in Y_0$, which contradicts the assumption that $X(U, V) \not\equiv 0$.

The proof of the theorem has therefore been completed.

2.2. The local maximum theorems for generalized Bieberbach–Eilenberg, bounded and Grunsky–Shah functions. Let $l = 1, 2, 3$ be fixed. Let the functional J^l have complex Gâuteaux derivative on C_m^l, i.e., let the following asymptotic formula

$$J^l(F + \varepsilon F^0) = J^l(F) + \varepsilon \sum_{k=1}^{m} J_k^l(F_k^0) + o(\varepsilon) \tag{33}$$

hold, where $F \in C_m^l$, $F^0 \in \mathscr{A}_m$, $J_k^l(F_k^0) = J^l(F; F_k^0)$, $k = 1, \ldots, m$, with that $J_1^l, \ldots, J_m^l$ are continuous linear functionals with respect to $F_1^0, \ldots, F_k^0$, respectively, depending also on F.

Let

$$\varphi^l(w) = \varphi^{l,0}(w), \quad \varphi_k^l(w) = \varphi_k^{l,0}(w), \quad \Phi^l(w) = \varphi^l(w) + \psi^l(w),$$
$$\psi^l(w) = \psi^{l,0}(w), \quad \psi_k^l(w) = \psi_k^{l,0}(w), \quad \Phi_k^l(w) = \varphi_k^l(w) + \psi_k^l(w).$$

So, if $u_k = F_k(\zeta_k)$, $\zeta_k \in \Delta \setminus \{0\}$ for all $k = 1, \ldots, m$, then, for the functions $F^* = F^*(\varepsilon)$ defined in Theorem 1.2.4,

$$\operatorname{Re} J^l(F^*) = \operatorname{Re} J^l(F) + \varepsilon \operatorname{Re}\Big[e^{i\alpha} \sum_{k=1}^{m} M_k^l \circ F_k(\zeta_k)\Big] + o(\varepsilon),$$

where

$$M_k^l \circ F_k(\zeta_k) = R_k^l \circ F_k(z_k) - \left(\frac{F_k(\zeta_k) - a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 Q_k^l(\zeta_k) K_k^l \circ F_k(\zeta_k) - \tag{34}$$
$$- \left[\left(\frac{F_k(\zeta_k) - a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 Q_k^l(\zeta_k) L_k^l \circ F_k(\zeta_k)\right]^-,$$

with that, for $k = 1, \ldots, m$,

$$Q_k^l(\zeta) = I_k^l(\zeta) + \operatorname{Re}\{J_k^l[z_k F_k'(z_k)]\} + [I_k^l(1/\bar{\zeta})]^-, \quad I_k^l(\zeta) = J_k^l[z_k F_k'(z_k)\zeta/(z_k - \zeta)],$$

$$R_k^l \circ F_k(z_k)$$
$$= J_k^l[F_k(z_k)\Phi^l \circ F_k(z_k)] \qquad \text{if } l = 1,$$
$$= J_k^l[F_k(z_k)\varphi^l \circ F_k(z_k)] + \{J_k^l[F_k(z_k)\psi^l \circ F_k(z_k)]\}^- \qquad \text{if } l = 2, 3,$$

$$K_k^l \circ F_k(\zeta_k) = F_k(\zeta_k)\,\Phi_k^l \circ F_k(\zeta_k), \quad L_k^l \circ F_k(\zeta_k) = 0 \quad \text{if } l = 1,$$
$$= F_k(\zeta_k)\,\varphi_k^l \circ F_k(\zeta_k), \quad = F_k(\zeta_k)\,\psi_k^l \circ F_k(\zeta_k) \quad \text{if } l = 2, 3$$

when $p_k \neq 0$ and $q_k \neq 0$, or

$$K_k^l \circ F_k(\zeta_k) = F_k(\zeta_k)\,\varphi_k^l \circ F_k(\zeta_k), \quad L_k^l \circ F_k(\zeta_k) = 0 \quad \text{if } l = 1, 2, 3$$

when $p_k \neq 0$ and $q_k = 0$, or

$$K_k^l \circ F_k(\zeta_k) = F_k(\zeta_k)\,\psi_k^l \circ F_k(\zeta_k), \quad L_k^l \circ F_k(\zeta_k) = 0 \quad \text{if } l = 1,$$
$$= 0, \quad = F_k(\zeta_k)\,\psi_k^l \circ F_k(\zeta_k) \quad \text{if } l = 2, 3$$

when $p_k = 0$ and $q_k \neq 0$. Because α is arbitrary, we have for F a local maximum of $\operatorname{Re} J^l$

$$\sum_{k=1}^{m} M_k^l \circ F_k(\zeta_k) = 0$$

and, consequently, for all $k = 1, \ldots, m$,

$$\left(\frac{F_k(\zeta_k) - a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 Q_k^l(\zeta_k)\, K_k^l \circ F_k(\zeta_k) +$$
$$+ \left\{\left(\frac{F_k(\zeta_k) - a_{0k}}{\zeta_k F_k'(\zeta_k)}\right)^2 Q_k^l(\zeta_k)\, L_k^l \circ F_k(\zeta_k)\right\}^{-} = P_k^l \circ F_k(\zeta_k),$$

where

$$P_k^l \circ F_k(\zeta_k) = R_k^l \circ F_k(z_k) + \sum_{\substack{j=1 \\ j \neq k}}^{m} M_j^l \circ F_k(\zeta_k). \tag{35}$$

In particular,

$$\left(\frac{\zeta_k F_k'(\zeta_k)}{F_k(\zeta_k) - a_{0k}}\right)^2 \check{P}_k^l \circ F_k(\zeta_k) = Q_k^l(\zeta_k),$$

where

$$\check{P}_k^l \circ F_k(\zeta_k) = \left[R_k^l \circ F_k(z_k) + \sum_{\substack{j=1 \\ j \neq k}}^{m} M_j^l \circ F_j(\zeta_j)\right] / K_k^l \circ F_k(\zeta_k) \tag{36}$$

when $l = 1$, $p_k \neq 0$ and $q_k \neq 0$ or $l = 1, 2, 3$, $p_k \neq 0$ and $q_k = 0$ or $l = 1$, $p_k = 0$ and $q_k \neq 0$, or

$$\left(\frac{\zeta_k F_k'(\zeta_k)}{F_k(\zeta_k) - a_{0k}}\right)^2 [\hat{P}_k^l \circ F_k(\zeta_k)]^{-} = [Q_k^l(\zeta_k)]^{-},$$

when

$$\hat{P}_k^l \circ F_k(\zeta_k) = [R_k^l \circ F_k(z_k) + \sum_{\substack{j=1 \\ j \neq k}}^{m} M_j^l \circ F_j(\zeta_j)] / [L_k^l \circ F_k(\zeta_k)]^{-} \tag{37}$$

when $l = 2, 3$, $p_k = 0$ and $q_k \neq 0$.

Let

$$X^l(U) = \sum_{k=1}^{m} R_k^l \circ F_k(z_k),$$

$$Y^l(F) = C \setminus \bigcup_{k=1}^{m} \left[F_k(\Delta) \cup (1/F_k(\Delta))\right] \qquad \text{if } l = 1,$$

$$= \Delta \setminus \bigcup_{k=1}^{m} F_k(\Delta) \qquad \text{if } l = 2,$$

$$= C \setminus \bigcup_{k=1}^{m} \left[F_k(\Delta) \cup (-1/\overline{F_k(\Delta)})\right] \qquad \text{if } l = 3.$$

The following theorems hold:

THEOREM 2.2.7. *Let $l = 1, 2, 3$ be fixed. If J^l is a functional having a complex Gâteaux derivative on C_m^l as in* (33), *and if F is a local maximum with respect to* Re J^l, *then all $z_1, \ldots, z_m \in \Delta \setminus \{0\}$ satisfy the differential equation*

$$\sum_{k=1}^{m} M_k^l \circ F_k(z_k) = 0,$$

where the functions M_k^l, $k = 1, \ldots, m$, are defined in (34). *What is more,* Im $J_k^l[z_k F_k'(z_k)] = 0$ *for $z_k \in \Delta$, $k = 1, \ldots, m$, and $Q_k^l(z) \leqslant 0$ for $z \in \partial\Delta$, $k = 1, \ldots, m$.*

THEOREM 2.2.8. *Let $l = 1, 2, 3$ be fixed. If J^l and F are as those in the preceding theorem, then, for all $z \in \Delta \setminus \{0\}$,*

$$\left(\frac{F_k(z) - a_{0k}}{zF_k'(z)}\right)^2 Q_k^l(z)\, K_k^l \circ F_k(z) + \left[\left(\frac{F_k(z) - a_{0k}}{zF_k'(z)}\right)^2 Q_k^l(z)\, L_k^l \circ F_k(z)\right]^- = P_k^l \circ F_k(z)$$

for all $k = 1, \ldots, m$ and, in particular,

$$\left(\frac{zF_k'(z)}{F_k(z) - a_{0k}}\right)^2 \check{P}_k^l \circ F_k(z) = Q_k^l(z)$$

when $l = 1$, $p_k \neq 0$ and $q_k \neq 0$ or $l = 1, 2, 3$, $p_k \neq 0$ and $q_k = 0$ or $l = 1$, $p_k = 0$ and $q_k \neq 0$, or

$$\left(\frac{zF_k'(z)}{F_k(z) - a_{0k}}\right)^2 [\hat{P}_k^l \circ F_k(z)]^- = [Q_k^l(z)]^-$$

when $l = 2, 3$, $p_k = 0$ and $q_k \neq 0$, where the functions $P_k^l \circ F_k$, $\check{P}_k^l \circ F_k$ and $\hat{P}_k^l \circ F_k$ are defined by equations (35), (36) *and* (37), *respectively, in which $\zeta_k = z$, z is arbitrary, whereas $\zeta_1, \ldots, \zeta_{k-1}, \zeta_{k+1}, \ldots, \zeta_m$ are fixed points of $\Delta \setminus \{0\}$.*

If $X^l(U)$ is not identically zero with respect to the variables $u_1, \ldots, u_m$, the set $Y^l(F)$ has no interior points.

If the mapping $\check{P}_k^l$ ($[\hat{P}_k^l]^-$) is analytic off isolated singularities and not identically zero, then the set $\partial F_k(\Delta)$ lies on the trajectory of the quadratic differential $\check{P}_k^l(w)(w-a_{0k})^{-2}\,dw^2$ ($[\hat{P}_k^l(w)]^-(w-a_{0k})^{-2}\,dw^2$).

The proofs of Theorems 2.2.7 and 2.2.8, as quite analogous to those of Theorems 2.1.5 and 2.1.6, are omitted.

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